Uniqueness of a differential polynomial and a differential monomial sharing a small function

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Abstract

In this paper, we study the uniqueness of a differential polynomial and monomial sharing a small function with finite weight and obtain two theorems which extend and improves the results of Harina P. Waghamore and Husna V [6].

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1 Introduction and main results

Let f be a non-constant meromorphic function in the complex plane $\mathbb{C}$. We assume that the reader is familiar with the standard notions of the Nevanlinna value distribution theory such as $T(r,f), m(r,f), N(r,f)$ (see e.g., [4, 5, 9]). By $S(r,f)$, as usual, we shall mean a quantity that satisfies

$$S(r,f) = o(T(r,f)) \ as \ r \to \infty,$$

possibly outside an exceptional set of finite logarithmic measure. A meromorphic function a is said to be a small function of f, if $T(r,a) = S(r,f)$ as $r \to \infty$.

A more general concept is the weighted sharing of meromorphic functions. For a non negative integer k, we denote by $E_k(a,f)$ the set of all zeros of $f-a$, where a zero of of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a,f) = E_k(a,g)$, then f and g are said to share the function a with weight k. We write "f and g share (a,k)" to mean that "f and g share the function a with weight k". Since $E_k(a,f) = E_k(a,g)$ implies that $E_p(a,f) = E_p(a,g)$ for any integer $p\,(0 \leq p < k)$, if f and g share (a,k), then f and g share (a,p), $0 \leq p < k$. Moreover, we note that f and g share the function a IM (ignoring multiplicity) or CM (counting multiplicity) if and only if f and g share $(a,0)$ or (a,∞), respectively.

For any constant a, we define

$$\Theta(a,f) = 1 - \limsup_{r\to\infty} \frac{\overline{N}(r, \frac{1}{f-a})}{T(r,f)}.$$

$$\delta_k(a,f) = 1 - \limsup_{r\to\infty} \frac{N_k(r, \frac{1}{f-a})}{T(r,f)}.$$

Clearly,

$$0 \leq \delta(a,f) \leq \delta_k(a,f) \leq \delta_{k-1}(a,f) ... \leq \delta_1(a,f) = \Theta(a,f).$$

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Definition. Let $n_{0j}, n_{1j}, ..., n_{kj}$ be nonnegative integers. The expression

$$M_j[f] = (f)^{n_{0j}}(f^{(1)})^{n_{1j}}...(f^{(k)})^{n_{kj}}$$

is called a differential monomial generated by f of degree $d_{M_j} = d(M_j) = \sum_{i=0}^{k} n_{ij}$ and weight $\Gamma_{M_j} = \sum_{i=0}^{k}(i+1)n_{ij}$.

The sum $P[f] = \sum_{j=1}^{t} b_j M_j[f]$ is called a differential polynomial generated by f of degree $\overline{d}(P) = \max\{d(M_j) : 1 \leq j \leq t\}$ and weight $\Gamma_P = \max\left\{\Gamma_{M_j} : 1 \leq j \leq t\right\}$, where $T(r, b_j) = S(r, f)$ for $j = 1, 2, ..., t$.

The numbers $\underline{d}(P) = \min\{d(M_j) : 1 \leq j \leq t\}$ and k (the highest order of the derivative of f in $P[f]$) are called respectively the lower degree and order of $P[f]$.

$P[f]$ is said to be homogeneous if $\overline{d}(P) = \underline{d}(P)$. Moreover, $P[f]$ is called a linear differential polynomial generated by f if $\overline{d}(P) = 1$. Otherwise, $P[f]$ is called a non-linear differential polynomial.

We denote by $Q = \max\{\Gamma_{M_j} - d(M_j) : 1 \leq j \leq t\} = \max\{n_{1j} + 2n_{2j} + ... + kn_{kj} : 1 \leq j \leq t\}$.

Also for the sake of convenience for a differential monomial $M[f]$ we denote by $\lambda = \Gamma_M - d_M$.

In 1996, R. Bruck [3] first considered the uniqueness problems of an entire function sharing one value with its derivative and proved the following result.

Theorem A. Let f be a entire function which is not constant. If f and f' share the value 1 CM and if $N(r, \frac{1}{f'}) = S(r, f)$, then $\frac{f'-1}{f-1} \equiv c$ for some nonzero constant $c \in \mathbb{C} \setminus \{0\}$.

In 2003, Yu [10] considered the problem of an entire or meromorphic function sharing one small function with its derivative and proved the following two theorems.

Theorem B. Let f be a non-constant entire function and $a \equiv a(z)$ be a meromorphic function such that $a \not\equiv 0, \infty$ and $T(r, a) = o(T(r, f))$ as $r \to +\infty$. If $f - a$ and $f^{(k)} - a$ share the value 0 CM and $\delta(0, f) > \frac{3}{4}$, then $f \equiv f^{(k)}$.

Theorem C. Let f be a non-constant, non-entire meromorphic function and $a \equiv a(z)$ be a meromorphic function such that $a \not\equiv 0, \infty$ and $T(r, a) = o(T(r, f))$ as $r \to +\infty$. If
(i) f and a have no common poles,
(ii) $f - a$ and $f^{(k)} - a$ share the value 0 CM,
(iii) $4\delta(0, f) + 2\Theta(\infty, f) > 19 + 2k$,
then $f \equiv f^{(k)}$ where k is a positive integer.

In 2004, Lahiri [8] improved the results of Zhang [11] with weighted shared value obtained the following two theorems.

Theorem D. Let f be a non-constant meromorphic function and k be a positive integer. If f and $f^{(k)}$ share (1,2) and

$$2\overline{N}(r, f) + N_2(r, \frac{1}{f^{(k)}}) + N_2(r, \frac{1}{f}) < (\lambda + o(1))T(r, f^{(k)})$$

for $r \in I$, where $0 < \lambda < 1$ and I is a set of infinite linear measure, then $\frac{f^{(k)}-1}{f-1} \equiv c$ for some

constant $c \in \mathbb{C} \setminus \{0\}$.

Theorem E. Let f be a non-constant meromorphic function and k be a positive integer. If f and $f^{(k)}$ share (1,1) and

$$2\overline{N}(r,f) + N_2(r,\frac{1}{f^{(k)}}) + 2\overline{N}(r,\frac{1}{f}) < (\lambda + o(1))T(r,f^{(k)})$$

for $r \in I$, where $0 < \lambda < 1$ and I is a set of infinite linear measure, then $\frac{f^{(k)}-1}{f-1} \equiv c$ for some constant $c \in \mathbb{C} \setminus \{0\}$.

