Some properties of the pseudo-Chebyshev polynomials of half-integer degree

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Abstract

New sets of orthogonal functions, derived from the first and second kind Chebyshev polynomials, considering half-integer indexes, have been recently introduced. In this article several properties of these new sets are considered and the links with the classical Chebyshev polynomials are underlined.

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1 Introduction

The Chebyshev polynomials of the first and second kind were introduced by Pafnuty L. Chebyshev in the XIX Century. They can be derived as the real and imaginary part of the exponential function $e^{i\,n\,\theta} = (\cos\theta + i \sin\theta)^n$ (see equation (7)), putting $x = \cos\theta$, and using the Euler formula (see [15] for details).
The Chebyshev polynomials of the first kind are important in approximation theory and Gaussian quadrature rules. In fact, by using their roots - called Chebyshev nodes - the resulting interpolation polynomial minimizes the Runge phenomenon. Furthermore, the relevant approximation is the best approximation to a continuous function under the maximum norm.
Linked with such polynomials are the *Chebyshev polynomials of the second kind*, which appear in computing the powers of 2×2 non-singular matrices [11]. Generalizations of these polynomials have been also introduced, in particular for computing powers of higher order matrices (see e.g. [12, 13]).
An excellent book is [10]. The importance of these polynomial sets in applications is shown in [2]. Recently the Chebyshev polynomials of the first and second kind have been used in order to represent the real and imaginary part of complex Appell polynomials [17].

In a recent article, [14] new sets of functions related to the classical Chebyshev polynomials have been introduced, in connections with the complex version of the Bernoulli spiral. Actually, the real and imaginary part of the Bernoulli spirals define the Rodhonea (or Grandi) curves of fractional indexes, which often appear in natural shapes [7]. This allows to define a set functions corresponding to the first and second kind Chebyshev polynomials with fractional degree, called pseudo-Chebyshev polynomials (or pseudo-Chebyshev functions), as they are irrational functions. It was shown that, in the case of half-integer degree, the relevant pseudo-Chebyshev polynomials are orthogonal in the interval $(-1, 1)$ with respect to the same weights of the Chebyshev polynomials of the same type.

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See also [6], where the basic properties of these functions are obtained, by a different approach. Only a few properties have been proved in [14], but now we are in condition to show that many properties of the classical Chebyshev polynomials still hold in the case of the half-integer index.

We start considering the first kind pseudo-Chebyshev polynomials of half-integer degree. In the second part of the article, we will consider similar properties for the second kind pseudo-Chebyshev functions.

Remark 1.1. It is worth to note that the results obtained in what follows can be derived even by using different methods. In particular, the same achievements follow by exploiting difference equations and representing the pseudo-Chebyshev functions as multiples of Jacobi polynomials. However, dealing with elementary functions, which are essentially related to the trigonometric ones, our approach seems to be more natural. In fact we extend well known equations, related to Chebyshev polynomials, simply changing the integer degree into the half-integer one. Proofs are obtained by methods proportional to the topics to be studied, precisely by using induction or direct calculation. In such a way, we avoid to shoot flies with cannons.

For typographical reasons, we use sometimes the functions:

$$W_k(x) := \sqrt{1-x^2}\; U_k(x) = \sin((k+1)\,\arccos(x))\,. \tag{1}$$

Since in the case of half-integer degree we are dealing with irrational functions, there is no problem to introduce a further square root in the above definition.

Remark 1.2. It is suitable to remind that, as a consequence of the even or odd symmetries of the cosine or sine functions, it is possible to define the Chebyshev polynomials of the first kind [15] also for negative integer index by putting: $T_{-k}(x) := T_k(x)$, and the polynomials of the second kind by $U_{-k}(x) := -U_{k-2}(x)$.

2 The case of half-integer degree

2.1 Definitions

We will consider, in what follows, only the case of the half-integer degree. In reference [14] can be found definitions and some properties in the more general situation of rational degrees.
Note that, as in the case of classical Chebyshev polynomials we will limit to consider our functions in the interval $(-1, 1)$.
We put, by definition:

$$T_{k+\frac{1}{2}}(x) = \cos\left((k+\tfrac{1}{2})\,\arccos(x)\right) \tag{2}$$

$$W_{k-\frac{1}{2}}(x) = \sqrt{1-x^2}\; U_{k-\frac{1}{2}}(x) = \sin\left((k+\tfrac{1}{2})\,\arccos(x)\right)\,. \tag{3}$$

Note that the $W_{k-\frac{1}{2}}(x)$ are recognized as the fourth kind pseudo-Chebyshev functions in [4], where their properties are also recalled.

