Revisiting Meir-Keeler type fixed operators on Branciari distance space

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Abstract

In this paper, we are revisiting Meir-Keeler type fixed operators in the setting of Branciari distance space. The presented results improve and generalize several existing results in the literature. We consider an example to illustrate our result.

2010 Mathematics Subject Classification. **46T99**. 47H10, 54H25
Keywords. fixed point, metric space.

1 Introduction and preliminaries

Definition 1.1. [9, 10] Let X be a non-empty set and $d : X \times X \to [0, \infty)$ be a mapping such that for all $x, y \in X$ and for all distinct points $u, v \in X$, each of them different from x and y, one has

(i) $d(x, y) = 0$ if and only if $x = y$,

(ii) $d(x, y) = d(y, x)$,

(iii) $d(x, y) \leq d(x, u) + d(u, v) + d(v, y)$ (quadrilateral inequality).

Then (X, d) is called an Branciari distance space (or BDS).

Firstly, we note that Branciari distance space has been called in several ways in distinct sources, e.g. Branciari metric space, rectangular metric space, generalized metric space and so on. As shown in [18], the standard metric and the function d defined in Definition 1.1 are not compatible. Hence, it can not be considered as a generalized metric. In other words, there is no special condition which reduces the function d defined in Definition 1.1 to a metric. That is why, in this note, we prefer to use "distance" instead of "metric".

In what follows we recollect the concepts of convergence, Cauchy sequence, completeness and continuity in the setting of Branciari distance space (X, d). Let $\{x_n\}$ be a sequence in X. We say that $\{x_n\}$ converges to some $x \in X$ if $d(x_n, x) \to 0$ as $n \to \infty$. The sequence $\{x_n\}$ is called Cauchy if $d(x_n, x_m) \to 0$ as $n, m \to \infty$. Moreover, if every Cauchy sequence in X is convergent in X, then (X, d) is called a complete Branciari distance space.

We say that a self-mapping T on Branciari distance space (X, d) is continuous if for every sequence $\{x_n\}$ in X such that $d(x_n, x) \to 0$ as $n \to \infty$, we have $d(Tx_n, Tx) \to 0$ as $n \to \infty$.

Remark 1.2. At first glance, the topology and topological properties in the setting of Branciari distance space seems very standard. On the other hand, all mentioned notions in BDS are quite

Tbilisi Mathematical Journal 12(4) (2019), pp. 97–110.
Tbilisi Centre for Mathematical Sciences.
Received by the editors: 24 January 2019.
Accepted for publication: 15 October 2019.

distinct from the corresponding notions in the standard metric space. In particular, (p1) Branciari distance function is not necessarily continuous, (p2) the topology of BDS is not necessarily Haussdorf (accordingly, the limit is not necessarily unique), (p3) an open ball in the setting of BDS is not necessarily an open set, (p4) a convergent sequence in BDS needs not to yield Cauchy property. The most interesting observation was reported by Suzuki [18]: The mentioned topologies are incompatible. For more details, see also [6]-[17]

Example 1.3. Let $X = A \cup B$ where $A = \{a, b, c\}$ and $B = \left\{\frac{1}{n}, n \in \mathbb{N}\right\}$. Define $d : X \times X \to [0, \infty)$ satisfying $d(x, y) = d(y, x)$ as follows

$$d(x,y) = \begin{cases} y, & \text{if } x \in \{b,c\}, y \in B \\ 0, & \text{if } x = y \\ 1, & otherwise \end{cases}$$

We observe that, for example,

$$1 = d(\frac{1}{2}, \frac{1}{4}) > \frac{1}{2} + \frac{1}{4} = d(\frac{1}{2}, b) + d(b, \frac{1}{4}).$$

which shows that $d(x, y)$ is not a metric space, but it is easy to see that (X, d) is a BDS. Plus, we can observe that:

$$\lim_{n\to\infty} d\left(\frac{1}{n}, b\right) = \lim_{n\to\infty} d\left(\frac{1}{n}, c\right) = \lim_{n\to\infty} \frac{1}{n} = 0,$$

hence the sequence $\left\{\frac{1}{n}\right\}$ converges to both b and c.

The following lemma will be needed in the sequel.

Lemma 1.4. [14] Let $\{x_n\}$ be a Cauchy sequence in a Branciari distance space (X, d) such that $\lim_{n\to\infty} d(x_n, x) = 0$, where $x \in X$. Then $\lim_{n\to\infty} d(x_n, y) = d(x, y)$ for all $y \in X$. In particular, the sequence $\{x_n\}$ does not converge to y if $y \neq x$.

2 Basic observations

Inspired by the interesting papers [19, 20] of Suzuki, we shall consider the following condition in the setting Branciari distance space (X, d). Let $T : X \to X$ be a self-mapping and $p : X \times X \to [0, \infty)$ a mapping.

$(P_p^1 : L)$ $x \neq y$ and $\max\left\{d(x, Tx), d(x, T^2x)\right\} \leq d(x, y)$ imply $p(x, y) \leq L(x, y)$, where

$$L(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty)\right\}.$$

$(P_p^2 : c)$ $x \neq y_n$, $\lim_{n\to\infty} d(x, y_n) = 0$ and

$$\lim_{n\to\infty} d(y_n, Ty_n) = 0 \text{ imply } \lim_{n\to\infty} p(x, y_n) \leq cd(x, Tx),$$

where $c \in [0, 1)$.

