# On a fourth order rational difference equation

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**Abstract**

In this paper, we determine and study the behavior of all admissible solutions of the difference equation

$$x_{n+1} = \frac{x_n x_{n-2}}{a x_{n-2} + b x_{n-3}}, \quad n = 0, 1, \ldots,$$

where $a, b$ are positive real numbers and the initial conditions $x_{-3}, x_{-2}, x_{-1}, x_0$ are real numbers. We show when $a = b = 1$ that, every admissible solution converges to 0.

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## 1 Introduction

In [2] and [3] respectively, we have discused the global behavior of all solutions of the difference equations

$$x_{n+1} = \frac{a x_n x_{n-2}}{b x_n + c x_{n-3}}, \quad n = 0, 1, \ldots,$$

and

$$x_{n+1} = \frac{a x_n x_{n-2}}{-b x_n + c x_{n-3}}, \quad n = 0, 1, \ldots,$$

where $a, b, c$ are positive real numbers and the initial conditions $x_{-3}, x_{-2}, x_{-1}, x_0$ are real numbers.

In this paper, we discuss the global behavior of all solutions of the difference equation

$$x_{n+1} = \frac{x_n x_{n-2}}{a x_{n-2} + b x_{n-3}}, \quad n = 0, 1, \ldots, \tag{1.1}$$

where $a, b$ are positive real numbers and the initial conditions $x_{-3}, x_{-2}, x_{-1}, x_0$ are real numbers.

Recently, there has been a great interest in studying properties of nonlinear and rational difference equations. One can see [1], [4]- [26] and the references therein.

## 2 Forbidden set and solutions of equation (1.1)

In this section we derive the forbidden set and introduce an explicit formula for the solutions of the difference equation (1.1).

**Proposition 2.1.** The forbidden set $F$ of equation (1.1) is

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$$F = \bigcup_{n=0}^{\infty}\{(v_0,v_{-1},v_{-2},v_{-3}) : v_0 = v_{-1}(-\frac{b}{\sum_{i=0}^{n}(\frac{1}{b})^i})\}$$
$$\cup \bigcup_{m=0}^{\infty}\{(v_0,v_{-1},v_{-2},v_{-3}) : v_{-1} = v_{-2}(-\frac{b}{\sum_{i=0}^{m}(\frac{1}{b})^i})\}$$
$$\cup \bigcup_{l=0}^{\infty}\{(v_0,v_{-1},v_{-2},v_{-3}) : v_{-2} = v_{-3}(-\frac{b}{\sum_{i=0}^{l}(\frac{1}{b})^i})\}$$
$$\cup \{(v_0,v_{-1},v_{-2},v_{-3}) : v_0 = 0\}$$
$$\cup \{(v_0,v_{-1},v_{-2},v_{-3}) : v_{-1} = 0\}$$
$$\cup \{(v_0,v_{-1},v_{-2},v_{-3}) : v_{-2} = 0\}.$$

*Proof.* Suppose that $x_0x_{-1}x_{-2} = 0$ and $x_{-3} \neq 0$. Then we have the following:
If $x_0 = 0$ and $x_{-1}x_{-2} \neq 0$, then $x_4$ is undefined.
If $x_{-1} = 0$ and $x_0x_{-2} \neq 0$, then $x_6$ is undefined.
If $x_{-2} = 0$ and $x_0x_{-1} \neq 0$, then we have either $x_1$ is undefined when $x_{-3} = 0$ or $x_5$ is undefined when $x_{-3} \neq 0$.
Now suppose that $x_{-t} \neq 0$ for all $t \in \{0,1,2,3\}$. From equation (1.1), using the substitution $z_n = \frac{x_{n-1}}{x_n}$, we can obtain the second order difference equation

$$z_{n+1} = bz_{n-2} + a, \quad z_{-2} = \frac{x_{-3}}{x_{-2}}, z_{-1} = \frac{x_{-2}}{x_{-1}}, z_0 = \frac{x_{-1}}{x_0}.$$

This equation can be written

$$z_{3m+i} = bz_{3m+i-3} + a \tag{2.1}$$

where

$$z_{-2} = \frac{x_{-3}}{x_{-2}}, z_{-1} = \frac{x_{-2}}{x_{-1}}, z_0 = \frac{x_{-1}}{x_0}, \ i = 1,2,3 \text{ and } m = 0,1,....$$