In particular:

$$\begin{aligned} T_{\frac{1}{2}}(x) &= \cos\left(\tfrac{1}{2}\,\arccos(x)\right) = \sqrt{\frac{1+x}{2}}\,, \\ W_{-\frac{1}{2}}(x) &= \sin\left(\tfrac{1}{2}\,\arccos(x)\right) = \sqrt{\frac{1-x}{2}}\,. \end{aligned} \tag{4}$$

$$\begin{aligned} T_{\frac{3}{2}}(x) &= \cos\left(\tfrac{3}{2}\,\arccos(x)\right) = \cos\left(\arccos(x) + \tfrac{1}{2}\,\arccos(x)\right) = \\ &= \cos(\arccos(x))\cos\left(\tfrac{1}{2}\,\arccos(x)\right) - \sin(\arccos(x))\sin\left(\tfrac{1}{2}\,\arccos(x)\right) = \\ &= x\,\sqrt{\frac{1+x}{2}} - \sqrt{1-x^2}\,\sqrt{\frac{1-x}{2}} = (2x-1)\,\sqrt{\frac{1+x}{2}}\,. \end{aligned} \tag{5}$$

Note that the last equation can be written as

$$T_{\frac{3}{2}}(x) = 2x\;T_{\frac{1}{2}}(x) - T_{-\frac{1}{2}}(x)\,, \tag{6}$$

according to the symmetry property $T_{-1/2}(x) = T_{1/2}(x)$. This is the first step for recognizing the recurrence relation (Section 3.1).

$$\begin{aligned} T_{\frac{5}{2}}(x) &= \cos\left(\tfrac{5}{2}\,\arccos(x)\right) = \cos\left(2\,\arccos(x) + \tfrac{1}{2}\,\arccos(x)\right) = \\ &= \cos(2\,\arccos(x))\cos\left(\tfrac{1}{2}\,\arccos(x)\right) - \sin(2\,\arccos(x))\sin\left(\tfrac{1}{2}\,\arccos(x)\right) = \\ &= T_2(x)\,\sqrt{\frac{1+x}{2}} - W_1(x)\,\sqrt{\frac{1-x}{2}}\,. \end{aligned} \tag{7}$$

Remark 2.1. – It is worth to note that the third and fourth kind Chebyshev polynomials $V_n(x)$ and $W_n(x)$ (see e.g. [1]) have a similar definition, but they does not coincide with the pseudo-Chebyshev functions. More information is given in Sect. 5.

2.2 Mixed type recurrences

We have, in general:

$$\begin{aligned} T_{k+\frac{1}{2}}(x) &= \cos\left(k\,\arccos(x) + \tfrac{1}{2}\,\arccos(x)\right) = \\ &= T_k(x)\,\sqrt{\frac{1+x}{2}} - W_{k-1}(x)\,\sqrt{\frac{1-x}{2}} \end{aligned} \tag{8}$$

that is:

$$T_{k+\frac{1}{2}}(x) = T_k(x)\,T_{\frac{1}{2}}(x) - W_{k-1}(x)\,W_{-\frac{1}{2}}(x)\,. \tag{9}$$

In similar way, for the second kind, we find:

$$U_{k+\frac{1}{2}}(x) = U_{k-1}(x)\,T_{\frac{1}{2}}(x) + U_{-\frac{1}{2}}(x)\,T_k(x)\,, \tag{10}$$

or

$$W_{k+\frac{1}{2}}(x) = W_{k-1}(x)\,T_{\frac{1}{2}}(x) + W_{-\frac{1}{2}}(x)\,T_k(x)\,. \tag{11}$$

3 The case of the first kind $T_{k+1/2}$

This section is devoted to derive several properties of the first kind pseudo-Chebyshev functions (their first few graphs in Figure 1) extending the validity of the corresponding classical ones to the case of an half-integer degree.