Lemma 2.1. [19] Let I be an index set. Let $p_1, p_2, q_i : X \times X \to [0,\infty)$, $i \in I$ and $c \in [0,1)$. Define functions p_3, p_4 and p_5 by

$$\begin{aligned} p_3(x,y) &= \sup\{q_i(x,y) : i \in I\}, \\ p_4(x,y) &= \max\{p_1(x,y), p_2(x,y)\}, \\ p_5(x,y) &= \alpha p_1(x,y) + \beta p_2(x,y), \end{aligned}$$

for $x, y \in X$ and $\alpha, \beta \in [0,1)$ with $\alpha + \beta = 1$. Then we have

(i) If $p_1 \le p_2$ and p_2 satisfies $(P^1_{p_2} : L)$ then p_1 also satisfies $(P^1_{p_1} : L)$.

(ii) If $p_1 \le p_2$ and p_2 satisfies $(P^2_{p_2} : c)$then p_1 also satisfies $(P2 : p_1, c)$.

(iii) If $p_3(x,y) < \infty$ holds for $x, y \in X$ and q_i satisfies $(P^1_{q_i} : L)$ for $i \in I$, then p_3 also satisfies $(P^1_{p_3} : L)$.

(iv) If p_1 and p_2 satisfy $(P^1_{p_1} : L)$ and $(P^1_{p_2} : L)$ then p_5 also satisfies $(P^1_{p_5} : L)$.

(v) If p_1 and p_2 satisfy $(P^2_{p_1}, c)$ and $(P^2_{p_2} : c)$ then p_4 and p_5 satisfy $(P^2_{p_4} : c)$ and $(P^2_{p_5} : c)$, respectively.

Example 2.2. Let (X,d) be a BDS and a function $N : X \times X \to [0,\infty)$ defined by

$$N(x,y) = \max\left\{ d(x,y), \frac{d(x,Tx) + d(y,Ty)}{2} \right\}.$$

It is easy to see that d satisfies $(P^1_d : L)$ and $(P^2_d : 0)$.

Further, due to the inequality

$$\frac{d(x,Tx) + d(y,Ty)}{2} \le \max\{d(x,Tx), d(y,Ty)\},$$

we have $d \le N \le L$. Hence, from (i), we conclude that $(P^1_N : L)$ holds. If $x \ne y_n$, $\lim_{n\to\infty} d(x,y_n) = 0$ and $\lim_{n\to\infty} d(y_n, Ty_n) = 0$ then $\limsup_{n\to\infty} N(x,y_n) = \frac{1}{2} d(x,Tx)$, thus N satisfies $(P^2_N : \frac{1}{2})$.

Lemma 2.3. Let (X,d) be a Branciari distance space, $T : X \to X$ be a given map and a function $p : X \times X \to [0,\infty)$ such that $p(x,y) = 0$ implies $d(Tx,Ty) = 0$. The following statements are equivalent.

(a) For any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$x \ne y, \varepsilon \le p(x,y) < \varepsilon + \delta \Longrightarrow d(Tx,Ty) < \varepsilon.$$

(b) For any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$x \ne y, p(x,y) < \varepsilon + \delta \Longrightarrow d(Tx,Ty) < \varepsilon.$$

(c) For any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\varepsilon \le p(x,y) < \varepsilon + \delta \Longrightarrow d(Tx,Ty) < \varepsilon.$$

(d) For any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$p(x,y) < \varepsilon + \delta \Longrightarrow d(Tx,Ty) < \varepsilon.$$

We skip the proof of this lemma since it is verbatim of the proof of Lemma 2.4. in [19].
We state the analog of Lemma 4.3 in [19] in the setting of Branciari distance space.

Lemma 2.4. Let (X, d) be a complete Branciari distance space, a map T on X a function $p : X \times X \to [0, \infty)$ and the following statements:

(a) $p(x, y) > 0$ implies $d(Tx, Ty) < p(x, y)$;

(b) $x \neq y$ implies $d(Tx, Ty) < p(x, y)$;

(c) $x \neq y$ and $p(x, y) >$ implies $d(Tx, Ty) < p(x, y)$.

Then (b)$\Rightarrow$ (a) $\Leftrightarrow$ (c).

Proof. It is sufficient to prove that (c) $\Rightarrow$ (a), since the others are obvious. Let $x, y \in X$ such that $p(x, y) > 0$. If $x = y$ then $d(Tx, Ty) = 0 < p(x, y)$. In the other case for $x \neq y$ we have $d(Tx, Ty) < p(x, y)$. Therefore, (a) holds. Q.E.D.

Lemma 2.5. If the condition

$$p(x, y) = 0 \Rightarrow x = y$$

is added to the conditions of Lemma 2.4, then (a), (b), (c) are equivalent.

Proof. If $p(x, y) = 0 \Rightarrow x = y$ then $x \neq y \Rightarrow p(x, y) > 0$. So (a) $\Rightarrow$ (b). Q.E.D.