Now consider the function $f(x) = bx + a$ and suppose that we start from an initial point $(x_0, x_{-1}, x_{-2}, x_{-3})$.
The backward orbits, $\frac{x_{n-1}}{x_n} = u_n$ satisfy the equation

$$u_n = f^{-1}(u_{n-3}) = \frac{1}{b}u_{n-3} - \frac{a}{b} \text{ with } u_{-3+i} = \frac{x_{-4+i}}{x_{-3+i}}, \ i = 1,2,3. \tag{2.2}$$

Equation (2.2), by setting $n = 3m + i$, $i \in \{1,2,3\}$ can be written as

$$u_{3m+i} = \frac{1}{b}u_{3m+i-3} - \frac{a}{b}, \quad u_{-3+i} = \frac{x_{-4+i}}{x_{-3+i}}, i = 1,2,3.$$

If for a certain $i \in \{1,2,3\}$ we have $u_{-3+i} = \frac{x_{-4+i}}{x_{-3+i}} = -\frac{a}{b}$, then we obtain

$$u_{3(m-1)+i} = \frac{x_{3(m-1)+i-1}}{x_{3(m-1)+i}} = -\frac{a}{b}\sum_{j=0}^{m}(\frac{a}{b})^j.$$

Then $x_{3(m-1)+i} = x_{3(m-1)+i-1}(-\frac{b}{a\sum_{j=0}^{m}(\frac{1}{b})^j})$.
On the other hand, we can observe that if we start from an initial point $(x_0, x_{-1}, x_{-2}, x_{-3})$ such that $z_{-3+i} = -\frac{a}{b}\sum_{j=0}^{m_0}(\frac{1}{b})^j$ for any $i \in \{1,2,3\}$ and for a certain $m_0 \in \mathbb{N}$, then according to equation (2.1) we obtain

$$z_{3m_0+i-3} = \frac{x_{3m_0+i-4}}{x_{3m_0+i-3}} = -\frac{a}{b}.$$

This implies that $ax_{3m_0+i-3} + bx_{3m_0+i-4} = 0$. Therefore, $x_{3m_0+i}$ is undefined.
This completes the proof. Q.E.D.

**Theorem 2.2.** Let $x_{-3}, x_{-2}, x_{-1}$ and $x_0$ be real numbers such that $(x_0, x_{-1}, x_{-2}, x_{-3}) \notin F$. If $b \neq 1$, then the solution $\{x_n\}_{n=-3}^{\infty}$ of equation (1.1) is

$$x_n = \begin{cases} x_0 \frac{1-b}{b^{\frac{n-1}{3}+1}\theta_1+a} \prod_{j=0}^{\frac{n-1}{3}-1} \frac{(1-b)^3}{\prod_{t=1}^{3}(b^{j+1}\theta_t+a)} & , n = 1,4,7,..., \\ x_0 \frac{1-b}{b^{\frac{n-2}{3}+1}\theta_1+a} \frac{1-b}{b^{\frac{n-2}{3}+1}\theta_2+a} \prod_{j=0}^{\frac{n-1}{3}-1} \frac{(1-b)^3}{\prod_{t=1}^{3}(b^{j+1}\theta_t+a)} & , n = 2,5,8,..., \\ x_0 \prod_{j=0}^{\frac{n-1}{3}} \frac{(1-b)^3}{\prod_{t=1}^{3}(b^{j+1}\theta_t+a)} & , n = 3,6,9,..., \end{cases} \tag{2.3}$$

where $\theta_t = \frac{1-b-a\alpha_t}{\alpha_t}$ and $\alpha_t = \frac{x_{-3+t}}{x_{-4+t}}$, $t = 1,2,3$.