3.1 Recurrence relation

Theorem 3.1. The first kind pseudo-Chebyshev functions satisfy the recurrence relation:

$$\begin{cases} T_{k+\frac{1}{2}}(x) = 2\,x\,T_{k-\frac{1}{2}}(x) - T_{k-\frac{3}{2}}(x)\,, \\ \\ T_{\pm\frac{1}{2}}(x) = \sqrt{\frac{1+x}{2}}\,. \end{cases} \tag{12}$$

Proof. - We use induction. Equation (12) holds for $k = 1$, according to equation (6). Recalling equation (9), the induction hypothesis writes:

$$T_{k-\frac{1}{2}}(x) = T_{k-1}(x)\,T_{1/2}(x) - W_{k-2}(x)\,W_{-1/2}(x)$$

Therefore, by using the recursions of the classical Chebyshev polynomials, we have:

$$T_{k+\frac{1}{2}}(x) = T_k(x)\,T_{1/2}(x) - W_{k-1}(x)\,W_{-1/2}(x) =$$

$$= \left[2\,x\,T_{k-1}(x) - T_{k-2}\right]\,\sqrt{\frac{1+x}{2}} - \left[2\,x\,W_{k-2}(x) - W_{k-3}(x)\right]\,\sqrt{\frac{1-x}{2}} =$$

$$= 2\,x\,\left[T_{k-1}(x)\,T_{1/2}(x) - W_{k-2}(x)\,W_{-1/2}(x)\right] - \left[T_{k-2}(x)\,T_{1/2}(x) - W_{k-4}(x)\,W_{-1/2}(x)\right] =$$

$$= 2\,x\,T_{k-\frac{1}{2}}(x) - T_{k-\frac{3}{2}}(x)\,.$$

3.2 Composition identities

Theorem 3.2. The first kind pseudo-Chebyshev functions satisfy the composition identities:

$$T_{k+\frac{1}{2}}\left[T_{h+\frac{1}{2}}(x)\right] = T_{(k+\frac{1}{2})(h+\frac{1}{2})}(x)\,. \tag{13}$$

$$T_{k+\frac{1}{2}}\left(\frac{x+x^{-1}}{2}\right) = \left(\frac{x^{k+\frac{1}{2}} + x^{-(k+\frac{1}{2})}}{2}\right)\,. \tag{14}$$

Proof. - Equation (13) immediately follows from the definition (2). Note that this nesting property still holds for the multi-variable Chebyshev polynomials (see e.g. [13]).
For proving equation (14), we use induction. Indeed, for $k = 0$ we find:

$$T_{\frac{1}{2}}\left(\frac{x+x^{-1}}{2}\right) = \sqrt{\frac{1+\frac{1}{2}(x+x^{-1})}{2}} = \frac{x+1}{2\sqrt{x}} = \frac{x^{1/2}+x^{-1/2}}{2}\,.$$

Supposing valid equation (14) up to the index $k-1$, and recalling the recursion (12), we have:

$$T_{k+\frac{1}{2}}\left(\frac{x+x^{-1}}{2}\right) = 2\left(\frac{x+x^{-1}}{2}\right) T_{k-\frac{1}{2}}\left(\frac{x+x^{-1}}{2}\right) - T_{k-\frac{3}{2}}\left(\frac{x+x^{-1}}{2}\right) =$$

$$= \left(x+x^{-1}\right)\left(\frac{x^{k-1/2}+x^{-(k-1/2)}}{2}\right) - \left(\frac{x^{k-3/2}+x^{-(k-3/2)}}{2}\right) =$$

$$= \left(\frac{x^{k+1/2}+x^{-(k+1/2)}}{2}\right).$$

3.3 Product identities

The first kind pseudo-Chebyshev functions satisfy the product identities:

Theorem 3.3.

$$T_{k+\frac{1}{2}}(x)\, T_{h+\frac{1}{2}}(x) = \tfrac{1}{2}\left[T_{k+h+1}(x) + T_{k-h}(x)\right]. \tag{15}$$

Proof - Equation (15) is a straightforward consequence of the product-to-sum formula for the cosine.