3 Main results

Theorem 3.1. Let (X, d) be a Branciari distance space, $T : X \to X$ be a given continuous map and a function $p : X \times X \to [0, \infty)$ satisfying (P_p^1, L). Suppose that
(i) For any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that

$$x \neq y \text{ and } p(x, y) < \varepsilon + \delta(\varepsilon) \text{ imply } d(Tx, Ty) \leq \varepsilon.$$

(ii) $x \neq y$ and $p(x, y) > 0$ imply $d(Tx, Ty) < p(x, y)$.
Then T has a unique fixed point u. Moreover $\{T^n x\}$ converges to u for all $x \in X$.

Remark 3.2. Under the assumption of Theorem 3.1, we have that $p(x, y) = 0$ which implies $Tx = Ty$. Indeed, for the case $x = y$, it is obvious that $Tx = Ty$. In the other case, $x \neq y$, assumption (i) of Theorem 3.1 yields that $d(Tx, Ty) \leq \varepsilon$ for any arbitrary $\varepsilon > 0$. Hence $Tx = Ty$.

Proof. Let now $x_0 \in X$. Define a sequence $\{x_n\} \subset X$ by $x_{n+1} = Tx_n$ for $n \in \mathbb{N}$. If $x_{n_0} = x_{n_0+1}$ for some $n_0 \in \mathbb{N}$, we put $u = x_{n_0}$ and it is then obvious that u is a fixed point of T. Hence the proof is completed. From now on, we suppose that

$$x_n \neq Tx_n = x_{n+1} \text{ for all } n \in \mathbb{N}, \tag{1.1}$$

so we have $d(x_n, x_{n+1}) > 0$ for all $n \in \mathbb{N}$.
Next we assert that for $p(x_n, x_{n+1}) = p(x_n, Tx_n) > 0$, we have

$$d(x_{n+1}, x_{n+2}) < p(x_n, x_{n+1}) \leq L(x_n, x_{n+1}) = d(x_n, x_{n+1}).$$

Indeed, on account of $x_n \neq x_{n+1}$ and $p(x_n, x_{n+1}) > 0$, the property $(P_p^1 : L)$ yields

$$p(x_n, x_{n+1}) \leq L(x_n, x_{n+1}) = \max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\}.$$

If there exists $n \in \mathbb{N}$ such that $\max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\} = d(x_{n+1}, x_{n+2})$, then by (ii) we find that

$$d(x_{n+1}, x_{n+2}) < p(x_n, x_{n+1}) \leq \max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\} = d(x_{n+1}, x_{n+2}),$$

which is a contradiction. As a result, we get $L(x_n, x_{n+1}) = d(x_n, x_{n+1})$. Moreover, again by taking (ii) into account, we find that

$$d(x_{n+1}, x_{n+2}) < p(x_n, x_{n+1}) \leq d(x_n, x_{n+1}). \tag{1.2}$$

Therefore, the sequence $\{d(x_n, x_{n+1})\}$ is strictly decreasing. Then, there exists $\varepsilon \geq 0$ such that $\lim_{n\to\infty} d(x_n, x_{n+1}) = \varepsilon$. Indeed, we have $\varepsilon < d(x_{n+1}, x_{n+2})$. We claim that $\varepsilon = 0$. Suppose, on the contrary, that $\varepsilon > 0$. Then we can find $N \in \mathbb{N}$ such that there is $\delta = \delta(\varepsilon) > 0$ satisfying:

$$d(x_n, x_{n+1}) < \varepsilon + \delta(\varepsilon) \tag{1.3}$$

for all $n \geq N$. Thus, from (1.2) we have that $p(x_n, x_{n+1}) < \varepsilon + \delta(\varepsilon)$ which implies $d(Tx_n, Tx_{n+1}) \leq \varepsilon$ by assuming (ii). Taking (1.2) into account, we get

$$\varepsilon < d(x_{n+1}, x_{n+2}) < d(x_n, x_{n+1}) \leq \varepsilon, \tag{1.4}$$

which is a contradiction. Hence, $\varepsilon = 0$, that is,

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{1.5}$$

We shall also show that

$$\lim_{n\to\infty} d(x_n, x_{n+2}) = 0.$$

Keeping (1.1) in mind, we ask whether there exists $n \in \mathbb{N}$ such that $x_n = x_{n+2}$ with $x_n \neq x_{n+1}$. In other words, there is a natural number n such that the distances from x_{n+1} to both x_n an x_{n+2} are the same, i.e. $< d(x_n, x_{n+1}) = d(x_{n+2}, x_n)$. Combining this observation with (1.2), we find that

$$0 < d(x_n, x_{n+1}) < d(x_{n+1}, x_{n+2}) = d(x_{n+1}, x_n),$$

which is a contradiction. Consequently, we suppose that

$$x_{n+1} = Tx_n \neq x_n \neq T^2 x_n = x_{n+2} \text{ for any } n \in \mathbb{N}.$$