*Proof.* We can write the given solution (2.3) as

$$x_{3m+i} = x_{3m} \prod_{t=1}^{i} \frac{1-b}{b^{m+1}\theta_t + a}, \quad i = 1,2,3 \tag{2.4}$$

where

$$x_{3m} = x_0 \prod_{j=0}^{m-1} \frac{(1-b)^3}{\prod_{t=1}^{3}(b^{j+1}\theta_t + a)}, \quad m = 0,1,....$$

We can see with $m = 0$ in (2.4) that

$$\begin{aligned} x_1 &= x_0 \frac{1-b}{b\theta_1 + a} = x_0 \frac{1-b}{b\frac{1-b-a\alpha_t}{\alpha_1} + a} \\ &= x_0 \frac{\alpha_1(1-b)}{b(1-b) - ba\alpha_1 + a\alpha_1} = x_0 \frac{\alpha_1}{b + a\alpha_1} = \frac{x_0 x_{-2}}{ax_{-2} + bx_{-3}}, \end{aligned}$$

$$\begin{aligned} x_2 &= x_0 \frac{(1-b)^2}{(b\theta_1 + a)(b\theta_2 + a)} = x_0 \frac{(1-b)^2}{(b\frac{1-b-a\alpha_1}{\alpha_1} + a)(b\frac{1-b-a\alpha_2}{\alpha_2} + a)} \\ &= x_0 \frac{(1-b)^2\alpha_1\alpha_2}{(1-b)^2(b + a\alpha_1)(b + a\alpha_2)} = x_0 \frac{\frac{x_{-2}}{x_{-3}}}{b + a\frac{x_{-2}}{x_{-3}}} \frac{\frac{x_{-1}}{x_{-2}}}{b + a\frac{x_{-1}}{x_{-2}}} \\ &= \frac{x_0 x_{-2}}{ax_{-2} + bx_{-3}} \frac{x_{-1}}{ax_{-1} + bx_{-2}} = \frac{x_1 x_{-1}}{ax_{-1} + bx_{-2}} \end{aligned}$$

and

$$x_3 = x_0 \frac{(1-b)^3}{(b\theta_1+a)(b\theta_2+a)(b\theta_3+a)} = x_0 \frac{(1-b)^3}{(b\frac{1-b-a\alpha_1}{\alpha_1}+a)(b\frac{1-b-a\alpha_2}{\alpha_2}+a)(b\frac{1-b-a\alpha_3}{\alpha_3}+a)}$$
$$= x_0 \frac{(1-b)^3\alpha_1\alpha_2\alpha_3}{(1-b)^3(b+a\alpha_1)(b+a\alpha_2)(b+a\alpha_3)} = x_0 \frac{\frac{x_{-2}}{x_{-3}}}{b+a\frac{x_{-2}}{x_{-3}}} \frac{\frac{x_{-1}}{x_{-2}}}{b+a\frac{x_{-1}}{x_{-2}}} \frac{\frac{x_0}{x_{-1}}}{b+a\frac{x_0}{x_{-1}}}$$
$$= \frac{x_0x_{-2}}{ax_{-2}+bx_{-3}} \frac{x_{-1}}{ax_{-1}+bx_{-2}} \frac{x_0}{ax_0+bx_{-1}} = \frac{x_2x_0}{ax_0+bx_{-1}}.$$

Now assume that $m \geq 1$. Then

$$\frac{x_{3m+3}x_{2m+1}}{ax_{3m+1}+bx_{3m}} = \frac{x_{3m}(\prod_{t=1}^{3} \frac{1-b}{b^{m+1}\theta_t+a})x_{3m}\frac{1-b}{b^{m+1}\theta_1+a}}{ax_{3m}\frac{1-b}{b^{m+1}\theta_1+a}+bx_{3m}}$$
$$= x_{3m}\frac{(1-b)(\prod_{t=1}^{3} \frac{1-b}{b^{m+1}\theta_t+a})}{a(1-b)+b^{m+2}\theta_1+ba}$$
$$= x_{3m}(\prod_{t=1}^{3} \frac{1-b}{b^{m+1}\theta_t+a})\frac{1-b}{b^{m+2}\theta_1+a} = x_{3(m+1)}\frac{1-b}{b^{m+2}\theta_1+a} = x_{3(m+1)+1}.$$

This completes the proof. Q.E.D.

## 3 Global behavior of equation (1.1)

In this section, we investigate the global behavior of equation (1.1) with $b \neq 1$, using the explicit formula of its solution.

During this section, we assume that all solutions are admissible ($(x_0, x_{-1}, x_{-2}, x_{-3}) \notin F$).

