Blow up, exponential growth of solution for a reaction-diffusion equation with multiple nonlinearities

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Abstract

In this paper, we consider a reaction diffusion equation with multiple nonlinearities. We prove a blow up and exponential growth of solution with negative initial energy. Our new results generalizes and improves earlier results.

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1 Introduction

In this paper, we consider the following initial-boundary value problem

$$\begin{cases} u_t - \Delta u_t - \operatorname{div}(|\nabla u|^{m-2}\nabla u) + |u|^{q-2}u_t = |u|^{p-2}u, & x \in \Omega,\ t > 0, \\ u(x,t) = 0, & x \in \partial\Omega,\ t \geq 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1}$$

where p, q, $m > 2$ are real numbers and Ω is a bounded domain in R^n $(n \geq 1)$ with smooth boundary $\partial\Omega$ so that the divergence theorem can be applied.

This type of equations are used in many branches of applied science and engineering, for example model of chemical reactions, model of heat transfer, model of a population and biological species and so on we refer to [1, 2, 3] and the reference therein.

In the case of $m = 2$, Zang et al. [4, 5] considered the following problem

$$u_t - \triangle u_t - \triangle u + |u|^{q-2}u_t = |u|^{p-2}u, \tag{2}$$

where $q > 2$, $p > 2$. They studied the blow up and exponential growth properties of the problem (2).

In the case of $m = 2$, Korpusov and Svesnikov [6] discussed the sufficent close-to-necessary conditions for blow up of solutions to the following equation

$$u_t - \triangle u_t - \triangle u + |u|^{q-2}u_t = u(u+\alpha)(u-\beta), \tag{3}$$

where α, $\beta > 0$.

Equation (2) without $|u|^{k-2}u_t$ term become the following problem

$$u_t - \triangle u_t - \triangle u = f(u). \tag{4}$$

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There are a lot of works to discuss the problem (4) (see [7, 8, 9, 10, 11, 12]). They have given the existence, blow up and asymptotic behavior of solution.

In the absence of $\triangle u_t$ and $|u|^{q-2} u_t$ term, Eq. (1) reduced to the following equation

$$u_t - \operatorname{div}(|\nabla u|^{m-2} \nabla u) = f(u) \tag{5}$$

where $m \geq 2$. The existence of global/nonglobal solutions for a (5) was discussed by many authors [13, 14, 15, 16, 17, 18, 19].

This paper is organized as follows: In section 2, we present some notations and stament of assumptions. Section 3 and section 4 are studied to the blow up and the growth of solutions when the initial energy $E(0)$ is negative (this means that our initial data are large enough), respectively. Our new result about blow up substantially improve several results in the literature. The blow up of solution is discussed under some conditions by adopting the method of [20, 21, 22]

2 Preliminaries

In this paper, we denote $\|.\| = \|.\|_{L^2(\Omega)}$, $\|.\|_p = \|.\|_{L^p(\Omega)}$ and $(u, v) = \int_\Omega u(x)v(x)dx$.

For the numbers p and q, we suppose that

$$\begin{cases} 2 < q < p < +\infty & \text{if } n = 1, 2, \\ 2 < q < p \leq \frac{2(n-1)}{n-2} & \text{if } n \geq 3. \end{cases} \tag{6}$$

Similar to [11], we call $u(x,t)$ a solution of problem (1) on $\Omega \times [0,T)$, if

$$\begin{cases} u \in L^\infty(0,T; W_0^{1,m}(\Omega)), \\ u_t \in L^m(0,T; L^m(\Omega)), \\ |u|^{q-2} u_t \in L^2(\Omega \times [0,T)) \end{cases} \tag{7}$$

satisfying the initial condition $u(x,0) = u_0(x)$ and

$$\int_\Omega \left(|\nabla u|^{m-2} \nabla u \nabla v + \nabla u_t \nabla v + u_t v + |u|^{q-2} u_t v - |u|^{p-2} uv \right) dx = 0, \tag{8}$$

for all $v \in C(0,T; W_0^{1,m}(\Omega))$.