On account of $(P_p^1 : L)$, by choosing $x = x_n$ and $y = x_{n+2}$ we derive that

$$p(x_n, T^2 x_n) = p(x_n, x_{n+2}) \leq L(x_n, x_{n+2})$$

where

$$\begin{aligned} L(x_n, x_{n+2}) &= \max\{d(x_n, x_{n+2}), d(x_n, Tx_n), d(x_{n+2}, Tx_{n+2})\} \\ &= \max\{d(x_n, x_{n+2}), d(x_n, x_{n+1}), d(x_{n+2}, x_{n+3})\} \\ &= \max\{d(x_n, x_{n+2}), d(x_n, x_{n+1})\}. \end{aligned}$$

Accordingly, we obtain

$$0 < p(x_n, x_{n+2}) \leq \max\{d(x_n, x_{n+2}), d(x_n, x_{n+1})\}. \tag{1.6}$$

On the other hand, by combining the observations above together with the assumption (ii), we have

$$0 < d(x_{n+1}, x_{n+3}) = d(Tx_n, Tx_{n+2}) < p(x_n, x_{n+2}) \leq \max\{d(x_n, x_{n+2}), d(x_n, x_{n+1})\}. \tag{1.7}$$

By letting $a_n = d(x_n, x_{n+2})$ and $b_n = d(x_n, x_{n+1})$, the inequality (1.7) can be written as

$$a_{n+1} \leq \max\{a_n, b_n\}, \tag{1.8}$$

for all $n \in \mathbb{N}$. Besides, by taking (1.2) into account, we conclude that

$$b_{n+1} < b_n \leq \max\{a_n, b_n\}. \tag{1.9}$$

Therefore, for all $n \in \mathbb{N}$

$$\max\{a_{n+1}, b_{n+1}\} \leq \max\{a_n, b_n\}. \tag{1.10}$$

Then the sequence $\{\max\{a_n, b_n\}\}$ is monotone and nonincreasing. Thus, there exists $\ell \geq 0$ such that $\max\{a_n, b_n\} \to \ell$ as $n \to \infty$. Assume that $\ell > 0$. By (1.2)

$$\limsup_{n\to\infty} a_n = \limsup_{n\to\infty} \max\{a_n, b_n\} = \lim_{n\to\infty} \max\{a_n, b_n\} = \max\{\ell, 0\} = \ell.$$

Then, we have

$$0 < \ell < a_n \text{ for all } n \in \mathbb{N}.$$

For this $\ell > 0$, there exists $\delta(\ell) > 0$ such that, for chosen $n_0 \in \mathbb{N}$ we have

$$\ell < a_{n_0} < \ell + \delta(\ell). \tag{1.11}$$

By $(P_p^1 : L)$ we have

$$p(x_{n_0}, x_{n_0+2}) \leq L(x_{n_0}, x_{n_0+2}) = d(x_{n_0}, x_{n_0+2}) < \ell + \delta(\ell).$$

Combining this observation with (i) and (ii) we obtain

$$a_{n_0+1} = d(x_{n_0+1}, x_{n_0+3}) = d(Tx_{n_0}, Tx_{n_0+2}) \leq \ell.$$

Therefore, we have

$$\ell < a_{n_0+2} < a_{n_0+1} \leq \ell, \tag{1.12}$$

which is a contradiction. Hence, $\ell = 0$, that is

$$\lim_{n\to\infty} d(x_n, x_{n+2}) = 0. \tag{1.13}$$

On what follows we shall demonstrate that $x_n \neq x_m$ for al $n \neq m$ by *Reductio ad Absurdum*. Suppose that $x_n = x_m$ for some $m, n \in \mathbb{N}$ with $m \neq n$. We already have $x_n \neq x_{n+1} \Leftrightarrow d(x_n, x_{n+1}) > 0$, hence without loss of generality, we may assume that $m > n+1$. Due to the assumption $x_n = x_m$, we have $d(x_n, x_{n+1}) = d(x_m, x_{m+1})$. Thus, the inequality (1.2) implies that

$$0 < d(x_n, x_{n+1}) = d(x_m, x_{m+1}) < d(x_{m-1}, x_m) < ... < d(x_n, x_{n+1}).$$

This is a contradiction. Therefore, we conclude that $x_n \neq x_m$ for all $m, n \in \mathbb{N}$.

Now, we shall use the method of induction to show that the sequence $\{x_n\}$ is Cauchy in BDS. In other words, for a fixed $\varepsilon_2 > 0$ we will show that

$$d(x_m, x_{m+l}) < \varepsilon_2 + \delta(\varepsilon_2), \tag{1.14}$$

for sufficiently big enough natural numbers m, l. On account of (1.5) and (1.13) one can choose $N \in \mathbb{N}$ so that the following inequalities hold

$$d(x_m, x_{m+1}) < \delta(\varepsilon_2)/2 \text{ and } d(x_m, x_{m+2}) < \delta(\varepsilon_2)/2, \text{ for any } m \geq N. \tag{1.15}$$

The inequality (1.14) is satisfied trivially for $l = 1$ and $l = 2$. We assume that the condition (1.14) is satisfied for some $k \in \mathbb{N}$ and we shall show that it holds for $l = k + 1$.