Let

$$\beta_j = \frac{(1-b)^3}{\prod_{t=1}^{3} b^{j+1}\theta_t+a}, \quad j = 0, 1, ....$$

The we can write

$$x_{3m+i} = x_{3m}\prod_{t=1}^{i} \frac{1-b}{b^{m+1}\theta_t+a}, \quad i = 1, 2, 3 \tag{3.1}$$

where

$$x_{3m} = x_0\prod_{j=0}^{m-1} \beta_j, \quad m = 0, 1, ....$$

**Theorem 3.1.** Let $\{x_n\}_{n=-3}^{\infty}$ be a solution of equation (1.1). Then we have the following:

1. If $b > 1$, then $\{x_n\}_{n=-3}^{\infty}$ converges to 0.
2. If $b < 1$, then we have the following:
   (a) If $1 - b < a$, then $\{x_n\}_{n=-3}^{\infty}$ converges to 0.
   (b) If $1 - b > a$, then the subsequences $\{x_{3m+i}\}_{m=-1}^{\infty}$ and $i = 1, 2, 3$ are unbounded.

*Proof.* 1. If $b > 1$, then $b^j \to \infty$ as $j \to \infty$. This implies that $\beta_j$ converges to 0 as $j \to \infty$.
It follows for a given $0 < \varepsilon < 1$ that, there exists $j_0 \in \mathbb{N}$ such that $\mid \beta_j \mid < \varepsilon$ for all $j \geq j_0$.
Therefore,

$$\begin{aligned} \mid x_{3m} \mid &= \mid x_0 \mid\mid \prod_{j=0}^{m-1} \beta_j \mid \\ &= \mid x_0 \mid\mid \prod_{j=0}^{j_0-1} \beta_j \mid\mid \prod_{j=j_0}^{m-1} \beta_j \mid \\ &< \mid x_0 \mid\mid \prod_{j=0}^{j_0-1} \beta_j \mid \varepsilon^{m-j_0}. \end{aligned}$$

As $m$ tends to infinity, $x_{3m}$ converges to 0.
Now for each $i \in \{1, 2, 3\}$ we have

$$\mid x_{3m+i} \mid = \mid x_{3m} \mid\mid \prod_{t=1}^{i} \frac{1-b}{b^{j+1}\theta_t + a} \mid \to 0 \text{ as } m \to \infty.$$

Therefore, $\{x_n\}_{n=-3}^{\infty}$ converges to 0.

2. If $b < 1$ (that is $\beta_j$ converges to $(\frac{1-b}{a})^3$), then we have the following:

   (a) If $1 - b < a$, then $\beta_j$ converges to $(\frac{1-b}{a})^3 < 1$ as $j \to \infty$. This implies that, there exists $j_1 \in \mathbb{N}$ such that $\beta_j < \varepsilon_1$, where $(\frac{1-b}{a})^3 < \varepsilon_1 < 1$ for all $j \geq j_1$. Therefore, the solution $\{x_n\}_{n=-3}^{\infty}$ converges to 0 as in (1).

   (b) If $1 - b > a$, then $\beta(j)$ converges to $(\frac{1-b}{a})^3 > 1$ as $j \to \infty$. Then for a given $1 < \varepsilon_2 < (\frac{1-b}{a})^3$ there exists $j_2 \in \mathbb{N}$ such that $\beta_j > \varepsilon_2 > 1$ for all $j \geq j_2$.
   For large values of $m$ we have

$$\begin{aligned} \mid x_{3m} \mid &= \mid x_0 \mid\mid \prod_{j=0}^{m-1} \beta_j \mid \\ &= \mid x_0 \mid\mid \prod_{j=0}^{j_2-1} \beta_j \mid\mid \prod_{j=j_2}^{m-1} \beta_j \mid \\ &> \mid x_0 \mid\mid \prod_{j=0}^{j_2-1} \beta_j \mid \varepsilon_2^{m-j_2}. \end{aligned}$$

   As $m \to \infty$, $\mid x_{3m} \mid \to \infty$.
   Now for each $i \in \{1, 2, 3\}$ we have

$$\mid x_{3m+i} \mid = \mid x_{3m} \mid\mid \prod_{t=1}^{i} \frac{1-b}{b^{j+1}\theta_t + a} \mid \to \infty \text{ as } m \to \infty.$$

   Therefore, the subsequences $\{x_{3m+i}\}_{m=-1}^{\infty}$, $i \in \{1, 2, 3\}$ are unbounded.