We define the energy function as follows

$$E(t) = E(u) = -\frac{1}{p} \|u\|_p^p + \frac{1}{m} \|\nabla u\|_m^m, \tag{9}$$

$$E(0) = -\frac{1}{p} \|u_0\|_p^p + \frac{1}{m} \|\nabla u_0\|_m^m,$$

where $u \in W_0^{1,m}(\Omega)$. Multiplying (1) by u_t and integrating over Ω, we have

$$E'(t) = -\|u_t\|^2 - \|\nabla u_t\|^2 - \int_\Omega |u|^{q-2} u_t^2 dx < 0, \tag{10}$$

and then

$$E(t) \leq E(0). \tag{11}$$

3 Blow up of solutions

In this part, we deal with the blow up results of the solution for the problem (1).

Theorem 3.1. Suppose that (6) holds. Let $u_0 \in W_0^{1,m}(\Omega)$ and u is a local solution of the problem (1), and $E(0) < 0$. Then the solution of the system (1) blows up in finite time.

Proof. Let

$$H(t) = -E(t). \tag{12}$$

By the definition of $H(t)$ and (10), we get

$$H'(t) = -E'(t) \geq 0 \tag{13}$$

Consequently, since $E(0) < 0$, we get

$$H(0) = -E(0) > 0. \tag{14}$$

By the integrate (13), we get

$$0 < H(0) \leq H(t). \tag{15}$$

By using (9) and (12), we get

$$H(t) - \frac{1}{p}\|u\|_p^p = -\frac{1}{m}\|\nabla u\|_m^m < 0, \tag{16}$$

then, we have

$$0 < H(0) \leq H(t) \leq \frac{1}{p}\|u\|_p^p. \tag{17}$$

We define

$$\Psi(t) = H^{1-\sigma}(t) + \frac{\varepsilon}{2}\|u\|^2 + \frac{\varepsilon}{2}\|\nabla u\|^2, \tag{18}$$

where $\varepsilon > 0$ small to be chosen later and $0 < \sigma < (p-q)/p$.

Our goal is to show that $\Psi(t)$ satisfies a differential inequality of the form

$$\Psi'(t) \geq \xi \Psi^{\zeta}(t), \ \zeta > 1.$$

This, of course, will lead to a blow up in finite time.

By taking a derivative of (18) and using Eq. (1), we obtain

$$\begin{aligned}
\Psi'(t) &= (1-\sigma) H^{-\sigma}(t) H'(t) + \varepsilon \left(\int_\Omega u u_t dx + \int_\Omega \nabla u \nabla u_t dx \right) \\
&= (1-\sigma) H^{-\sigma}(t) H'(t) + m\varepsilon H(t) + m\varepsilon E(t) \\
&\quad -\varepsilon \|\nabla u\|_m^m + \varepsilon \|u\|_p^p - \varepsilon \int_\Omega |u|^{q-2} u u_t dx \\
&\geq (1-\sigma) H^{-\sigma}(t) H'(t) + m\varepsilon H(t) \\
&\quad +\varepsilon \left(1 - \frac{m}{p}\right) \|u\|_p^p - \varepsilon \int_\Omega |u|^{q-2} u u_t dx.
\end{aligned} \tag{19}$$

In order to estimate the last term in (19), we make use of the following Young's inequality

$$XY \leq \delta^{-1} X^2 + \delta Y^2, \tag{20}$$

for $\delta > 0$. So, we find

$$\begin{aligned} \int_\Omega |u|^{q-2} u u_t dx &\leq \int_\Omega |u|^{\frac{q-2}{2}} u_t |u|^{\frac{q-2}{2}} u dx \\ &\leq \delta^{-1} \int_\Omega |u|^{q-2} u_t^2 dx + \delta \int_\Omega |u|^q dx. \end{aligned} \tag{21}$$

and therefore, (19) becomes

$$\begin{aligned} \Psi'(t) &\geq (1-\sigma) H^{-\sigma}(t) H'(t) + m\varepsilon H(t) + \varepsilon \left(1 - \frac{m}{p}\right) \|u\|_p^p \\ &\quad -\varepsilon\delta \|u\|_q^q - \delta^{-1} \int_\Omega |u|^{q-2} u_t^2 dx. \end{aligned} \tag{22}$$

By taking δ such that $\delta^{-1} = MH^{-\sigma}(t)$ for M enough large constants to be fixed later, and by using (10), we have

$$\begin{aligned} \Psi'(t) &\geq (1-\sigma-M\varepsilon) H^{-\sigma}(t) H'(t) + \varepsilon m H(t) \\ &\quad +\varepsilon(1 - \frac{m}{p}) \|u\|_p^p - \varepsilon M^{-1} H^{\sigma}(t) \|u\|_q^q. \end{aligned} \tag{23}$$

Thanks to the embeddings $L^p \hookrightarrow L^q \hookrightarrow L^2$ $(p > q > 2)$, taking into account (17), we obtain,

$$H^{\sigma}(t) \|u\|_q^q \leq c_1 \|u\|_p^{p\sigma} \|u\|_q^q \leq c_2 \|u\|_p^{p\sigma+q}. \tag{24}$$

for some positive constants c_1 and c_2.