If $d(x_m, x_{m+k}) \leq \varepsilon_2$, then by the quadrilateral inequality, we have

$$\begin{aligned} d(x_m, x_{m+k+1}) &\leq d(x_m, x_{m+k}) + d(x_{m+k}, x_{m+k+2}) + d(x_{m+k+1}, x_{m+k+2}) \\ &< \varepsilon_2 + d(x_m, x_{m+2}) + d(x_m, x_{m+1}) < \varepsilon_2 + \delta(\varepsilon_2)/2 + \delta(\varepsilon_2)/2 \\ &= \varepsilon_2 + \delta(\varepsilon_2) \end{aligned} \tag{1.16}$$

If $d(x_m, x_{m+k}) > \varepsilon_2$, together with (1.15),

$$d(x_m, x_{m+1}) < \delta(\varepsilon_2)/2 < \delta(\varepsilon_2) < \varepsilon_2 < d(x_m, x_{m+k})$$

By $(P_p^1 : L)$, we have

$$\begin{aligned} p(x_m, x_{m+k}) &\leq L(x_m, x_{m+k}) = \max\{d(x_m, x_{m+k}), d(x_m, x_{m+1}), d(x_{m+k}, x_{m+k+1})\} \\ &< \max\{\varepsilon_2 + \delta(\varepsilon_2), \delta(\varepsilon_2)/2, \delta(\varepsilon_2)/2\} = \varepsilon_2 + \delta(\varepsilon_2) \end{aligned}$$

which implies, by (i), that $d(Tx_m, Tx_{m+k}) \leq \varepsilon_2$. So, we have

$$\begin{aligned} d(x_m, x_{m+k+1}) &\leq d(x_m, x_{m+2}) + d(x_{m+2}, x_{m+1}) + d(x_{m+1}, x_{m+k+1}) \\ &= d(x_m, x_{m+2}) + d(x_{m+2}, x_{m+1}) + d(Tx_m, Tx_{m+k}) \\ &< \delta(\varepsilon_2)/2 + \delta(\varepsilon_2)/2 + \varepsilon_2 = \delta(\varepsilon_2) + \varepsilon_2 \end{aligned}$$

Consequently, (1.14) holds for $l = k + 1$. Hence we derive that $d(x_m, x_{m+l}) < \varepsilon_2 + \delta(\varepsilon_2)$ for all $m \geq N$ and $l \geq 1$. Since ε_2 is arbitrary, we conclude that $\limsup_{n\to\infty} d(x_n, x_m) = 0$ which shows that the sequence $\{x_n\}$ is Cauchy in the complete Branciari distance space (X, d), so it converges to a limit $u \in X$. In addition, by Lemma 1.4, this limit is unique.

Now, we show that u is a fixed point of T. Since T is continuous we have

$$\lim_{n\to\infty} d(Tx_n, Tu) = \lim_{n\to\infty} d(x_{n+1}, Tu) = 0,$$

that is, the sequence Tx_n converges to Tu as well. On account of Lemma 1.4, the limit is unique. Hence, we conclude that $Tu = u$.

As a last step, we shall show the uniqueness of a fixed point by using the method of *Reductio ad Absurdum.* Let v be another fixed point of T in X such that $u \neq v$. As it is discussed in Remark 3.2, if $p(u, v) = 0$ then $u = Tu = Tv = v$. Hence, we suppose that $p(u, v) > 0$. Thus,

$$d(u, v) = d(Tu, Tv) < p(u, v) \leq L(u, v) = d(u, v).$$

Therefore $d(u, v) = 0$ implies $u \neq v$, which is a contradiction, hence the fixed point of T is unique. Q.E.D.

In the next theorem using the idea given by Bisht in [8] we will weaken the conditions for the existence of fixed points.

Theorem 3.3. Let (X, d) be a Branciari distance space, $T : X \to X$ be a given map and a function $p : X \times X \to [0, \infty)$ satisfying $(P_p^1 : L)$. Suppose that:
(i) For any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that

$$x \neq y \text{ and } p(x, y) < \varepsilon + \delta(\varepsilon) \text{ imply } d(Tx, Ty) \leq \varepsilon.$$

(ii) $x \neq y$ and $p(x, y) > 0$ imply $d(Tx, Ty) < p(x, y)$.
(iii) T^2 is continuous.
Then T has a unique fixed point u. Moreover $\{T^n x\}$ converges to u for all $x \in X$.

Proof. Let $x_0 \in X$. Following the proof of Theorem 3.1 we see that the sequence $x_n = T^n x_0$ is convergent to a limit $u \in X$ and this limit is unique. Since we proved that $\lim_{n\to\infty} d(x_n, u) = 0$ we get from the continuity of T^2 that

$$\lim_{n\to\infty} d(T^2 x_n, Tu) = \lim_{n\to\infty} d(x_{n+2}, T^2 u) = 0. \tag{1.17}$$

On the other hand, by Lemma 1.4 we have

$$\lim_{n\to\infty} d(x_n, T^2 u) = d(u, T^2 u). \tag{1.18}$$

From (1.17) and (1.18) we derive that $d(u, T^2u)$=0. We claim that $Tu = u$. On the contrary, if $Tu \neq u$ and $p(u, Tu) > 0$, from (ii) we have

$$d(u, Tu) = d(T^2 u, Tu) < p(u, Tu) \leq d(u, Tu),$$

which is a contradiction. Thus u is a fixed point of T. The uniqueness of the fixed point can be proven as in the proof of Theorem

Q.E.D.