Q.E.D.

**Theorem 3.2.** Assume that $\{x_n\}_{n=-3}^{\infty}$ is a solution of equation (1.1). If $1-b=a$, then the solution $\{x_n\}_{n=-3}^{\infty}$ converges to a finite limit.

*Proof.* Suppose that $\{x_n\}_{n=-3}^{\infty}$ is a solution of equation (1.1) such that $1-b=a$. As

$$\lim_{j\to\infty}\beta_j=\frac{(1-b)^3}{a^3}=1, \quad (\text{where} \quad b<1),$$

there exists $j_3\in\mathbb{N}$ such that, $\beta_j>0$, for all $j\geq j_3$.
Hence

$$\begin{aligned} x_{3m}&=x_0\prod_{j=0}^{m-1}\beta_j=x_0\prod_{j=0}^{j_3-1}\beta_j\prod_{j=j_3}^{m-1}\beta_j\\ &=x_0\prod_{j=0}^{j_3-1}\beta_j\exp\Big(\sum_{j=j_3}^{m-1}\ln(\beta_j)\Big).\end{aligned}$$

We shall test the convergence of the series $\sum_{j=j_3}^{\infty}\mid\ln(\beta_j)\mid$.
Since $\lim_{j\to\infty}\mid\frac{\ln(\beta_{j+1})}{\ln(\beta_j)}\mid=\frac{0}{0}$, using L'Hospital's rule we obtain
$\lim_{j\to\infty}\mid\frac{\ln(\beta_{j+1})}{\ln(\beta_j)}\mid=b<1$.
It follows from D' Alembert's test that the series $\sum_{j=j_3}^{\infty}\mid\ln(\beta_j)\mid$ is convergent.
This ensures that there is a real number $L$ such that

$$\lim_{m\to\infty}x_{3m}=L.$$

Now for each $i\in\{1,2,3\}$, we get

$$\lim_{m\to\infty}x_{3m+i}=\lim_{m\to\infty}x_{3m}\prod_{t=1}^{i}\frac{1-b}{b^{m+1}\theta_t+a}=L.$$

Therefore, the solution $\{x_n\}_{n=-3}^{\infty}$ converges to a finite limit. Q.E.D.

**Theorem 3.3.** Assume that $\{x_n\}_{n=-3}^{\infty}$ is a solution of equation (1.1). If $\alpha_t=\frac{1-b}{a}$ for all $t\in\{1,2,3\}$, then we have the following:

1. If $1-b<a$, then $\{x_n\}_{n=-3}^{\infty}$ converges to 0.
2. If $1-b>a$, then $\{x_n\}_{n=-3}^{\infty}$ is unbounded.
3. If $1-b=a$, then $x_n=x_0$ for all $n\geq 0$.

*Proof.* The condition $\alpha_t=\frac{1-b}{a}$ for all $t\in\{1,2,3\}$ implies that $\theta_t=0$ for all $t\in\{1,2,3\}$.
It is sufficient to see that

$$x_{3m}=x_0\Big(\frac{1-b}{a}\Big)^{3m}, \quad m=0,1,...,$$

and the rest of the proof is a direct consequence.

Q.E.D.

## 4 Case $a = b = 1$

We end this work by considering the case when $a = b = 1$.

**Proposition 4.1.** Assume that $a = b = 1$. Then the forbidden set $G$ of equation (1.1) is

$$\begin{aligned} G = & \bigcup_{n=0}^{\infty} \{(v_0, v_{-1}, v_{-2}, v_{-3}) : v_0 = v_{-1}(-\frac{1}{n+1})\} \\ & \cup \bigcup_{m=0}^{\infty} \{(v_0, v_{-1}, v_{-2}, v_{-3}) : v_{-1} = v_{-2}(-\frac{1}{m+1})\} \\ & \cup \bigcup_{l=0}^{\infty} \{(v_0, v_{-1}, v_{-2}, v_{-3}) : v_{-2} = v_{-3}(-\frac{1}{l+1})\} \\ & \cup \{(v_0, v_{-1}, v_{-2}, v_{-3}) : v_0 = 0\} \\ & \cup \{(v_0, v_{-1}, v_{-2}, v_{-3}) : v_{-1} = 0\} \\ & \cup \{(v_0, v_{-1}, v_{-2}, v_{-3}) : v_{-2} = 0\}. \end{aligned}$$

We assume that for all solutions $\{x_n\}_{n=-3}^{\infty}$ in this section, $(x_0, x_{-1}, x_{-2}, x_{-3}) \notin G$.