Since $0 < \frac{q}{p} < 1$, now applying the following algebraic inequality

$$x^l \leq (x+1) \leq (1 + \frac{1}{z})(x+z), \quad \forall x \geq 0,\ 0 \leq l \leq 1,\ z > 0,$$

especially, by the selection of σ, taking $x = \|u\|_p^p$, $l = (p\sigma + q)/p$, $z = H(0)$, we get

$$\begin{aligned} \|u\|_p^{p\sigma+q} &\leq (1 + \frac{1}{H(0)})(\|u\|_p^p + H(0)) \\ &\leq c_3 \|u\|_p^p. \end{aligned} \tag{25}$$

Taking into account (23) and (25), we have

$$\Psi'(t) \geq (1-\sigma-M\varepsilon) H^{-\sigma}(t) H'(t) + \varepsilon m H(t) + \varepsilon(1 - \frac{m}{p} - c_3 M^{-1}) \|u\|_p^p. \tag{26}$$

For large enough M such that $1 - \frac{m}{p} - c_3 M^{-1} = c_4 > 0$, once M is fixed, we pick ε small enough such that $1 - \sigma - M\varepsilon > 0$, (26) become

$$\Psi'(t) \geq c_5 \left(H(t) + \|u\|_p^p \right). \tag{27}$$

for $c_5 > 0$.

Then, we arrive at

$$\Psi(t) \geq \Psi(0) > 0, \ \forall t \geq 0. \tag{28}$$

We now need to estimate $\Psi(t)^{\frac{1}{1-\sigma}}$. By using (18), the algebraic inequality and thanks to the embedding $L^m \hookrightarrow L^2$ $(m > 2)$, we obtain

$$\begin{aligned} \|\nabla u\|_2^{\frac{2}{1-\sigma}} &\leq \|\nabla u\|_m^{\frac{2}{1-\sigma}} \\ &\leq C\left(\|\nabla u\|_m^m\right)^{\frac{2}{m(1-\sigma)}} \\ &\leq \left(1+\frac{1}{H(0)}\right)\left(\|\nabla u\|_m^m + H(0)\right) \\ &\leq C\left(\|\nabla u\|_m^m + H(t)\right). \end{aligned} \tag{29}$$

From definition of $H(t)$, we get

$$\|\nabla u\|_m^m = -mH(t) - \frac{m}{p}\|u\|_p^p. \tag{30}$$

Similarly, as the embedding $L^p \hookrightarrow L^2$ $(p > 2)$, we get

$$\|u\|_2^{\frac{2}{1-\sigma}} \leq \|u\|_p^{\frac{2}{1-\sigma}},$$

and by using algebraic inequality for $x = \|u\|_p^p$, $l = \frac{2}{p(1-\sigma)} < 1$, $z = H(0)$ and since $\sigma < \frac{p-q}{p} < \frac{p-2}{p}$, we get

$$\begin{aligned} \|u\|_p^{\frac{2}{(1-\sigma)}} &= \left(\|u\|_p^p\right)^{\frac{2}{p(1-\sigma)}} \\ &\leq (1+\frac{1}{H(0)})\left(\|u\|_p^p + H(0)\right) \\ &\leq C\|u\|_p^p. \end{aligned} \tag{31}$$

By (29) - (31) and using the following inequality

$$(a+b+c)^\lambda \leq C\left(a^\lambda + b^\lambda + c^\gamma\right), \quad a,b,c > 0, \ \gamma \geq 1$$

we obtain

$$\begin{aligned} \Psi^{\frac{1}{1-\sigma}}(t) &= \left[H^{1-\sigma}(t) + \frac{\varepsilon}{2}\left(\|u\|^2 + \|\nabla u\|^2\right)\right]^{\frac{1}{1-\sigma}} \\ &\leq C\left(H(t) + \|u\|_p^p\right). \end{aligned} \tag{32}$$

By combining of (27) and (32), we arrive at

$$\Psi'(t) \geq \xi\Psi^{\frac{1}{1-\sigma}}(t), \tag{33}$$

where ξ is a positive constant.