In the same paper Lahiri [8] also obtained the following result which is an improvement of Theorem C.

Theorem F. Let f be a non-constant meromorphic function and k be a positive integer. Also let $a \equiv a(z)\,(\not\equiv 0,\infty)$ be a meromorphic function such that $T(r,a) = S(r,f)$. If

(i) a has no zero (pole) which is also a zero (pole) of f or $f^{(k)}$ with the same multiplicity.

(ii) $f - a$ and $f^{(k)} - a$ share (0,2) CM,

(iii) $2\delta_{2+k}(0,f) + (4+k)\Theta(\infty,f) > 5 + k$,

then $f \equiv f^{(k)}$.

In 2005, Zhang [12] improved the above results and proved the following theorems.

Theorem G. Let f be a non-constant meromorphic function and $k(\geq 1), l(\geq 0)$ be integers. Also, let $a \equiv a(z)\,(\not\equiv 0,\infty)$ be a meromorphic function such that $T(r,a) = S(r,f)$. Suppose that $f - a$ and $f^{(k)} - a$ share $(0,l)$. If $l \geq 2$
and

$$2\overline{N}(r,f) + N_2(r,\frac{1}{f^{(k)}}) + N_2(r,\frac{1}{(f/a)'}) < (\lambda + o(1))T(r,f^{(k)}), \tag{1.1}$$

or $l = 1$ and

$$2\overline{N}(r,f) + N_2(r,\frac{1}{f^{(k)}}) + 2\overline{N}(r,\frac{1}{(f/a)'}) < (\lambda + o(1))T(r,f^{(k)}), \tag{1.2}$$

or $l = 0$, i.e., $f - a$ and $f^{(k)} - a$ share the value 0 IM and

$$4\overline{N}(r,f) + 3N_2(r,\frac{1}{f^{(k)}}) + 2\overline{N}(r,\frac{1}{(f/a)'}) < (\lambda + o(1))T(r,f^{(k)}) \tag{1.3}$$

for $r \in I$, where $0 < \lambda < 1$ and I is a set of infinite linear measure, then $\frac{f^{(k)}-a}{f-a} \equiv c$ for some constant $c \in \mathbb{C} \setminus \{0\}$.

Theorem H. Let f be a non-constant meromorphic function and $k(\geq 1), l(\geq 0)$ be integers. Also let $a \equiv a(z)(\not\equiv 0,\infty)$ be a meromorphic function such that $T(r,a) = S(r,f)$. Suppose that $f - a$ and $f^{(k)} - a$ share $(0,l)$. If $l \geq 2$ and

$$(3+k)\Theta(\infty,f) + 2\delta_{2+k}(0,f) > k + 4, \tag{1.4}$$

or $l = 1$ and

$$(4+k)\Theta(\infty,f) + 3\delta_{2+k}(0,f) > k + 6, \tag{1.5}$$

or $l = 0$ i.e., $f - a$ and $f^{(k)} - a$ share the value 0 IM and

$$(6 + 2k)\Theta(\infty, f) + 5\delta_{2+k}(0, f) > 2k + 10, \tag{1.6}$$

then $f \equiv f^{(k)}$.

In 2016, Harina P. Waghamore and Husna V [6] considered Theorems G and H to the uniqueness of more generalised form of a function namely f^n and $(f^{(k)})^m$sharing a small function for two arbitrary positive integer n and m and extended the above theorems as follows

Theorem I. Let f be a non-constant meromorphic function and $k(\geq 1), n(\geq 1), m(\geq 2), l(\geq 0)$ be integers. Also let $a \equiv a(z)(\not\equiv 0, \infty)$ be a meromorphic function such that $T(r, a) = S(r, f)$. Suppose that $f^n - a$ and $(f^{(k)})^m - a$ share $(0, l)$.
If $l \geq 2$ and

$$\frac{2}{m}\overline{N}(r, f) + \frac{2}{m}\overline{N}(r, \frac{1}{f^{(k)}}) + N_2(r, \frac{1}{(f/a)'}) < (\lambda + o(1))T(r, f^{(k)}) \tag{1.7}$$

or $l = 1$ and

$$\frac{2}{m}\overline{N}(r, f) + \frac{2}{m}\overline{N}(r, \frac{1}{f^{(k)}}) + 2N(r, \frac{1}{(f/a)'}) < (\lambda + o(1))T(r, f^{(k)}) \tag{1.8}$$

or $l = 0$ i.e., $f - a$ and $(f^{(k)})^m - a$ share the value 0 IM and

$$\frac{4}{m}\overline{N}(r, f) + \frac{6}{m}\overline{N}(r, \frac{1}{f^{(k)}}) + 2\overline{N}(r, \frac{1}{(f/a)'}) < (\lambda + o(1))T(r, f^{(k)}) \tag{1.9}$$

for $r \in I$, where $0 < \lambda < 1$ and I is a set of infinite linear measure, then $\frac{(f^{(k)})^m - a}{f^n - a} \equiv C$ for some constant $c \in \mathbb{C} \setminus \{0\}$.

Theorem J. Let f be a non-constant meromorphic function and $k(\geq 1), n(\geq 1), m(\geq 2), l(\geq 0)$ be integers. Also let $a \equiv a(z)(\not\equiv 0, \infty)$ be a meromorphic function such that $T(r, a) = S(r, f)$. Suppose that $f^n - a$ and $(f^{(k)})^m - a$ share $(0, l)$.
If $l \geq 2$ and

$$(3 + 2k)\Theta(\infty, f) + 2\Theta(0, f) + 2\delta_{1+k}(0, f) > 2k + 7 - n \tag{1.10}$$

or $l = 1$ and

$$(4 + 2k)\Theta(\infty, f) + 4\Theta(0, f) + 2\delta_{1+k}(0, f) > 2k + 10 - n \tag{1.11}$$

or $l = 0$ i.e., $f - a$ and $(f^{(k)})^m - a$ share the value 0 IM and

$$(6 + 4k)\Theta(\infty, f) + 6\Theta(0, f) + \delta_{1+k}(0, f) > 16 + 4k - n, \tag{1.12}$$

then $f^n \equiv (f^{(k)})^m$.

In this paper, our interest is to investigate the uniqueness of a differential polynomial and monomial. In general this is not true, but under essential conditions, we prove the result,which extend and improves the previous results.