3.4 Differential equation

The first kind pseudo-Chebyshev functions $T_{k+1/2}(x)$ satisfy the differential equation:

Theorem 3.4.

$$(1-x^2)\, y'' - x\, y' + \left(k+\tfrac{1}{2}\right)^2 y = 0\,. \tag{16}$$

Proof - Note that

$$D_x T_{k+\frac{1}{2}}(x) = \left(k+\tfrac{1}{2}\right)\, U_{k-\frac{1}{2}}(x)\,,$$

$$D_x^2 T_{k+\frac{1}{2}}(x) = -\left(k+\tfrac{1}{2}\right)^2 (1-x^2)^{-1}\, T_{k+\frac{1}{2}}(x) + x\,(1-x^2)^{-1}\, \left(k+\tfrac{1}{2}\right)\, U_{k-\frac{1}{2}}(x)\,,$$

$$(1-x^2)\, D_x^2 T_{k+\frac{1}{2}}(x) - x\, D_x T_{k+\frac{1}{2}}(x) = -\left(k+\tfrac{1}{2}\right)^2 T_{k+\frac{1}{2}}(x)\,,$$

so that equation (16) follows.

3.5 Orthogonality for half-integer degree

Theorem 3.5. The first kind pseudo-Chebyshev functions $T_{k+1/2}(x)$ verify the orthogonality property:

$$\int_{-1}^{1} T_{h+\frac{1}{2}}(x)\, T_{k+\frac{1}{2}}(x) \frac{1}{\sqrt{1-x^2}}\, dx = 0\,, \qquad (h \neq k)\,, \tag{17}$$

where h, k are integer numbers,

$$\int_{-1}^{1} T^2_{k+\frac{1}{2}}(x) \frac{1}{\sqrt{1-x^2}}\, dx = \frac{\pi}{2}\,. \tag{18}$$

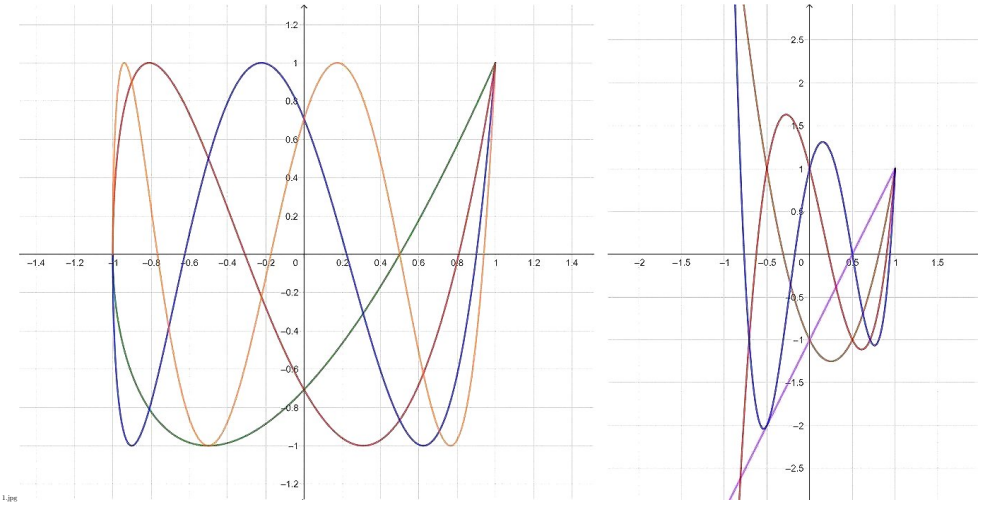


Figure 1: *Pseudo* $T_{n/2}(x),\ n = 3, 5, 7, 9.$ *Third kind* $V_k(x),\ \ k = 1, 2, 3, 4.$

4 The case of the second kind $U_{k+1/2}$

In this section we derive the main properties of the second kind pseudo-Chebyshev functions (their first few graphs in Figure 2) extending the validity of the corresponding classical ones to the case of an half-integer degree.

We recall that the classical Chebyshev polynomials of the second kind, and their extensions to the multivariate case, are used in computing powers of non-singular matrices (see e.g. [11, 12]).