A simple integration of (33) over $(0,t)$ yields $\Psi^{\frac{\sigma}{1-\sigma}}(t) \geq \frac{1}{\Psi^{-\frac{\sigma}{1-\sigma}}(0)-\frac{\xi\sigma t}{1-\sigma}}$, which implies that the solution blows up in a finite time T^*, with

$$T^* \leq \frac{1-\sigma}{\xi\sigma\Psi^{\frac{\sigma}{1-\sigma}}(0)}.$$

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4 Exponential growth of solution

In this part, we state and prove exponential growth result.

Theorem 4.1. Suppose that (6) holds. Let $u_0 \in W_0^{1,m}(\Omega)$ and $E(0) < 0$. Then the solution of the problem (1) grows exponentially.

Proof. We define

$$G(t) = H(t) + \frac{\varepsilon}{2}\|u\|^2 + \frac{\varepsilon}{2}\|\nabla u\|^2, \tag{34}$$

where $H(t) = -E(t)$. By taking the time derivative of (34) and by (1), we get

$$\begin{aligned} G'(t) &= H'(t) + \varepsilon\left(\int_\Omega u u_t dx + \int_\Omega \nabla u \nabla u_t dx\right) \\ &= \|\nabla u_t\|^2 + \|u_t\|^2 + \int_\Omega |u|^{q-2} u_t^2 dx \\ &\quad -\varepsilon\|\nabla u\|_m^m + \varepsilon\|u\|_p^p - \varepsilon\int_\Omega |u|^{q-2} u u_t dx \\ &\geq \|u_t\|^2 + \int_\Omega |u|^{q-2} u_t^2 dx + \varepsilon\|u\|_p^p - \varepsilon\|\nabla u\|_m^m - \varepsilon\int_\Omega |u|^{q-2} u u_t dx. \end{aligned} \tag{35}$$

In order to estimate the last term in (35). Applying the above Young inequality, we find

$$\begin{aligned} \int_\Omega |u|^{q-2} u u_t dx &\leq \int_\Omega |u|^{\frac{q-2}{2}} u_t |u|^{\frac{q-2}{2}} u dx \\ &\leq \delta^{-1}\int_\Omega |u|^{q-2} u_t^2 dx + \delta\int_\Omega |u|^q dx. \end{aligned}$$

Therefore, we have

$$G'(t) \geq \|u_t\|^2 - \varepsilon\|\nabla u\|_m^m + \varepsilon\|u\|_p^p - \varepsilon\delta\|u\|_q^q + \left(1-\varepsilon\delta^{-1}\right)\int_\Omega |u|^{q-2} u_t^2 dx. \tag{36}$$

From the definition $H(t)$, we have

$$\|u\|_p^p = -pH(t) + \frac{p}{m}\|\nabla u\|_m^m.$$

Hence, (36) becomes

$$\begin{aligned} G'(t) &\geq \|u_t\|^2 - \varepsilon \|\nabla u\|_m^m + \varepsilon \left[pH(t) + \frac{p}{m} \|\nabla u\|_m^m \right] \\ &\quad -\varepsilon\delta \|u\|_q^q + \left(1 - \varepsilon\delta^{-1}\right) \int_\Omega |u|^{q-2} u_t^2 dx \\ &\geq \|u_t\|^2 + \left(1 - \varepsilon\delta^{-1}\right) \int_\Omega |u|^{q-2} u_t^2 dx \\ &\quad +\varepsilon a_1 \|\nabla u\|_m^m + \varepsilon p H(t) - \varepsilon\delta \|u\|_q^q \end{aligned} \tag{37}$$

where $a_1 = \frac{p}{m} - 1 > 0$.