Theorem 1.1. Let f be a non-constant meromorphic function and $M[f]$ be a differential monomial of degree d_M and weight Γ_M and $k(\geq 1)$ is the highest derivative in $M[f]$. Let $P[f]$ be a non-constant differential polynomial of f. Also, let $a(z)\,(\not\equiv 0, \infty)$ be a small function with respect to f. Suppose

$P[f]-a$ and $M[f]-a$ share $(0,l)$. If
$l \geq 2$ and

$$(Q+\lambda+3)\overline{N}(r,f)+d_M N_{2+k}\left(r,\frac{1}{f}\right)+\overline{d}(P)N_{3+k}\left(r,\frac{1}{f}\right)<(\beta+o(1))\,T(r,M) \tag{1.13}$$

or $l=1$ and

$$(2Q+\lambda+4)\overline{N}(r,f)+d_M N_{2+k}\left(r,\frac{1}{f}\right)+2\overline{d}(P)N_{2+k}\left(r,\frac{1}{f}\right)<(\beta+o(1))\,T(r,M) \tag{1.14}$$

or $l=0$ i.e., $P[f]-a$ and $M[f]-a$ share the value 0 IM and

$$(2Q+3\lambda+6)\overline{N}(r,f)+3d_M N_{2+k}\left(r,\frac{1}{f}\right)+2\overline{d}(P)N_{2+k}\left(r,\frac{1}{f}\right)<(\beta+o(1))\,T(r,M) \tag{1.15}$$

for $r\in I$, where $0<\beta<1$ and I is a set of infinite linear measure, then $\dfrac{M[f]-a}{P[f]-a}\equiv C$.

Theorem 1.2. Let f be a non-constant meromorphic function and $M[f]$ be a differential monomial of degree d_M and weight Γ_M and $k(\geq 1)$ is the highest derivative in $M[f]$. Let $P[f]$ be a non-constant differential polynomial of f. Also, let $a(z)(\not\equiv 0,\infty)$ be a small function with respect to f. Suppose $P[f]-a$ and $M[f]-a$ share $(0,l)$. If
$l \geq 2$ and

$$(3+\lambda)\,\Theta(\infty,f)+\left(\overline{d}(P)+d_M\right)\delta_{2+k}(0,f)>\lambda+d_M+3 \tag{1.16}$$

or $l=1$ and

$$(Q+4+\lambda)\,\Theta(\infty,f)+\left(2\overline{d}(P)+d_M\right)\delta_{2+k}(0,f)>Q+d_M+\lambda+\overline{d}(P)+4 \tag{1.17}$$

or $l=0$ i.e., $P[f]-a$ and $M[f]-a$ share the value 0 IM and

$$(2Q+6+2\lambda)\,\Theta(\infty,f)+\left(3\overline{d}(P)+2d_M\right)\delta_{2+k}(0,f)>2Q+2\lambda+2\overline{d}(P)+2d_M+6, \tag{1.18}$$

then $P[f]\equiv M[f]$.

From Theorem 1.2. we have

Corollary 1.2. Let f be a non-constant entire function and $M[f]$ be a differential monomial of degree d_M and weight Γ_M and $k(\geq 1)$ is the highest derivative in $M[f]$. Also let $P[f]$ be a non-constant differential polynomial of f. Also, let $a(z)(\not\equiv 0,\infty)$ be a small function with respect to f. If $P[f]-a$ and $M[f]-a$ share the value 0 CM and $\delta(0,f)>\frac{d_M}{\overline{d}(P)+d_M}$ or if $P[f]-a$ and $M[f]-a$ share the value 0 IM and $\delta(0,f)>\frac{2\overline{d}(P)+2d_M}{3\overline{d}(P)+2d_M}$.

Example 1.2. Let $f(z)=e^{\beta z}+a-\dfrac{a}{\beta^2}$; where $a\neq 0,\infty$ and $\beta\neq 0,\pm 1$. Let $M[f]=f^{(2)}$ and $P[f]=2f^{(2)}$. Then we have $d_M=1,\Gamma_M=3$ and $\lambda=1$, also $\overline{d}(P)=1$, $\underline{d}(P)=1$, $\Gamma_{M_j}=3$, $d(M_j)=1$ and $Q=1$. Also, since $\overline{N}(r,f)=S(r,f)$, $\Theta(\infty,f)=1$ and $\overline{N}\left(r,\frac{1}{f}\right)=\overline{N}\left(r,\dfrac{a}{\beta^2}-a;e^{\beta z}\right)\sim$

$T(r,f)$, $\delta_{2+k}(0,f)=0$.
Also, we know that $P[f]$ and $M[f]$ share $(a,l)\,(l\geq 0)$ but none of the inequalities (1.16), (1.17) and (1.18) satisfied and $P[f]\not\equiv M[f]$. Therefore, for $P[f]\equiv M[f]$, the conditions of Theorem 1.2 are essential.

2 Lemmas

Lemma 2.1 ([1]). Let f be a non-constant meromorphic function and $M[f]$ be a differential monomial of degree d_M and weight Γ_M. Then

$$T(r,M)\leq d_M T(r,f)+\lambda\overline{N}(r,\infty;f)+S(r,f).$$

Lemma 2.2 ([1]). For the differential monomial $M[f]$,

$$N_p(r,0;M[f])\leq d_M N_{p+k}(r,0;f)+\lambda\overline{N}(r,\infty;f)+S(r,f).$$

Lemma 2.3 ([2]). Let f be a meromorphic function and $P[f]$ be a differential polynomial in f. Then

$$\begin{aligned}
m\left(r,\frac{P[f]}{f^{\overline{d}(P)}}\right)&\leq(\overline{d}(P)-\underline{d}(P))m\left(r,\frac{1}{f}\right)+S(r,f),\\
m\left(r,\frac{P[f]}{f^{\underline{d}(P)}}\right)&\leq(\overline{d}(P)-\underline{d}(P))m\left(r,f\right)+S(r,f),\\
N\left(r,\frac{P[f]}{f^{\overline{d}(P)}}\right)&\leq(\overline{d}(P)-\underline{d}(P))N\left(r,\frac{1}{f}\right)+Q\left[\overline{N}(r,f)+\overline{N}\left(r,\frac{1}{f}\right)\right]+S(r,f),\\
N\left(r,P[f]\right)&\leq\overline{d}(P)N(r,f)+Q\overline{N}(r,f)++S(r,f),\\
T\left(r,P[f]\right)&\leq Q\overline{N}(r,f)+\overline{d}(P)T(r,f)+S(r,f),
\end{aligned}$$

where $Q=max\left\{n_{i1}+2n_{i2}+3n_{i3}+...+mn_{im};1\leq i\leq n\right\}$.