4.1 Recurrence relation

Theorem 4.1. The second kind pseudo-Chebyshev functions satisfy the recurrence relation:

$$\begin{cases} U_{k+\frac{1}{2}}(x) = 2\,x\,U_{k-\frac{1}{2}}(x) - U_{k-\frac{3}{2}}(x)\,, \\ \\ U_{-\frac{1}{2}}(x) = \frac{1}{\sqrt{2(1+x)}}\,, \qquad U_{\frac{1}{2}}(x) = \frac{2x+1}{\sqrt{2(1+x)}}\,. \end{cases} \tag{19}$$

Proof. - Note that, by using the property $U_{-k}(x) := -U_{k-2}(x)$, which is still valid for rational indexes, we find $U_{-3/2} = -U_{-1/2}$, so that equation (19) is valid for $k = 0$. Therefore, by using induction - as in Theorem 3.1 - and equation (10), the result follows.

4.2 Differential equation

The second kind pseudo-Chebyshev functions $U_{k-\frac{1}{2}}(x)$ satisfy the differential equation:

Theorem 4.2.

$$(1-x^2)\,y'' - 3\,x\,y' + \left(k-\tfrac{1}{2}\right)\left(k+\tfrac{3}{2}\right)\,y = 0\,. \tag{20}$$

Proof - Note that

$$D_x U_{k-\frac{1}{2}} = \frac{x\,\sin\left[\left(k+\frac{1}{2}\right)\arccos(x)\right]}{(1-x^2)^{3/2}} - \frac{\left(k+\frac{1}{2}\right)\cos\left[\left(k+\frac{1}{2}\right)\arccos(x)\right]}{1-x^2}\,,$$

$$D_x^2 U_{k-\frac{1}{2}} = -\frac{3\left(k+\frac{1}{2}\right)\,x\,\cos\left[\left(k+\frac{1}{2}\right)\arccos(x)\right]}{(1-x^2)^2} - \frac{\left(k+\frac{1}{2}\right)^2\sin\left[\left(k+\frac{1}{2}\right)\arccos(x)\right]}{(1-x^2)^{3/2}}$$

$$+\,\frac{\sin\left[\left(k+\frac{1}{2}\right)\arccos(x)\right]}{(1-x^2)^{3/2}} + \frac{3x^2\,\sin\left[\left(k+\frac{1}{2}\right)\arccos(x)\right]}{(1-x^2)^{5/2}}\,,$$

$$(1-x^2)\,D_x^2 U_{k-\frac{1}{2}} - 3\,x\,D_x U_{k-\frac{1}{2}} = -\left(k^2+k-\tfrac{3}{4}\right)\,U_{k-\frac{1}{2}}\,,$$

so that equation (19) follows.

4.3 Orthogonality for half-integer degree

Theorem 4.3. The second kind pseudo-Chebyshev functions $U_{k+1/2}(x)$ verify the orthogonality property:

$$\int_{-1}^{1} U_{h+\frac{1}{2}}(x)\,U_{k+\frac{1}{2}}(x)\sqrt{1-x^2}\,dx = 0\,, \qquad (h\neq k)\,, \tag{21}$$

where h, k are numbers,

$$\int_{-1}^{1} U_{k+\frac{1}{2}}^2(x)\,\sqrt{1-x^2}\,dx = \frac{\pi}{2}\,. \tag{22}$$

5 The third and fourth kind Chebyshev polynomials

It is worth to note that the third and fourth kind Chebyshev polynomials $V_n(x)$ and $W_n(x)$ (see e.g. [1]) have a definition similar to that of pseudo-Chebyshev polynomials, but they does not coincide with them (see Figures 1 and 2), since actually they are true polynomials, and satisfy an orthogonality property with respect to different weights.

The third and fourth kind Chebyshev polynomials have been studied and applied by several scholars (see e.g. [1, 5, 8]), because they are useful useful in quadrature rules, when the singularities occur only at one of the end points ($+1$ or -1) (see [10]). Furthermore, recently they have been applied in Numerical Analysis for solving high odd-order boundary value problems with homogeneous or nonhomogeneous boundary conditions [5].