Since $p > q > m > 2$ and by the embedding theorem, then by Hölder's inequality

$$\|u\|_q^q \leq C \|u\|_p^q \leq C \left(\|u\|_p^p \right)^{\frac{q}{p}}, \tag{38}$$

where C is a positive constant. Since $0 < \frac{q}{p} < 1$, now applying the following algebraic inequality

$$x^l \leq (x+1) \leq (1 + \frac{1}{z})(x+z), \quad \forall x \geq 0,\ 0 \leq l \leq 1,\ z > 0,$$

in particular, taking $x = \|u\|_p^p$, $l = q/p$, $z = H(0)$, we get

$$\left(\|u\|_p^p \right)^{\frac{q}{p}} \leq (1 + \frac{1}{H(0)})(\|u\|_p^p + H(0)),$$

then, from (38) and (17), we get

$$\|u\|_q^q \leq C \|u\|_p^q \leq C_1 \|u\|_p^p. \tag{39}$$

So, we have

$$G'(t) \geq \|u_t\|^2 + \left(1 - \varepsilon\delta^{-1}\right) \int_\Omega |u|^{q-2} u_t^2 dx + \varepsilon a_1 \|\nabla u\|_m^m + \varepsilon p H(t) - \varepsilon\delta C_1 \|u\|_p^p. \tag{40}$$

Taking $ma_2 = a_1 > 0$, and noting that $mH(t) = \frac{m}{p} \|u\|_p^p - \|\nabla u\|_m^m$, we get

$$\begin{aligned} G'(t) &\geq \|u_t\|^2 + \left(1 - \varepsilon\delta^{-1}\right) \int_\Omega |u|^{q-2} u_t^2 dx + \varepsilon (a_1 - a_2) \|\nabla u\|_m^m \\ &\quad +\varepsilon a_2 \|\nabla u\|_m^m + \varepsilon p H(t) - \varepsilon\delta C_1 \|u\|_p^p \\ &\geq \|u_t\|^2 + \left(1 - \varepsilon\delta^{-1}\right) \int_\Omega |u|^{q-2} u_t^2 dx + \varepsilon (a_1 - a_2) \|\nabla u\|_m^m + \varepsilon p H(t) \\ &\quad -\varepsilon\delta C_1 \|u\|_p^p + \varepsilon a_2 \left[\|\nabla u\|_m^m - \frac{m}{p} \|u\|_p^p \right] + \varepsilon a_2 \frac{m}{p} \|u\|_p^p \\ &= \|u_t\|^2 + \left(1 - \varepsilon\delta^{-1}\right) \int_\Omega |u|^{q-2} u_t^2 dx + \varepsilon (m-1) a_2 \|\nabla u\|_m^m \\ &\quad +\varepsilon (p - ma_2) H(t) + \varepsilon \left(\frac{m}{p} a_2 - \delta C_1 \right) \|u\|_p^p. \end{aligned} \tag{41}$$

Taking δ small enough such that $\frac{m}{p}a_2-\delta C_1>0$, then we pick ε small enough such that $1-\varepsilon\delta^{-1}>0$, also noting that

$$\begin{aligned} p-ma_2 &= p-a_1 \\ &= p-\frac{p}{m}-1 \\ &= \frac{p(m-1)}{m}-1>0, \end{aligned}$$

then

$$G'(t)\geq C\left(H(t)+\|u_t\|^2+\|\nabla u\|_m^m+\|u\|_p^p\right). \tag{42}$$

Thus, the functional $G(t)$ is strictly positive and increasing for all $t\geq 0$.

Conversely, from $G(t)$ function we obtain

$$\begin{aligned} G(t) &= H(t)+\frac{\varepsilon}{2}\|u\|^2+\frac{\varepsilon}{2}\|\nabla u\|^2 \\ &\leq C(H(t)+\|\nabla u\|^2) \end{aligned} \tag{43}$$

where C is positive contant.

Now, we estimate

$$\begin{aligned} \|\nabla u\|_2^2 &\leq C\|\nabla u\|_m^2\leq C\left(\|\nabla u\|_m^m\right)^{\frac{2}{m}} \\ &\leq \left(1+\frac{1}{H(0)}\right)\left(\|\nabla u\|_m^m+H(0)\right) \\ &\leq C\left(\|\nabla u\|_m^m+H(t)\right), \end{aligned} \tag{44}$$

then, we obtain

$$\|\nabla u\|_2^2\leq C\|\nabla u\|_m^2\leq C_1\|\nabla u\|_m^m.$$

So, we have

$$\begin{aligned} G(t) &\leq C\left(H(t)+\|\nabla u\|_m^m\right) \\ &\leq C\left(H(t)+\|\nabla u\|_m^m+\|u_t\|^2+\|u\|_p^p\right). \end{aligned} \tag{45}$$

From (45) and (42) we arrive at

$$G'(t)\geq rG(t) \tag{46}$$

where r is a positive constant.

Integration of (46) over $(0,t)$ gives us

$$G(t)\geq G(0)\exp(rt).$$

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