Another version of Theorem 3.1 can be formulated using the condition $(P_p^2 : c)$.

Theorem 3.4. Let (X,d) be a Branciari distance space, $T:X\to X$ be a given map and a function $p:X\times X\to[0,\infty)$ satisfying $(P_p^1:L)$. Suppose that:
(i) For any $\varepsilon>0$ there exists $\delta(\varepsilon)>0$ such that

$$x\neq y \text{ and } p(x,y)<\varepsilon+\delta(\varepsilon) \text{ imply } d(Tx,Ty)\leq\varepsilon.$$

(ii) $x\neq y$ and $p(x,y)>0$ imply $d(Tx,Ty)<p(x,y)$.
(iii) p satisfies (P_p^2,c) for some $c\in[0,1)$. Then T has a unique fixed point u. Moreover $\{T^nx\}$ converges to u for all $x\in X$.

Proof. Let $x_0\in X$. Following the proof of Theorem 3.1 we find a sequence $x_n=T^nx_0$ that is convergent to a limit $u\in X$. Moreover, the mentioned limit is unique and it is a fixed point of the given mapping T. Suppose, on the contrary, that $Tu\neq u$. Then, since $\{x_n\}$ is a Cauchy sequence, by Lemma 1.4, since $\lim\limits_{n\to\infty}d(x_n,x_{n+1})=0$ and $\lim\limits_{n\to\infty}d(x_n,u)=0$ it follows that $\lim\limits_{n\to\infty}d(x_n,Tu)=d(u,Tu)$. Therefore, from (ii) and $(P_p^2:c)$, for any $c\in[0,1)$ we get

$$d(u,Tu)=\lim_{n\to\infty}d(x_{n+1},Tu)\leq\limsup_{n\to\infty}\rho(x_n,u)\leq cd(u,Tu)$$

which is a contradiction. Hence, u is a fixed point of T. Following the lines in Theorem 3.1 the uniqueness of the fixed point results immediately Q.E.D.

By Lemma 2.3 and Lemma 2.4 we conclude that the Theorems 3.1, 3.3, 3.4 can be reformulate as follows.

Corollary 3.5. Let (X,d) be a Branciari distance space, $T:X\to X$ be a given map and a function $p:X\times X\to[0,\infty)$ satisfying $(P_p^1:L)$. Suppose that
(i) For any $\varepsilon>0$ there exists $\delta(\varepsilon)>0$ such that

$$p(x,y)<\varepsilon+\delta(\varepsilon) \text{ imply } d(Tx,Ty)\leq\varepsilon;$$

(ii) $p(x,y)>0$ imply $d(Tx,Ty)<p(x,y)$;
(iii) one of the following conditions is satisfied: T is continuous or T^2 is continuous or p satisfies $(P_p^2:c)$ for some $c\in[0,1)$. Then T has a unique fixed point u. Moreover $\{T^nx\}$ converges to u for all $x\in X$.

4 Further consequences and examples

Theorem 4.1. Let (X,d) be a complete Branciari distance space and $T:X\to X$ be a given map. Suppose that:
(i) For any $\varepsilon>0$ there exists $\delta(\varepsilon)>0$ such that

$$d(x,y)<\varepsilon+\delta(\varepsilon) \text{ implies } d(Tx,Ty)\leq\varepsilon.$$

(ii) $x\neq y$ implies $d(Tx,Ty)<d(x,y)$.
Then T has a unique fixed point u. Moreover $\{T^nx\}$ converges to u for all $x\in X$.

Proof. Since $x\neq y$ implies $d(x,y)>0$, assumption (ii) is equivalent with
(ii') $(d(x,y)>0$ implies $d(Tx,Ty)<d(x,y)$.
Thus, the result follows from Corollary 3.5 and Example 2.2. Q.E.D.

Theorem 4.2. Let (X, d) be a complete Branciari distance space and a mapping $T : X \to X$. Define the function $N : X \times X \to [0, \infty)$ by

$$N(x, y) = \max \left\{ d(x, y), \frac{d(x, Tx) + d(y, Ty)}{2} \right\},$$

for all $x, y \in X$. Suppose that:
(i) For any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that

$$N(x, y) < \varepsilon + \delta(\varepsilon) \text{ implies } d(Tx, Ty) \leq \varepsilon.$$

(ii) $x \neq y$ implies $d(Tx, Ty) < N(x, y)$.
Then T has a unique fixed point u. Moreover $\{T^n x\}$ converges to u for all $x \in X$.

Proof. If $N(x, y) = 0$ then $d(x, y) = 0$, therefore $x = y$. So, in the following we can suppose that $x \neq y$. Then, the assumption (ii) becomes
(ii') $N(x, y) > 0$ implies $d(Tx, Ty) < N(x, y)$
which is assumption (ii) by Corollary 3.5. On the other hand, as we have shown in the Example 2.2 the function N satisfies $(P_N^1 : L)$ and $(P_N^2 : \frac{1}{2})$. Therefore all the hypotheses in Corollary 3.5 are satisfied. Then, T has a unique fixed point. Q.E.D.

