On Hermite-Hadamard type inequalities for co-ordinated trigonometrically ρ-convex functions

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Abstract

In this study, we first introduce the co-ordinated trigonometrically ρ-convex functions. Then we establish some Hermite-Hadamard type inequalities for this kind of functions. We also give refinement of the obtained inequalities by using the result given by Budak. The inequalities obtained in this study provide generalizations of some result given in earlier works.

2010 Mathematics Subject Classification. **26D07**. 26D10, 26D15, 26A33
Keywords. convex function, trigonometrically ρ-convex functions, Hermite-Hadamard inequality.

1 Introduction

The inequalities discovered by C. Hermite and J. Hadamard for convex functions are considerable significant in the literature (see, e.g.,[14], [22], [31, p.137]). These inequalities state that if $f : I \to \mathbb{R}$ is a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

Both inequalities hold in the reversed direction if f is concave. We note that Hermite-Hadamard inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen's inequality.

The paper is organized as follows. After giving definition of co-ordinated convex functions and related Hermite-Hadamard inequality, we present trigonometrically ρ-convex functions and Hermite-Hadamrd type inequalities for this kind of convexity. In Section 2, we introduce the concept of co-ordinated trigonometrically ρ-convex functions. We also prove a lemma which will be frequently used next sections. Some Hermite-Hadamrd type inequalities for co-ordinated trigonometrically ρ-convex functions are obtained and some special cases of the results are also given in Section 3. In Section 4, by using the inequalities obtained by Budak, we establish a refinement of the inequalities obtained in Section 3. Finally, some conclusions and further directions of research are discussed in Section 5.

A formal definition for co-ordinated convex function may be stated as follows:

Definition 1.1. A function $f : \Delta \to \mathbb{R}$ is called co-ordinated convex on Δ, for all $(x, u), (y, v) \in \Delta$

Tbilisi Mathematical Journal 13(2) (2020), pp. 1–26.
Tbilisi Centre for Mathematical Sciences.
Received by the editors: 12 February 2019.
Accepted for publication: 25 November 2019.

and $t, s \in [0,1]$, if it satisfies the following inequality:

$$\begin{aligned} & f(tx+(1-t)\ y, su+(1-s)\ v) \\ \leq\ & ts\ f(x,u)+t(1-s)f(x,v)+s(1-t)f(y,u)+(1-t)(1-s)f(y,v). \end{aligned} \tag{1.2}$$

The mapping f is a co-ordinated concave on Δ if the inequality (1.2) holds in reversed direction for all $t, s \in [0,1]$ and $(x,u),(y,v) \in \Delta$.

In [16], Dragomir proved the following inequalities which is Hermite-Hadamard type inequalities for co-ordinated convex functions on the rectangle from the plane $\mathbb{R}^2$.

Theorem 1.2. Suppose that $f : \Delta \to \mathbb{R}$ is co-ordinated convex, then we have the following inequalities:

$$\begin{aligned} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) &\leq \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x, \frac{c+d}{2}\right)dx + \frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2}, y\right)dy\right] \\ &\leq \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx \\ &\leq \frac{1}{4}\left[\frac{1}{b-a}\int_a^b f(x,c)dx + \frac{1}{b-a}\int_a^b f(x,d)dx\right. \\ &+ \left.\frac{1}{d-c}\int_c^d f(a,y)dy + \frac{1}{d-c}\int_c^d f(b,y)dy\right] \\ &\leq \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}. \end{aligned} \tag{1.3}$$

The above inequalities are sharp. The inequalities in (1.3) hold in reverse direction if the mapping f is a co-ordinated concave mapping.

Over the years, the numerous studies have focused on to establish generalization of the inequality (1.1) and (1.3). For some of them, please see ([1], [4], [5], [6], [11]-[13], [23]-[26], [28]-[30], [34]-[40]).

1.1 Trigonometrically ρ-convex functions

First, we give the definition of trigonometrically ρ-convex functions and some related inequalities. Then we define the co-ordinated trigonometrically ρ-convex functions.