**Theorem 4.2.** Let $\{x_n\}_{n=-3}^{\infty}$ be a solution of equation (1.1) with $a = b = 1$. Then

$$x_n = \begin{cases} x_0 \frac{\alpha_1}{1+(\frac{n-1}{3}+1)\alpha_1} \prod_{j=0}^{\frac{n-1}{3}-1} \frac{\alpha_1\alpha_2\alpha_3}{\prod_{t=1}^{3}(1+(j+1)\alpha_t)} & , n = 1, 4, 7, ..., \\ x_0 \frac{\alpha_1}{1+(\frac{n-2}{3}+1)\alpha_1} \frac{\alpha_2}{1+(\frac{n-2}{3}+1)\alpha_2} \prod_{j=0}^{\frac{n-2}{3}-1} \frac{\alpha_1\alpha_2\alpha_3}{\prod_{t=1}^{3}(1+(j+1)\alpha_t)} & , n = 2, 5, 8, ..., \\ x_0 \prod_{j=0}^{\frac{n-3}{3}} \frac{\alpha_1\alpha_2\alpha_3}{\prod_{t=1}^{3}(1+(j+1)\alpha_t)} & , n = 3, 6, 9, ..., \end{cases} \tag{4.1}$$

where $\alpha_t = \frac{x_{-3+t}}{x_{-4+t}}, \quad t = 1, 2, 3.$

**Theorem 4.3.** If $a = b = 1$, then every admissible solution of equation (1.1) converges to 0.

*Proof.* Assume that $\{x_n\}_{n=-3}^{\infty}$ is a solution of equation (1.1) such that $(x_0, x_{-1}, x_{-2}, x_{-3}) \notin G$ and let $a = b = 1$.
Let

$$\gamma_j = \frac{\alpha_1\alpha_2\alpha_3}{\prod_{t=1}^{3}(1+(j+1)\alpha_t)}$$

We can write the solution (4.1) as

$$x_{3m+i} = x_{3m} \prod_{t=1}^{i} \frac{\alpha_t}{1+(m+1)\alpha_t} \tag{4.2}$$

where

$$x_{3m} = x_0 \prod_{j=0}^{m-1} \gamma_j, \ m = 0, 1, ....$$

As

$$\lim_{j\to\infty} \gamma_j = \lim_{j\to\infty} \frac{\alpha_1\alpha_2\alpha_3}{\prod_{t=1}^{3}(1+(j+1)\alpha_t)} = 0,$$

it follows that, for a given $0 < \varepsilon_4 < 1$, there exists $j_4 \in \mathbb{N}$ such that, $\mid \gamma_j \mid < \varepsilon_4$ for all $j \geq j_4$. Therefore,

$$\begin{aligned} \mid x_{3m} \mid &= \mid x_0 \mid\mid \prod_{j=0}^{m-1} \gamma_j \mid \\ &= \mid x_0 \mid\mid \prod_{j=0}^{j_4-1} \gamma_j \mid\mid \prod_{j=j_4}^{m-1} \gamma_j \mid \\ &< \mid x_0 \mid\mid \prod_{j=0}^{j_4-1} \gamma_j \mid \varepsilon_4^{m-j_4}. \end{aligned}$$

As $m$ tends to infinity, $x_{3m}$ converges to 0.
Now for each $i \in \{1, 2, 3\}$, we get

$$x_{3m+i} = x_{3m} \prod_{t=1}^{i} \frac{\alpha_t}{1 + (m+1)\alpha_t} \to 0 \quad \text{as} \quad m \to \infty.$$

Therefore, $\{x_n\}_{n=-3}^{\infty}$ converges to 0.

Q.E.D.

## References


[1] R. Abo-Zeid, *Global behavior of two third order rational difference equations with quadratic terms*, Math. Slovaca, 69 (1) (2019), 147 − 158.