In what follows, we state Reich type fixed point theorems in BDS.

Theorem 4.3. Let (X, d) be a complete Branciari distance space and a mapping $T : X \to X$ such that T is continuous or T^2 is continuous. Define the function $q_1 : X \times X \to [0, \infty)$ by

$$q_1(x, y) = \alpha d(x, y) + \beta d(x, Tx) + \gamma d(x, T^2 x),$$

for all $x, y \in X$, where α, β, γ are nonnegative constants with $\alpha + \beta + \gamma \leq 1$. Suppose that:
(i) For any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that

$$q_1(x, y) < \varepsilon + \delta(\varepsilon) \text{ implies } d(Tx, Ty) \leq \varepsilon.$$

(ii) $x \neq y$ implies $d(Tx, Ty) < q_1(x, y)$.
Then T has a unique fixed point u. Moreover $\{T^n x\}$ converges to u for all $x \in X$.

Proof. If $q_1(x, y) = 0$ then $d(x, y) = 0$, therefore $x = y$. So, in the following we can suppose that $x \neq y$. Then, the assumption (ii) becomes
(ii') $q_1(x, y) > 0$ implies $d(Tx, Ty) < q_1(x, y)$
which is assumption (ii) by Corollary 3.5. Now, we will prove that the function q_1 satisfies condition $(P_{q_1}^1 : L)$. For this, we assume that $\max \left\{ d(x, Tx), d(x, T^2 x) \right\} \leq d(x, y)$. Then,

$$q_1(x, y) = \alpha d(x, y) + \beta d(x, Tx) + \gamma d(x, T^2 x) \leq (\alpha + \beta + \gamma) d(x, y) \leq d(x, y) \leq L(x, y).$$

Therefore $(P_{q_1}^1 : L)$ holds and all the hypotheses in Corollary 3.5 are satisfied. Then, T has a unique fixed point. Q.E.D.

Example 4.4. $X = [0,1]$, and $d : X \times X \to [0,\infty)$, where $d(x,x) = 0$, $d(\frac{1}{2},1) = d(1,\frac{1}{2}) = \frac{1}{2}$, $d(\frac{1}{2},0) = d(0,\frac{1}{2}) = \frac{1}{4}$ and $d(x,y) = \max\{x,y\}$ otherwise. We observe that

$$d(0,1) = 1 > \frac{1}{4} + \frac{1}{2} = d(0,\frac{1}{2}) + d(\frac{1}{2},1)$$

Hence (X,d) is not a metric space, but a Branciari distance space..

Let $T : X \to X$ be a map defined by

$$Tx = \begin{cases} 0, & \text{if } x \in \left[0,\frac{1}{2}\right] \\ \frac{1}{2}, & \text{if } x \in \left(\frac{1}{2},1\right] \end{cases}$$

The map T is not continuous, but T^2 is continuous (since $T^2(x) = 0$ for all $x \in [0,1]$)

Since for $x \in \left[0,\frac{1}{2}\right]$,

$$d(x,Tx) = \max\{x,0\} = x \leq \max\{x,y\} = d(x,y)$$

and for any $x \in \left(\frac{1}{2},1\right]$,

$$d(x,Tx) = \max\left\{x,\frac{1}{2}\right\} = x \leq \max\{x,y\} = d(x,y)$$

we conclude that $d(x,Tx) = x \leq d(x,y)$ for $x \neq y$. Beside this, $d(x,T^2x) = \max\{x,0\} = x \leq d(x,y)$. Therefore, for any $x,y \in X$, $x \neq y$ we have $\max\left\{d(x,Tx), d(x,T^2x)\right\} = x \leq d(x,y)$.

We distinguish the following cases:

(a) If $x \neq y$ and $x,y \in \left(0,\frac{1}{2}\right)$ or $x,y \in \left(\frac{1}{2},1\right)$ then $d(Tx,Ty) = 0 < q_1(x,y)$.

(b) If $x \in \left[0,\frac{1}{2}\right]$ and $y \in \left(\frac{1}{2},1\right)$ then $d(Tx,Ty) = d(0,\frac{1}{2}) = \frac{1}{4}$ and

$$q_1 = \alpha d(x,y) + \beta d(x,Tx) + \gamma d(x,T^2x) = \alpha y + (\beta+\gamma)x$$

(c) If $x \in \left[0,\frac{1}{2}\right)$ and $y = 1$ then $d(Tx,T1) = d(0,\frac{1}{2}) = \frac{1}{4}$ and

$$q_1 = \alpha d(x,1) + \beta d(x,Tx) + \gamma d(x,T^2x) = \alpha + (\beta+\gamma)x$$

(d) If $x = \frac{1}{2}$ and $y = 1$ then $d(T\frac{1}{2},T1) = d(0,\frac{1}{2}) = \frac{1}{4}$ and

$$q_1 = \alpha d(\frac{1}{2},1) + \beta d(\frac{1}{2},T\frac{1}{2}) + \gamma d(\frac{1}{2},T^2\frac{1}{2}) = \alpha\frac{1}{2} + (\beta+\gamma)\frac{1}{4}.$$

Consequently, for $x \neq y$, we have

$$d(Tx,Ty) < q_1(x,y)$$

(considering, for example $\alpha = \frac{1}{2}, \beta = \frac{1}{4}$ and $\gamma = \frac{1}{6}$).Thus the hypotheses of Theorem 4.3 are satisfied by T and, hence, T has a fixed point, $x = 0$.