Definition 1.3. [2] A function $f : I \to \mathbb{R}$ is said to be trigonometrically ρ-convex, if for any arbitrary closed subinterval $[a,b]$ of I such that $0 < \rho\,(b-a) < \pi$, we have

$$f(x) \leq \frac{\sin\left[\rho\,(b-x)\right]}{\sin\left[\rho\,(b-a)\right]}f(a) + \frac{\sin\left[\rho\,(x-a)\right]}{\sin\left[\rho\,(b-a)\right]}f(b) \tag{1.4}$$

for all $x \in [a,b]$. If we take $x = (1-t)a + tb$, $t \in [0,1]$ in (1.4), then the condition (1.4) becomes

$$f\left((1-t)\,a + tb\right) \leq \frac{\sin\left[\rho\,(1-t)\,(b-a)\right]}{\sin\left[\rho\,(b-a)\right]}f(a) + \frac{\sin\left[\rho t\,(b-a)\right]}{\sin\left[\rho\,(b-a)\right]}f(b). \tag{1.5}$$

If the inequality (1.4) holds with "$\geq$", then the function will be called trigonometrically ρ-concave on I.

For some properties and results concerning the class of trigonometrically ρ-convex functions, see ([2], [7]-[9], [17]-[21], [27], [32], [33]).

The following Hermite-Hadamard inequality for trigonometrically ρ-convex function is proved by M. S. S. Ali and S. S. Dragomir with two different methods in [3] and [15], respectively.

Theorem 1.4. Suppose that $f : I \to \mathbb{R}$ is trigonometrically ρ-convex on I. Then for any $a, b \in I$ with $0 < b - a < \frac{\pi}{\rho}$, we have

$$\frac{2}{\rho} f\left(\frac{a+b}{2}\right) \sin\left[\frac{\rho\,(b-a)}{2}\right] \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{\rho} \tan\left[\frac{\rho\,(b-a)}{2}\right]. \tag{1.6}$$

Moreover in [15], Dragomir prove the following another version of Hermite Hadamard type inequalities for trigonometrically ρ-convex functions.

Theorem 1.5. Assume that $f : I \to \mathbb{R}$ is trigonometrically ρ-convex on I. Then for any $a, b \in I$ with $0 < b - a < \frac{\pi}{\rho}$, we have

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x) \sec\left[\rho\left(x - \frac{a+b}{2}\right)\right] dx \leq \frac{f(a)+f(b)}{2} \sec\left[\frac{\rho\,(b-a)}{2}\right] \tag{1.7}$$

In [10], Budak prove the following inequality which refines the first inequality in (1.6).

Theorem 1.6. Suppose that $f : [a, b] \to \mathbb{R}$ is a positive function with $0 < b - a < \frac{\pi}{\rho}$. If f is trigonometrically ρ-convex on $[a, b]$, then Λ is monotonically increasing on $[0, 1]$ and we have the following refinement inequality

$$\frac{2}{\rho} f\left(\frac{a+b}{2}\right) \sin\left[\frac{\rho\,(b-a)}{2}\right] \leq \Lambda(t) \leq \int_a^b f(x)dx \tag{1.8}$$

where

$$\Lambda(t) = \frac{1}{2}\int_a^b \left[\cos\left(\frac{\rho\,(1-t)\,(b-x)}{2}\right) + \cos\left(\frac{\rho\,(1-t)\,(x-a)}{2}\right)\right] f\left(tx + (1-t)\frac{a+b}{2}\right) dx.$$

2 Co-ordinated trigonometrically ρ-convex functions

In this section we introduce the concept of co-ordinated trigonometrically ρ-convex functions.

Definition 2.1. A function $f : \Delta =: [a, b] \times [c, d] \to \mathbb{R}$ is said to co-ordinated trigonometrically ρ-convex on Δ, for $0 < \rho_1\,(b-a) < \pi$ and $0 < \rho_2\,(d-c) < \pi$, if the inequality

$$\begin{aligned} f(x,y) \quad \leq \quad & \frac{\sin\left[\rho_1\,(b-x)\right]}{\sin\left[\rho_1\,(b-a)\right]} \frac{\sin\left[\rho_2\,(d-y)\right]}{\sin\left[\rho_2\,(d-c)\right]} f(a,c) + \frac{\sin\left[\rho_1\,(b-x)\right]}{\sin\left[\rho_1\,(b-a)\right]} \frac{\sin\left[\rho_2\,(y-c)\right]}{\sin\left[\rho_2\,(d-c)\right]} f(a,d) \\ & + \frac{\sin\left[\rho_1\,(x-a)\right]}{\sin\left[\rho_1\,(b-a)\right]} \frac{\sin\left[\rho_2\,(d-y)\right]}{\sin\left[\rho_2\,(d-c)\right]} f(b,c) + \frac{\sin\left[\rho_1\,(x-a)\right]}{\sin\left[\rho_1\,(b-a)\right]} \frac{\sin\left[\rho_2\,(y-c)\right]}{\sin\left[\rho_2\,(d-c)\right]} f(b,d). \end{aligned} \tag{2.1}$$

holds.