Lemma 2.4 ([7]). For the differential polynomial $H[f]$,

$$N_p\left(r,\frac{1}{H[f]}\right)\leq\overline{d}(H)N_{p+k}\left(r,\frac{1}{f}\right)+Q\overline{N}(r,f)+S(r,f).$$

Lemma 2.5 ([8]). Let f be a non-constant meromorphic function, k be a positive integer, then

$$N_p\left(r,\frac{1}{f^{(k)}}\right)\leq N_{p+k}\left(r,\frac{1}{f}\right)+k\overline{N}(r,f)+S(r,f).$$

Lemma 2.6 ([1]). For any non-constant meromorphic function f,

$$N\left(r,\infty;\frac{M}{f^{d_M}}\right)\leq d_M N(r,0;f)+\lambda\overline{N}(r,\infty;f)+S(r,f).$$

Lemma 2.7. Let f be a non-constant meromorphic function and $a(z)$ be a small function of f. Let us define $F=\frac{P[f]}{a}$ and $G=\frac{M[f]}{a}$. Then, $FG\not\equiv 1$.
Proof. On contrary suppose $FG=1$.i.e.,

$$P[f]M[f]=a^2.$$

From above it is clear that the function f can't have any zeros and poles.
Therefore

$$\overline{N}\left(r,\frac{1}{f}\right)=S(r,f)=\overline{N}(r,f).$$

By using the first fundamental theorem and lemma 2.3, have

$$\begin{aligned}(\overline{d}(P)+d_M)T(r,f)&=T\left(r,f^{\overline{d}(P)+d_M}\right)\\&\leq T\left(r,\frac{f^{\overline{d}(P)}f^{d_M}}{P[f]M[f]}\right)+S(r,f)\\&\leq T\left(r,\frac{P[f]}{f^{\overline{d}(P)}}\right)+T\left(r,\frac{M[f]}{f^{d_M}}\right)+S(r,f)\\&\leq(\overline{d}(P)-\underline{\mathrm{d}}(P))T(r,f)+S(r,f)\\ i.e.,\ (d_M+\underline{\mathrm{d}}(P))T(r,f)&\leq S(r,f),\end{aligned}$$

which is not possible.
Hence, $FG\not\equiv 1$.
Hence the proof of lemma.

3 Proof of Theorem 1.1

Let $F=\frac{P[f]}{a}$, $G=\frac{M[f]}{a}$, then $F-1=\frac{P[f]-a}{a}$, $G-1=\frac{M[f]-a}{a}$. Since $P[f]-a$ and $M[f]-a$ share $(0,l)$, it follows that F and G share $(1,l)$ except the zeros and poles of $a(z)$.
Define

$$H=\left(\frac{F''}{F'}-\frac{2F'}{F-1}\right)-\left(\frac{G''}{G'}-\frac{2G'}{G-1}\right),\tag{3.1}$$

we have the following two cases to investigate
Case 1. $H\equiv 0$. Integration yields

$$\frac{1}{F-1}\equiv C\frac{1}{G-1}+D,\tag{3.2}$$

where C and D are constants and $C\neq 0$. If there exists a pole z_0 of f with multiplicity p which is not zero or poles of a, then z_0 is the pole of F with multiplicity $p\,d+\Gamma-d$ and the poles of G with multiplicity $pd_{M_j}+(\Gamma_{M_j}-d_{M_j})$. Because $d_{M_j}=d(M_j)=\sum_{i=0}^{k}n_{ij}$ and weight $\Gamma_{M_j}=\sum_{i=0}^{k}(i+1)n_{ij}$. This contradicts with (3.2). So

$$\overline{N}(r,f)\leq\overline{N}(r,a)+\overline{N}(r,\frac{1}{a})=S(r,f),\tag{3.3}$$

$$\overline{N}(r,F)=S(r,f)\ \ \overline{N}(r,G)=S(r,f)$$

(3.2) also shows F and G share the value 1 CM. Next we prove $D=0$. We first assume that $D\neq 0$, then

$$\frac{1}{F-1}\equiv\frac{D(G-1+\frac{C}{D})}{G-1}\tag{3.4}$$

So,

$$\overline{N}(r,\frac{1}{G-1+\frac{C}{D}})=\overline{N}(r,F)=S(r,f) \tag{3.5}$$

If $\frac{C}{D}\neq 1$, by the second fundamental theorem and (3.3), (3.5) and $S(r,G)=S(r,f)$, we have

$$\begin{aligned} T(r,G) &\leq \overline{N}(r,G)+\overline{N}(r,\frac{1}{G})+\overline{N}\left(r,\frac{1}{G-1+\frac{C}{D}}\right)+S(r,G)\\ &\leq \overline{N}(r,\frac{1}{G})+S(r,f)\leq T(r,G)+S(r,f)\\ \textit{So,}\qquad T(r,G) &= \overline{N}(r,\frac{1}{G})+S(r,f), \qquad (3.6)\\ \textit{i.e.,}\qquad T(r,M) &= \overline{N}(r,\frac{1}{M})+S(r,f)\\ T(r,M) &\leq d_M N_{1+k}\left(r,\frac{1}{f}\right)+\lambda\overline{N}(r,f)+S(r,f), \end{aligned}$$

this contradicts with conditions (1.13), (1.14) and (1.15) of this theorem.