[2] R. Abo-Zeid, *Global behavior of a fourth order difference equation with quadratic term*, Bol. Soc. Mat. Mexicana, DOI: 10.1007/$s$40590 − 017 − 0180 − 8.

[3] R. Abo-Zeid, *Global behavior of a fourth order difference equation*, Acta Comment. Univ. Tartu. Math., 18 (2) (2014), 211 − 220.

[4] R. Abo-Zeid, *Global behavior of a higher order difference equation*, Math. Slovaca, 64 (4) (2014), 931 − 940.

[5] R. Abo-Zeid, *Global behavior of a third order rational difference equation*, Math. Bohem., 139 (1) (2014), 25 − 37.

[6] R. Abo-Zeid, *On the solutions of two third order recursive sequences*, Arm. J. Math., 6 (2) (2014), 64 − 66.

[7] R. Abo-Zeid, *Attractivity of two nonlinear third order difference equations*, J. Egyptian Math. Soc., 21 (2013), 241 − 247.

[8] R. Abo-Zeid and C. Cinar, *Global behavior of the difference equation* $x_{n+1} = \frac{Ax_{n-1}}{B - Cx_n x_{n-2}}$, Bol. Soc. Paran. Mat. 31 (1) (2013), 43 − 49.

[9] R. Abo-Zeid and M. A. Al-Shabi, *Global behavior of a third order difference equation*, Tamkang J. Math., 43 (3) (2012), 375 − 383.

[10] R. Abo-Zeid, *Global asymptotic stability of a higher order difference equation*, Bull. Allahabad Math. Soc., 25 (2), (2010), 341 − 351.

[11] R. Abo-Zeid, *Global asymptotic stability of a second order rational difference equation*, J. Appl. Math. & Inform., 28 (3) (2010), 797 − 804.

[12] R.P. Agarwal, *Difference Equations and Inequalities, First Edition*, Marcel Dekker, 1992.

[13] E. Camouzis and G. Ladas, *Dynamics of Third-Order Rational Difference Equations: With Open Problems and Conjectures*, Chapman and Hall/HRC Boca Raton, 2008.

[14] E.A. Grove and G. Ladas, *Periodicities in Nonlinear Difference Equations*, Chapman and Hall/CRC, 2005.

[15] M. Gümüş and Ö. Öcalan, *The qualitative analysis of a rational system of diffrence equations*, J. Fract. Calc. Appl., 9 (2) (2018), 113 − 126.

[16] G. Karakostas, *Convergence of a difference equation via the full limiting sequences method*, Diff. Equ. Dyn. Sys., 1 (4) (1993), 289 − 294.

[17] V.L. Kocic, G. Ladas, *Global Behavior of Nonlinear Difference Equations of Higher Order with Applications*, Kluwer Academic, Dordrecht, 1993.

[18] N. Kruse and T. Nesemann, *Global asymptotic stability in some discrete dynamical systems*, J. Math. Anal. Appl., 253 (1) (1999), 151 − 158.

[19] M.R.S. Kulenović and G. Ladas, *Dynamics of Second Order Rational Difference Equations: With Open Problems and Conjectures*, Chapman and Hall/HRC Boca Raton, 2002.

[20] H. Levy and F. Lessman, *Finite Difference Equations*, Dover, New York, USA, 1992.

[21] H. Sedaghat, *Form Symmetries and Reduction of Order in Difference Equations*, CRC Press, Boca Raton, 2011.

[22] H. Sedaghat, *Every homogeneous equation of degree one admits a reduction in order*, J. Difference Equ. Appl., 15 (2009), 621 − 624.

[23] H. Sedaghat, *Global behaviours of rational difference equations of orders two and three with quadratic terms*, J. Difference Equ. Appl., 15 (3) (2009), 215 − 224.

[24] H. Sedaghat, *On third order rational equations with quadratic terms*, J. Difference Equ. Appl. 14 (8) 2008, 889 − 897.

[25] S. Stević, *On positive solutions of a (k +1)th order difference equation*, Appl. Math. Let., 19 (5) (2006), 427 − 431.

[26] S. Stević, *More on a rational recurrence relation*, Appl. Math. E-Notes, 4 (2004), 80 − 84.