If the inequality (2.1) holds with "$\geq$", then the function will be called co-ordinated trigonometrically ρ-concave on Δ.

If we take $x = (1-t)a + tb$ and $y = (1-s)c + sd$ for $t, s, \in [0,1]$, then the inequality (2.1) can be written as

$$\begin{aligned} & f((1-t)\,a + tb, (1-s)\,c + sd) \\ \leq\ & \frac{\sin\left[\rho_1\,(1-t)\,(b-a)\right]}{\sin\left[\rho_1\,(b-a)\right]} \frac{\sin\left[\rho_2\,(1-s)\,(d-y)\right]}{\sin\left[\rho_2\,(d-c)\right]} f(a,c) \\ & + \frac{\sin\left[\rho_1\,(1-t)\,(b-a)\right]}{\sin\left[\rho_1\,(b-a)\right]} \frac{\sin\left[\rho_2 s\,(d-y)\right]}{\sin\left[\rho_2\,(d-c)\right]} f(a,d) \\ & + \frac{\sin\left[\rho_1 t\,(b-a)\right]}{\sin\left[\rho_1\,(b-a)\right]} \frac{\sin\left[\rho_2\,(1-s)\,(d-y)\right]}{\sin\left[\rho_2\,(d-c)\right]} f(b,c) + \frac{\sin\left[\rho_1\,(b-a)\right]}{\sin\left[\rho_1\,(b-a)\right]} \frac{\sin\left[\rho_2 s\,(d-y)\right]}{\sin\left[\rho_2\,(d-c)\right]} f(b,d). \end{aligned} \tag{2.2}$$

Now we give the following useful lemma:

Lemma 2.2. If $f : \Delta = [a,b] \times [c,d] \to \mathbb{R}$ is co-ordinated ρ-convex functions on Δ, then for $0 < b-a < \frac{\pi}{\rho_1}$ and $0 < d-c < \frac{\pi}{\rho_2}$, we have the following inequality

$$\begin{aligned} & \cos\left[\rho_1\left(x - \frac{a+b}{2}\right)\right] \cos\left[\rho_2\left(y - \frac{c+d}{2}\right)\right] f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\ \leq\ & \frac{1}{4}\left[f(x,y) + f(x, c+d-y) + f(a+b-x, y) + f(a+b-x, c+d-y)\right] \\ \leq\ & \frac{f(a,c) + f(a,d) + f(b,c) + f(b,d)}{4} \frac{\cos\left[\rho_1\left(x - \frac{a+b}{2}\right)\right]}{\cos\left[\frac{\rho_1(b-a)}{2}\right]} \frac{\cos\left[\rho_2\left(y - \frac{c+d}{2}\right)\right]}{\cos\left[\frac{\rho_2(d-c)}{2}\right]} \end{aligned} \tag{2.3}$$

for all $(x,y) \in \Delta$.

Proof. Since f is co-ordinated ρ-convex functions on Δ, by the inequality (2.2) with $t = s = \frac{1}{2}$, we

have

$$\begin{aligned}
& f\left(\frac{u_1+u_2}{2},\frac{v_1+v_2}{2}\right)\\
\leq\ & \frac{\sin\left[\frac{\rho_1(u_2-u_1)}{2}\right]}{\sin\left[\rho_1\left(u_2-u_1\right)\right]}\frac{\sin\left[\frac{\rho_2(v_2-v_1)}{2}\right]}{\sin\left[\rho_1\left(v_2-v_1\right)\right]}f(u_1,v_1)+\frac{\sin\left[\frac{\rho_1(u_2-u_1)}{2}\right]}{\sin\left[\rho_1\left(u_2-u_1\right)\right]}\frac{\sin\left[\frac{\rho_2(v_2-v_1)}{2}\right]}{\sin\left[\rho_1\left(v_2-v_1\right)\right]}f(u_1,v_2)\\
& +\frac{\sin\left[\frac{\rho_1(u_2-u_1)}{2}\right]}{\sin\left[\rho_1\left(u_2-u_1\right)\right]}\frac{\sin\left[\frac{\rho_2(v_2-v_1)}{2}\right]}{\sin\left[\rho_1\left(v_2-v_1\right)\right]}f(u_2,v_1)+\frac{\sin\left[\frac{\rho_1(u_2-u_1)}{2}\right]}{\sin\left[\rho_1\left(u_2-u_1\right)\right]}\frac{\sin\left[\frac{\rho_2(v_2-v_1)}{2}\right]}{\sin\left[\rho_1\left(v_2-v_1\right)\right]}f(u_2,v_2)\\
=\ & \frac{\sin\left[\frac{\rho_1(u_2-u_1)}{2}\right]}{\sin\left[\rho_1\left(u_2-u_1\right)\right]}\frac{\sin\left[\frac{\rho_2(v_2-v_1)}{2}\right]}{\sin\left[\rho_1\left(v_2-v_1\right)\right]}\left[f(u_1,v_1)+f(u_1,v_2)+f(u_2,v_1)+f(u_2,v_2)\right]\\
=\ & \frac{1}{\cos\left[\frac{\rho_1(u_2-u_1)}{2}\right]\cos\left[\frac{\rho_2(v_2-v_1)}{2}\right]}\frac{f(u_1,v_1)+f(u_1,v_2)+f(u_2,v_1)+f(u_2,v_2)}{4}.
\end{aligned}$$