If $\frac{C}{D}=1$, from (3.4) we know

$$\frac{1}{F-1}\equiv C\frac{G}{G-1}$$

then

$$\left(F-1-\frac{1}{C}\right)G=-\frac{1}{C}.$$

Noticing that $F=\frac{P[f]}{a}$, $G=\frac{M[f]}{a}$, we have

$$\frac{1}{P[f](P[f]-(1+\frac{1}{C})a)}\equiv\frac{-C}{a^2}.\frac{M[f]}{P[f]} \tag{3.7}$$

By Lemmas 2.1, 2.3 and (3.3) and (3.7), then

$$\begin{aligned} 2\overline{d}(p)T(r,f) &= T(r,P[f](P[f]-(1+\frac{1}{C})a))+S(r,f) \qquad (3.8)\\ &= T\left(r,\frac{1}{P[f](P[f]-(1+\frac{1}{C})a}\right)+S(r,f)\\ &\leq T\left(r,\frac{M[f]}{P[f]}\right)+S(r,f)\\ &\leq T(r,M[f])+T\left(r,\frac{1}{P[f]}\right)+S(r,f)\\ &\leq d_M T(r,f)+\lambda\overline{N}(r,f)+Q\overline{N}(r,f)+\overline{d}(p)T(r,f)+S(r,f)\\ (\overline{d}(p)-Q-\lambda-d_M)T(r,f) &\leq S(r,f), \end{aligned}$$

which is impossible. Hence $D=0$, and $\frac{G-1}{F-1}\equiv C$, i.e., $\frac{M[f]-a}{P[f]-a}\equiv C$. This is just the conclusion of this theorem.

Case 2. $H \not\equiv 0$. From (3.1) it is easy to see that $m(r, H) = S(r, f)$.
Subcase 2.1. $l \geq 1$. From (3.1) we have

$$\begin{aligned} N(r,H) \leq \overline{N}(r,F) + \overline{N}_{(l+1}(r,\frac{1}{F-1}) + \overline{N}_{(2}(r,\frac{1}{F}) + \overline{N}_{(2}(r,\frac{1}{G}) + \overline{N}_0(r,\frac{1}{F'}) \\ + \overline{N}_0(r,\frac{1}{G'}) + \overline{N}(r,a) + \overline{N}(r,\frac{1}{a}), \end{aligned} \tag{3.9}$$

where $N_0(r, \frac{1}{F'})$ denotes the counting function of the zeros of F' which are not the zeros of F and $F-1$, and $\overline{N}_0(r, \frac{1}{F'})$ denotes its reduced form. In the same way, we can define $N_0(r, \frac{1}{G'})$ and $\overline{N}_0(r, \frac{1}{G'})$, Let z_0 be a simple zero of $F-1$ but $a(z_0) \neq 0, \infty$, then z_0 is also the simple zero of $G-1$. By calculating z_0 is the zero of H, So

$$N_{1)}(r,\frac{1}{F-1}) \leq N(r,\frac{1}{H}) + N(r,a) + N(r,\frac{1}{a}) \leq N(r,H) + S(r,f) \tag{3.10}$$

Noticing that $N_{1)}(r, \frac{1}{G}) = N_{1)}(r, \frac{1}{F}) + S(r, f)$
we have

$$\begin{aligned} \overline{N}(r,\frac{1}{G-1}) &= N_{1)}(r,\frac{1}{F-1}) + \overline{N}_{(2}(r,\frac{1}{F-1}) \\ &\leq \overline{N}(r,F) + \overline{N}_{(l+1}(r,\frac{1}{F-1}) + \overline{N}_{(2}(r,\frac{1}{F-1}) + \overline{N}_{(2}(r,\frac{1}{F}) \\ &+ \overline{N}_{(2}(r,\frac{1}{G}) + \overline{N}_0(r,\frac{1}{F'}) + \overline{N}_0(r,\frac{1}{G'}) + S(r,f) \end{aligned} \tag{3.11}$$

By the second fundamental theorem and (3.11) and noticing

$$\overline{N}(r,F) = \overline{N}(r,G) + S(r,f),$$

then

$$\begin{aligned} T(r,G) &\leq \overline{N}(r,G) + \overline{N}(r,\frac{1}{G}) + \overline{N}(r,\frac{1}{G-1}) - N_0(r,\frac{1}{G'}) + S(r,G) \\ &\leq 2\overline{N}(r,F) + \overline{N}(r,\frac{1}{G}) + \overline{N}_{(2}(r,\frac{1}{G}) + \overline{N}_{(2}(r,\frac{1}{F}) + \overline{N}_{(l+1}(r,\frac{1}{F-1}) \\ &+ \overline{N}_{(2}(r,\frac{1}{F-1}) + \overline{N}_0(r,\frac{1}{F'}) + S(r,f). \end{aligned} \tag{3.12}$$

While $l \geq 2$,

$$\overline{N}_{(2}(r,\frac{1}{F}) + \overline{N}_{(l+1}(r,\frac{1}{F-1}) + \overline{N}_{(2}(r,\frac{1}{F-1}) + \overline{N}_0(r,\frac{1}{F'}) \leq N_2(r,\frac{1}{F'}), \tag{3.13}$$

so

$$T(r,G) \leq 2\overline{N}(r,F) + N_2(r,\frac{1}{G}) + N_2(r,\frac{1}{F'}) + S(r,f)$$

i.e.,

$$T(r,M) \leq 2\overline{N}(r,f) + N_2(r,\frac{1}{M}) + N_2(r,\frac{1}{(\frac{P[f]}{a})'}) + S(r,f)$$

$$T(r,M) \leq 3\overline{N}(r,f) + d_M N_{2+k}(r,\frac{1}{f}) + \lambda\overline{N}(r,f) + \overline{d}(P)N_{3+k}(r,\frac{1}{f}) + Q\overline{N}(r,f) + S(r,f)$$

$$T(r,M) \leq (Q+\lambda+3)\overline{N}(r,f) + d_M N_{2+k}\left(r,\frac{1}{f}\right) + \overline{d}(P)N_{3+k}\left(r,\frac{1}{f}\right) + S(r,f),$$

this contradicts with (1.13).
While $l = 1$, (3.13) turns into

$$\overline{N}_{(2}(r,\frac{1}{F}) + \overline{N}_{(l+1}(r,\frac{1}{F-1}) + \overline{N}_{(2}(r,\frac{1}{F-1}) + \overline{N}_0(r,\frac{1}{F'}) \leq 2\overline{N}(r,\frac{1}{F'})$$

Similarly as above , we have

$$T(r,M) \leq (2Q+\lambda+4)\overline{N}(r,f) + d_M N_{2+k}\left(r,\frac{1}{f}\right) + 2\overline{d}(P)N_{2+k}\left(r,\frac{1}{f}\right) + S(r,f).$$

This contradicts with (1.14).