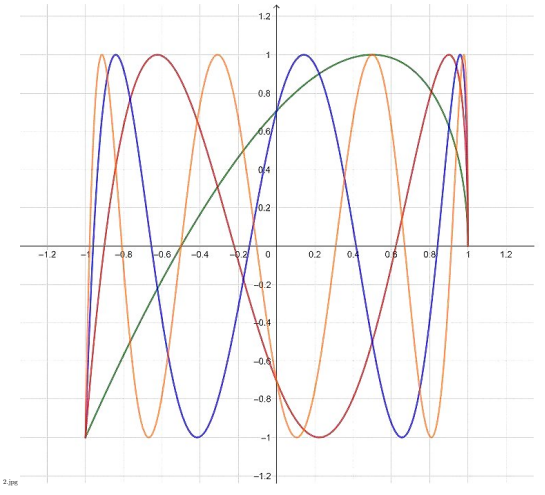
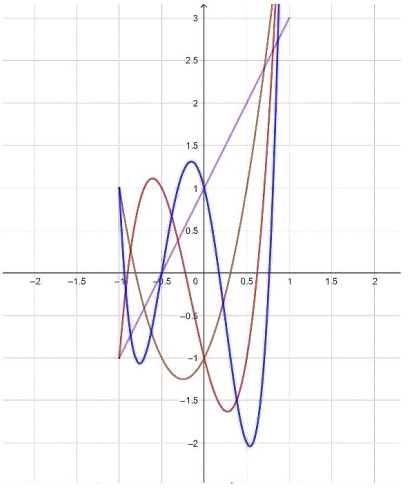


Figure 2: *Pseudo* $U_{n/2}(x)$, $n = 1, 3, 5, 7$.



Fourth kind $W_k(x)$, $k = 1, 2, 3, 4$.

5.1 Links with the pseudo-Chebyshev functions

Actually, the definitions of the third and fourth kind Chebyshev polynomials are as follows:

$$V_n(x) = \frac{\cos\left[(n+1/2)\arccos x\right]}{\cos[(\arccos x)/2]} = \sqrt{\frac{2}{1+x}}\, T_{n+\frac{1}{2}}(x) = T_{\frac{1}{2}}^{-1}(x)\;\; T_{n+\frac{1}{2}}(x)\,, \tag{23}$$

$$W_n(x) = \frac{\sin\left[(n+1/2)\arccos x\right]}{\sin[(\arccos x)/2]} = 2\sqrt{\frac{1+x}{2}}\, U_{n-\frac{1}{2}}(x) = 2\, T_{\frac{1}{2}}(x)\, U_{n-\frac{1}{2}}(x)\,. \tag{24}$$

Recalling the representation in terms of Jacobi polynomials:

$$V_n(x) = \frac{2^{2n}}{\binom{2n}{n}}\; P_n^{\left(-\frac{1}{2},\frac{1}{2}\right)}\,, \qquad W_n(x) = \frac{2^{2n}}{\binom{2n}{n}}\; P_n^{\left(\frac{1}{2},-\frac{1}{2}\right)}\,, \tag{25}$$

we find the inverse formulas:

$$T_{n+\frac{1}{2}}(x) = \cos\left[(n+1/2)\,\arccos(x)\right] = \frac{2^{2n}}{\binom{2n}{n}}\,\sqrt{\frac{1+x}{2}}\; P_n^{\left(-\frac{1}{2},\frac{1}{2}\right)}\,, \tag{26}$$

$$U_{n-\frac{1}{2}}(x) = \frac{\sin\left[(n+1/2)\arccos x\right]}{\sin(\arccos x)} = \frac{2^{2n-1}}{\binom{2n}{n}}\,\sqrt{\frac{2}{1+x}}\; P_n^{\left(\frac{1}{2},-\frac{1}{2}\right)}\,. \tag{27}$$

5.2 Hypergeometric representations

We use the hypergeometric forms of Jacobi polynomials [9]:

$$P_n^{(\alpha,\beta)}(x) = \binom{n+\alpha}{n}\ {}_2F_1\left(-n, n+\alpha+\beta+1; \alpha+1 \,\middle|\, \frac{1-x}{2}\right), \tag{28}$$

Therefore, we find representations formulas for the pseudo-Chebyshev functions.