Theorem 4.5. Let (X,d) be a complete Branciari distance space and a mapping $T : X \to X$. We define the function $q_2 : X \times X \to [0,\infty)$ by

$$q_2(x,y) = \alpha d(x,y) + \beta d(x,Tx) + \gamma d(y,Ty),$$

for all $x, y \in X$, where α, β, γ are nonnegative constants with $\alpha + \beta + \gamma \leq 1$. Suppose that:
(i) For any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that

$$q_2(x, y) < \varepsilon + \delta(\varepsilon) \text{ implies } d(Tx, Ty) \leq \varepsilon.$$

(ii) $x \neq y$ implies $d(Tx, Ty) < q_2(x, y)$.
Then T has a unique fixed point u. Moreover $\{T^n x\}$ converges to u for all $x \in X$.

Proof. Firstly, we show that the function q_2 satisfies the condition $(P^1_{q_2} : L)$. Let $x, y \in X$ such that $x \neq y$. We assume that $\max\left\{d(x, Tx), d(x, T^2x)\right\} \leq d(x, y)$. Then,

$$\begin{aligned} q_2(x, y) &= \alpha d(x, y) + \beta d(x, Tx) + \gamma d(y, Ty) \leq (\alpha + \beta) d(x, y) + \gamma d(y, Ty) \\ &\leq \max\left\{d(x, y), d(y, Ty)\right\} \leq L(x, y). \end{aligned}$$

Therefore $(P^1_{q_2} : L)$ holds. On the other hand, if we assume that $x \neq y_n$, $\lim\limits_{n\to\infty} d(x, y_n) = 0$ and $\lim\limits_{n\to\infty} d(y_n, Ty_n) = 0$. Then,

$$\limsup_{n\to\infty} q_2(x, y_n) \leq \limsup_{n\to\infty} \left(\alpha d(x, y_n) + \beta d(x, Tx) + \gamma d(y_n, Ty_n)\right) = \beta d(x, Tx)$$

which shows that q_2 satisfies $(P^2_{q_2} : \beta)$.

Thus, all hypotheses in Corollary 3.5 are satisfied. Then, T has a unique fixed point.

Q.E.D.

Example 4.6. Let $X = \{a_1, a_2, a_3, a_4, a_5\}$, and $d : X \times X \to [0, \infty)$, such that $d(x, y) = d(y, x)$, $d(x, x) = 0$ for any $x, y \in X$ and

$$\begin{aligned} d(a_1, y) &= d(a_2, y) = 1 \text{ if } y \in \{a_3, a_4, a_5\}, \\ d(a_3, a_4) &= d(a_3, a_5) = d(a_4, a_5) = \tfrac{17}{8},\ d(a_1, a_2) = \tfrac{1}{8}. \end{aligned}$$

Since

$$d(a_3, a_4) = \frac{17}{8} > 1 + 1 = d(a_3, a_1) + d(a_1, a_4),$$

we observe that (X, d) is not a metric space. Also, it is easy to show that (X, d) is a BDS. Let $T : X \to X$ be the mapping defined by $Ta_1 = Ta_2 = a_2$, $Ta_3 = Ta_4 = Ta_5 = a_1$. Consequently, $T^2x = a_2$, for all $x \in X$ and

$$\begin{aligned} d(a_1, Ta_1) &= d(a_1, a_2) = \tfrac{1}{8},\ d(a_1, T^2a_1) = d(a_1, a_2) = \tfrac{1}{8}; \\ d(a_2, Ta_2) &= d(a_2, a_2) = 0,\ d(a_2, T^2a_2) = d(a_2, a_2) = 0; \\ d(a_3, Ta_3) &= d(a_3, a_1) = 1,\ d(a_3, T^2a_3) = d(a_3, a_2) = 1; \\ d(a_4, Ta_4) &= d(a_4, a_1) = 1,\ d(a_4, T^2a_4) = d(a_4, a_2) = 1; \\ d(a_5, Ta_5) &= d(a_5, a_1) = 1,\ d(a_5, T^2a_5) = d(a_5, a_2) = 1, \end{aligned}$$

so, for any $x, y \in X$, with $x \neq y$ we have $\max\left\{d(x, Tx), d(x, T^2x)\right\} \leq d(x, y)$. Also,

$$d(Tx, Ty) < q_2(x, y) = \frac{1}{8} d(x, y) + \frac{1}{4} d(x, Tx) + \frac{1}{4} d(y, Ty),$$

so, all the assumptions of Theorem 4.5 are satisfied and a_1 is a fixed point of T.

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