Subcase 2.2. $l = 0$. In this case, F and G share 1 IM except the zeros and poles of $a(z)$. Let z_0 be the zero of $F-1$ with multiplicity p and the zero of $G-1$ with multiplicity q.
We denote by $N_E^{1)}(r,\frac{1}{F})$ the counting function of the zeros of $F-1$ where $p-q=1$; by $N_E^{2)}(r,\frac{1}{F})$ the counting function of the zeros of $F-1$ where $p=q\geq 2$; by $\overline{N}_L(r,\frac{1}{F})$ the counting function of the zeros of $F-1$ where $p>q\geq 1$, each point in these counting functions is counted only once. In the same way, we can define $N_E^{1)}(r,\frac{1}{G})$, $N_E^{(2}(r,\frac{1}{G})$ and $\overline{N}_L(r,\frac{1}{G})$. It is easy to see that

$$N_E^{1)}(r,\frac{1}{F-1}) = N_E^{1)}(r,\frac{1}{G-1}) + S(r,f),$$

$$\overline{N}_E^{2)}(r,\frac{1}{F-1}) = \overline{N}_E^{(2}(r,\frac{1}{G-1}) + S(r,f),$$

$$\begin{aligned}\overline{N}(r,\frac{1}{F-1}) &= \overline{N}(r,\frac{1}{G-1}) + S(r,f)\\ &= N_E^{1)}(r,\frac{1}{F-1}) + N_E^{(2}(r,\frac{1}{F-1}) + \overline{N}_L(r,\frac{1}{F-1})\\ &\quad + \overline{N}_L(r,\frac{1}{G-1}) + S(r,f)\end{aligned} \tag{3.14}$$

From (3.1) we have now

$$\begin{aligned}N(r,H) \leq\ & \overline{N}(r,F) + \overline{N}_{(2}(r,\frac{1}{F}) + \overline{N}_{(2}(r,\frac{1}{G}) + \overline{N}_L(r,\frac{1}{F-1})\\ &+ \overline{N}_L(r,\frac{1}{G-1}) + \overline{N}_0(r,\frac{1}{F'}) + \overline{N}_0(r,\frac{1}{G'}) + S(r,f).\end{aligned} \tag{3.15}$$

In this case, (3.10) is replaced by

$$N_E^{1)}(r,\frac{1}{F-1}) \leq N(r,H) + S(r,f). \tag{3.16}$$

From (3.14), (3.15) and (3.16), we have

$$\begin{aligned}\overline{N}(r,\frac{1}{G-1}) &\le \overline{N}(r,F)+\overline{N}_{(2}(r,\frac{1}{F})+\overline{N}_{(2}(r,\frac{1}{G})+\overline{N}_{E}^{(2}(r,\frac{1}{F-1})+2\overline{N}_{L}(r,\frac{1}{F-1})\\ &\quad+2\overline{N}_{L}(r,\frac{1}{G-1})+\overline{N}_{0}(r,\frac{1}{F'})+\overline{N}_{0}(r,\frac{1}{G'})+S(r,f)\\ &\le \overline{N}(r,F)+2\overline{N}(r,\frac{1}{F'})+2\overline{N}_{L}(r,\frac{1}{G-1})+\overline{N}_{(2}(r,\frac{1}{G})+\overline{N}_{0}(r,\frac{1}{G'})+S(r,f)\end{aligned}$$

By the second fundamental theorem, then

$$\begin{aligned}T(r,G) &\le \overline{N}(r,G)+\overline{N}(r,\frac{1}{G})+\overline{N}(r,\frac{1}{G-1})-N_0(r,\frac{1}{G'})+S(r,G)\\ &\le 2\overline{N}(r,G)+2\overline{N}(r,\frac{1}{F'})+\overline{N}(r,\frac{1}{G})+2\overline{N}(r,\frac{1}{G'})+S(r,f)\end{aligned}$$

From Lemma 2.5 for $p=1,k=1$ we know

$$\overline{N}(r,\frac{1}{G'})\le N_2(r,\frac{1}{G})+\overline{N}(r,G)+S(r,G),$$

so,

$$T(r,G)\le 4\overline{N}(r,F)+3N_2(r,\frac{1}{G})+2\overline{N}(r,\frac{1}{F'})+S(r,f)$$

i.e.,

$$T(r,M)\le(2Q+3\lambda+6)\overline{N}(r,f)+3d_M N_{2+k}\left(r,\frac{1}{f}\right)+d_M N_{1+k}\left(r,\frac{1}{f}\right)+2\overline{d}(P)N_{2+k}\left(r,\frac{1}{f}\right)+S(r,f).$$

This contradicts with (1.15). The proof is complete.

4 Proof of Theorem 1.2

Let $F=\frac{P[f]}{a}$, $G=\frac{M[f]}{a}$, then $F-1=\frac{P[f]-a}{a}$, $G-1=\frac{M[f]-a}{a}$. Since $P[f]-a$ and $M[f]-a$ share $(0,l)$, it follows that F and G share $(1,l)$ except the zeros and poles of $a(z)$.

Also note that,

$$\overline{N}(r,F)=\overline{N}(r,f)+S(r,f)\ \text{ and }\ \overline{N}(r,G)=\overline{N}(r,f)+S(r,f).$$

Now we consider the following cases.

Case 3. $H\equiv 0$. By (3.1), we have

$$\frac{F''}{F'}-\frac{2F'}{F-1}=\frac{G''}{G'}-\frac{2G'}{G-1},$$

and so, we obtain

$$\frac{1}{F-1}=\frac{c}{G-1}+D, \tag{4.1}$$

where $C \neq 0$ and D are constants. Then, the following three cases arise:

Case (i). When $D \neq 0, -1$. Rewriting (4.1) as

$$\frac{G-1}{C} = \frac{F-1}{D+1-DF},$$

we have

$$\overline{N}(r,G) = \overline{N}\left(r, \frac{1}{F - \dfrac{(D+1)}{D}}\right).$$

In this case, by using second fundamental theorem and lemmas 2.1, 2.3, we get

$$\begin{aligned}
\overline{d}(P)T(r,f) + Q\overline{N}(r,f) &\leq \overline{N}(r,F) + \overline{N}(r,\frac{1}{F}) + \overline{N}\left(r, \frac{1}{F - \frac{(D+1)}{D}}\right) + S(r,f) \\
&\leq \overline{N}(r,f) + \overline{N}\left(r, \frac{1}{P[f]}\right) + \overline{N}(r,G) + S(r,f) \\
&\leq 2\overline{N}(r,f) + \overline{d}(P)N_{1+k}(r,\frac{1}{f}) + Q\overline{N}(r,f) + S(r,f) \\
\overline{d}(P)T(r,f) &\leq 2\overline{N}(r,f) + \overline{d}(P)N_{1+k}(r,\frac{1}{f}) + S(r,f) \\
2\Theta(\infty,f) + \overline{d}(p)\delta_{1+k}(0,f) &\leq 2,
\end{aligned}$$

which contradicts (1.16), (1.17) and (1.18).