Theorem 5.1. The first kind pseudo-Chebyshev functions can be defined trough the hypergeometric representation:

$$\begin{aligned} T_{n+\frac{1}{2}}(x) &= \binom{n-\frac{1}{2}}{n} \frac{2^{2n}}{\binom{2n}{n}} \sqrt{\frac{1+x}{2}}\ {}_2F_1\left(-n, n+1; \frac{1}{2} \,\middle|\, \frac{1-x}{2}\right) \\ &= -\frac{2^{n-1}\,\Gamma(n+\frac{1}{2})}{(2n-1)!!\,\sqrt{\pi}} \sqrt{\frac{1+x}{2}}\ {}_2F_1\left(-n, n+1; \frac{1}{2} \,\middle|\, \frac{1-x}{2}\right), \end{aligned} \tag{29}$$

The second kind pseudo-Chebyshev functions can be defined trough the hypergeometric representation:

$$\begin{aligned} U_{n+\frac{1}{2}}(x) &= \binom{n+\frac{1}{2}}{n} \frac{2^{2n-1}}{\binom{2n}{n}} \sqrt{\frac{2}{1+x}}\ {}_2F_1\left(-n, n+1; \frac{3}{2} \,\middle|\, \frac{1-x}{2}\right) \\ &= \frac{2^{n-1}\,\Gamma(n+\frac{3}{2})}{(2n-1)!!\,\sqrt{\pi}} \sqrt{\frac{2}{1+x}}\ {}_2F_1\left(-n, n+1; \frac{3}{2} \,\middle|\, \frac{1-x}{2}\right). \end{aligned} \tag{30}$$

6 Representation of the Dirichlet kernel

It is worth to note that the Dirichlet kernel $D_n(x)$ can be expressed in terms of the pseudo-Chebyshev functions.

Theorem 6.1. The representation formula of the Dirichlet kernel holds:

$$D_n(\arccos x) = W_n(x) = 2\,T_{\frac{1}{2}}(x)\,U_{n-\frac{1}{2}}(x)\ . \tag{31}$$

Proof - The Dirichlet kernel writes

$$D_n(x) = \frac{\sin[(n+1/2)x]}{\sin(x/2)}\,,$$

and therefore,

$$D_n(\arccos x) = W_n(x) = \frac{\sin[(n+1/2)\arccos x]}{\sin[(\arccos x)/2]} = \sqrt{1-x^2}\ U_{n-\frac{1}{2}}(x) \sqrt{\frac{2}{1-x}} =$$

$$= \sqrt{2\,(1+x)}\ U_{n-\frac{1}{2}}(x) = 2\ T_{\frac{1}{2}}(x)\ U_{n-\frac{1}{2}}(x)\ .$$

6.1 Summation of trigonometric series

A consequence of the preceding result is as follows. Consider a trigonometric series for a $L^1[-\pi,\pi]$, 2π-periodic function f, that is:

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(kx) + b_k \sin(kx)\,,$$

where a_k and b_k are the Fourier coefficients of f.
By Carleson's theorem [3], the above series converges in mean and even pointwise, up to a set of Lebesgue measure zero.
Since the partial sums of the above series writes

$$s_n(x,f) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos(kx) + b_k \sin(kx) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)\, D_n(t)\, dt\,, \tag{32}$$

the following result holds

Theorem 6.2. The partial sums of a Fourier series can be written in terms of the pseudo-Chebyshev functions as follows

$$\begin{aligned} s_n(x,f) &= \frac{1}{\pi}\int_{-1}^{1} f(x-\arccos\tau)\; W_n(\tau)\,\frac{1}{\sqrt{1-\tau^2}}\; d\tau = \\ &= \frac{2}{\pi}\int_{-1}^{1} f(x-\arccos\tau)\; T_{\frac{1}{2}}(\tau)\; U_{n-\frac{1}{2}}(\tau)\,\frac{1}{\sqrt{1-\tau^2}}\; d\tau = \\ &= \frac{1}{\pi}\int_{-1}^{1} f(x-\arccos\tau)\,\frac{U_{n-\frac{1}{2}}(\tau)}{U_{-\frac{1}{2}}(\tau)}\; d\tau\,. \end{aligned} \tag{33}$$

7 Conclusion

We have derived the main properties satisfied by the first and second kind pseudo-Chebyshev polynomials of half-integer degree, which actually are irrational functions. The relevant properties, and graphs of these new functions have been derived from the corresponding ones of the classical Chebyshev polynomials, simply considering the half-integer degrees instead of the integer ones.
Although this is a consequence of the continuous dependence of functions by their indexes, it seems to be quite unusual that irrational functions imitate in such a singular way the behavior of polynomial families borrowed from trigonometric functions.
Furthermore, a connection with the Dirichlet kernel and the partial sums of trigonometric series has been shown, in connection with the third and fourth kind Chebyshev polynomials.

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