Case (ii). When $D = 0$. Then from (4.1), we have

$$G = CF - (C-1). \tag{4.2}$$

Therefore, if $C \neq 1$, then

$$\overline{N}\left(r, \frac{1}{G}\right) = \overline{N}\left(r, \frac{1}{F - \frac{(C-1)}{C}}\right).$$

Now, by using second fundamental theorem of Nevanlinna and lemmas 2.1, 2.3 we get

$$\begin{aligned}
\overline{d}(P)T(r,f) + Q\overline{N}(r,f) &\leq \overline{N}(r,F) + \overline{N}(r,\frac{1}{F}) + \overline{N}\left(r, \frac{1}{F - \frac{(C-1)}{C}}\right) + S(r,f) \\
&\leq \overline{N}(r,F) + \overline{N}(r,\frac{1}{F}) + \overline{N}\left(r, \frac{1}{G}\right) + S(r,f) \\
&\leq \overline{N}(r,f) + \overline{N}\left(r, \frac{1}{P[f]}\right) + \overline{N}\left(r, \frac{1}{M[f]}\right) + S(r,f) \\
&\leq \overline{N}(r,f) + \overline{d}(P)N_{1+k}(r,\frac{1}{f}) + Q\overline{N}(r,f) + d_M N_{1+k}(r,\frac{1}{f}) + \lambda\overline{N}(r,f) + S(r,f)
\end{aligned}$$

$$(Q+\lambda+1)\Theta(\infty,f) + (\bar{d}+d_M)\delta_{1+k}(0,f) \leq Q+\lambda+1+d_M,$$

which contradicts (1.16), (1.17) and (1.18).

Thus, $C = 1$ and so in this case from (4.2), we obtain $F \equiv G$ and so

$$P[f] \equiv M[f].$$

Case (iii). When $D = -1$. Then from (4.1), we have

$$\frac{1}{F-1} = \frac{C}{G-1} - 1. \tag{4.3}$$

Therefore, if $C \neq -1$, then $\overline{N}\left(r, \frac{1}{G}\right) = \overline{N}\left(r, \frac{1}{F-\frac{C}{C+1}}\right)$, and by second fundamental theorem, we have

$$\overline{d}(p)T(r,f) + Q\overline{N}(r,f) \leq \overline{N}(r,f) + \overline{N}\left(r, \frac{1}{P[f]}\right) + \overline{N}\left(r, \frac{1}{M[f]}\right) + S(r,f),$$

which is same as Case (ii), which gives

$$(Q+\lambda+1)\Theta(\infty,f) + (d\overline{(p)} + d_M)\delta_{1+k}(0,f) \leq Q + \lambda + 1 + d_M,$$

which contradicts (1.16), (1.17) and (1.18).

Therefore

$$FG \equiv 1.$$

By Lemma 2.7, we get a contradiction to $FG \not\equiv 1$. Therefore, $F \equiv G$ i.e., $P[f] \equiv M[f]$

Case 4. $H \not\equiv 0$.

Subcase 4.1. $l \geq 1$ As similar as Subcase 2.1, From (3.9) and (3.10) we have

$$\begin{aligned}\overline{N}(r, \frac{1}{F-1}) + \overline{N}(r, \frac{1}{G-1}) &= N_{1)}(r, \frac{1}{F-1}) + \overline{N}_{(2}(r, \frac{1}{F-1}) + \overline{N}(r, \frac{1}{G-1}) \\ &\leq \overline{N}(r,F) + \overline{N}_{(2}(r, \frac{1}{F}) + \overline{N}_{(2}(r, \frac{1}{G}) + \overline{N}_{(l+1}(r, \frac{1}{G-1}) + \overline{N}_{(2}(r, \frac{1}{G-1}) \\ &\quad + \overline{N}(r, \frac{1}{G-1}) + \overline{N}_0(r, \frac{1}{F'}) + \overline{N}_0(r, \frac{1}{G'}) + S(r,f)\end{aligned}$$

While $l \geq 2$,

$$\overline{N}_{(l+1}(r, \frac{1}{G-1}) + \overline{N}_{(2}(r, \frac{1}{G-1}) + \overline{N}(r, \frac{1}{G-1}) \leq N(r, \frac{1}{G-1}) \leq T(r,G) + O(1).$$

So,

$$\begin{aligned}\overline{N}(r, \frac{1}{F-1}) + \overline{N}(r, \frac{1}{G-1}) &\leq \overline{N}(r,F) + \overline{N}_{(2}(r, \frac{1}{F}) + \overline{N}_{(2}(r, \frac{1}{G}) + \overline{N}_0(r, \frac{1}{F'}) \\ &\quad + \overline{N}_0(r, \frac{1}{G'}) + T(r,G) + S(r,f).\end{aligned}$$

By the second fundamental theorem, we have

$$\begin{aligned}
T(r,F)+T(r,G) &\leq \overline{N}(r,F)+\overline{N}(r,G)+\overline{N}(r,\frac{1}{F})+\overline{N}(r,\frac{1}{G})+\overline{N}(r,\frac{1}{F-1})+\overline{N}(r,\frac{1}{G-1})\\
&\quad - N_0(r,\frac{1}{F'})-N_0(r,\frac{1}{G'})+S(r,F)+S(r,G)\\
&\leq 3\overline{N}(r,F)+N_2(r,\frac{1}{F})+N_2(r,\frac{1}{G})+T(r,G)+S(r,f),\\
So,\ \ T(r,F) &\leq 3\overline{N}(r,F)+N_2(r,\frac{1}{F})+N_2(r,\frac{1}{G})+S(r,f),\\
&\leq 3\overline{N}(r,f)+N_2(r,\frac{1}{P[f]})+N_2(r,\frac{1}{M[f]})+S(r,f)\\
\overline{d}(p)T(r,f)+Q\overline{N}(r,f) &\leq 3\overline{N}(r,f)+\overline{d}(p)N_{2+k}(r,\frac{1}{f})+Q\overline{N}(r,f)+d_MN_{2+k}(r,\frac{1}{f})+\lambda\overline{N}(r,f)+S(r,f)\\
\overline{d}(p) &\leq (3+\lambda)-(3+\lambda)\Theta(\infty,f)+\overline{d}(p)-\overline{d}(p)\delta_{2+k}(0,f)+d_M-d_M\delta_{2+k}(0,f)
\end{aligned}$$

$$(3+\lambda)\Theta(\infty,f)+(\overline{d}(p+d_M))\delta_{2+k}(0,f)\leq\lambda+d_M+3.$$

This contradicts with (1.16).
While $l=1$,

$$\overline{N}_{(l+1}(r,\frac{1}{G-1})+\overline{N}(r,\frac{1}{G-1})\leq N(r,\frac{1}{G-1})\leq T(r,G)+O(1).$$

So by Lemma 2.5 for $p=1,k=1$, we have

$$\begin{aligned}
\overline{N}(r,\frac{1}{F-1})+\overline{N}(r,\frac{1}{G-1}) &\leq \overline{N}(r,F)+\overline{N}_{(2}(r,\frac{1}{F})+\overline{N}_{(2}(r,\frac{1}{G})+\overline{N}_{(2}(r,\frac{1}{F-1})+\overline{N}_0(r,\frac{1}{F'})\\
&\quad +\overline{N}_0(r,\frac{1}{G'})+T(r,G)+S(r,f).\\
&\leq \overline{N}(r,F)+\overline{N}_{(2}(r,\frac{1}{G})+\overline{N}(r,\frac{1}{F'})+\overline{N}_0(r,\frac{1}{G'})+T(r,G)+S(r,f)\\
&\leq 2\overline{N}(r,F)+\overline{N}_{(2}(r,\frac{1}{G})+N_2(r,\frac{1}{F})+\overline{N}_0(r,\frac{1}{G'})+T(r,G)+S(r,f)
\end{aligned}$$

As same as above, by the second fundamental theorem we have

$$T(r,F)+T(r,G)\leq 4\overline{N}(r,F)+2N_2(r,\frac{1}{F})+N_2(r,\frac{1}{G})+T(r,G)+S(r,f),$$

so

$$T(r,F)\leq 4\overline{N}(r,F)+2N_2(r,\frac{1}{F})+N_2(r,\frac{1}{G})+S(r,f),$$

$$\overline{d}(p)T(r,f)+Q\overline{N}(r,f)\leq 4\overline{N}(r,f)+2[\overline{d}(p)N_{2+k}\left(r,\frac{1}{f}\right)+Q\overline{N}(r,f)]+d_MN_{2+k}\left(r,\frac{1}{f}\right)$$
$$+\lambda\overline{N}(r,f)+S(r,f)$$
$$\overline{d}(p)\leq(Q+4+\lambda)-(Q+4+\lambda)\Theta(\infty,f)+2\overline{d}(p)-2\overline{d}(p)\delta_{2+k}(0,f)+d_M$$
$$-d_M\delta_{2+k}(0,f)$$
$$(Q+4+\lambda)\Theta(\infty,f)+(2\overline{d}(p+d_M))\delta_{2+k}(0,f)\leq Q+\lambda+d_M+\overline{d}(p)+4.$$

This contradicts with (1.17).

Subcase 4.2. $l=0$. From (3.14), (3.15) and (3.16) and Lemma 2.5 for $p=1,k=1$, noticing

$$N_E^{(2}(r,\frac{1}{G-1})+\overline{N}_L(r,\frac{1}{G-1})+\overline{N}(r,\frac{1}{G-1})\leq N(r,\frac{1}{G-1})\leq T(r,G)+S(r,f)$$

then

$$\overline{N}(r,\frac{1}{F-1})+\overline{N}(r,\frac{1}{G-1})=N_E^{1)}(r,\frac{1}{F-1})+N_E^{(2}(r,\frac{1}{F-1})+\overline{N}_L(r,\frac{1}{F-1})+\overline{N}_L(r,\frac{1}{G-1})$$
$$+\overline{N}(r,\frac{1}{G-1})$$
$$\leq\overline{N}(r,F)+\overline{N}_{(2}(r,\frac{1}{F})+\overline{N}_{(2}(r,\frac{1}{G})+2\overline{N}_L(r,\frac{1}{F-1})+\overline{N}_L(r,\frac{1}{G-1})$$
$$+\overline{N}_E^{(2}(r,\frac{1}{G-1})+\overline{N}_L(r,\frac{1}{G-1})+\overline{N}(r,\frac{1}{G-1})+\overline{N}_0(r,\frac{1}{F'})+\overline{N}_0(r,\frac{1}{G'})$$
$$+S(r,f)$$
$$\leq\overline{N}(r,F)+2\overline{N}(r,\frac{1}{F'})+\overline{N}(r,\frac{1}{G'})+T(r,G)+S(r,f)$$
$$\leq 4\overline{N}(r,F)+2N_2(r,\frac{1}{F})+N_2(r,\frac{1}{G})+T(r,G)+S(r,f)$$

As same as above, by the second fundamental theorem, we can obtain

$$T(r,F)+T(r,G)\leq 6\overline{N}(r,F)+3N_2(r,\frac{1}{F})+2N_2(r,\frac{1}{G})+T(r,G)+S(r,f)$$

So

$$T(r,F)\leq 6\overline{N}(r,F)+3N_2(r,\frac{1}{F})+2N_2(r,\frac{1}{G})+S(r,f)$$

$$\overline{d}(p)T(r,f)+Q\overline{N}(r,f)\leq 6\overline{N}(r,f)+3[\overline{d}(p)N_{2+k}\left(r,\frac{1}{f}\right)+Q\overline{N}(r,f)]$$
$$+2[d_MN_{2+k}\left(r,\frac{1}{f}\right)+\lambda\overline{N}(r,f)]+S(r,f)$$
$$\overline{d}(p)\leq(2Q+2\lambda+6)-(2Q+2\lambda+6)\Theta(\infty,f)+3\overline{d}(p)-3\overline{d}(p)\delta_{2+k}(0,f)$$
$$+2d_M-2d_M\delta_{2+k}(0,f)$$

$$(2Q+2\lambda+6)\Theta(\infty,f)+(3\overline{d}(p)+2d_M))\delta_{2+k}(0,f)\leq 2Q+2\lambda+2d_M+2\overline{d}(p)+6.$$

This contradicts with (1.18). Now the proof has been completed.

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