If we take $u_1=x$, $u_2=a+b-x$, $v_1=y$ and $v_2=c+d-y$, we have

$$f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\leq\frac{f(x,y)+f(x,c+d-y)+f(a+b-x,y)+f(a+b-x,c+d-y)}{4\cos\left[\rho_1\left(x-\frac{a+b}{2}\right)\right]\cos\left[\rho_2\left(y-\frac{c+d}{2}\right)\right]}. \tag{2.4}$$

Since $\cos\left[\rho_1\left(x-\frac{a+b}{2}\right)\right],\cos\left[\rho_2\left(y-\frac{c+d}{2}\right)\right]\geq 0$ for $(x,y)\in\Delta$ with $0<b-a<\frac{\pi}{\rho_1}$ and $0<d-c<\frac{\pi}{\rho_2}$, if we multiply (2.4) by

$$4\cos\left[\rho_1\left(x-\frac{a+b}{2}\right)\right]\cos\left[\rho_2\left(y-\frac{c+d}{2}\right)\right],$$

then we obtain the first inequality in (2.3).

On the other hand, as f is co-ordinated ρ-convex functions on Δ, by the inequality (2.1), we get the following inequalities

$$\begin{aligned}
& f(x,y)\\
\leq\ & \frac{\sin\left[\rho_1\left(b-x\right)\right]}{\sin\left[\rho_1\left(b-a\right)\right]}\frac{\sin\left[\rho_2\left(d-y\right)\right]}{\sin\left[\rho_2\left(d-c\right)\right]}f(a,c)+\frac{\sin\left[\rho_1\left(b-x\right)\right]}{\sin\left[\rho_1\left(b-a\right)\right]}\frac{\sin\left[\rho_2\left(y-c\right)\right]}{\sin\left[\rho_2\left(d-c\right)\right]}f(a,d) \qquad (2.5)\\
& +\frac{\sin\left[\rho_1\left(x-a\right)\right]}{\sin\left[\rho_1\left(b-a\right)\right]}\frac{\sin\left[\rho_2\left(d-y\right)\right]}{\sin\left[\rho_2\left(d-c\right)\right]}f(b,c)+\frac{\sin\left[\rho_1\left(x-a\right)\right]}{\sin\left[\rho_1\left(b-a\right)\right]}\frac{\sin\left[\rho_2\left(y-c\right)\right]}{\sin\left[\rho_2\left(d-c\right)\right]}f(b,d),
\end{aligned}$$

$$\begin{aligned}
& f(x,c+d-y))\\
\leq\ & \frac{\sin\left[\rho_1\left(b-x\right)\right]}{\sin\left[\rho_1\left(b-a\right)\right]}\frac{\sin\left[\rho_2\left(y-c\right)\right]}{\sin\left[\rho_2\left(d-c\right)\right]}f(a,c)+\frac{\sin\left[\rho_1\left(b-x\right)\right]}{\sin\left[\rho_1\left(b-a\right)\right]}\frac{\sin\left[\rho_2\left(d-y\right)\right]}{\sin\left[\rho_2\left(d-c\right)\right]}f(a,d \qquad (2.6)\\
& +\frac{\sin\left[\rho_1\left(x-a\right)\right]}{\sin\left[\rho_1\left(b-a\right)\right]}\frac{\sin\left[\rho_2\left(y-c\right)\right]}{\sin\left[\rho_2\left(d-c\right)\right]}f(b,c)+\frac{\sin\left[\rho_1\left(x-a\right)\right]}{\sin\left[\rho_1\left(b-a\right)\right]}\frac{\sin\left[\rho_2\left(d-y\right)\right]}{\sin\left[\rho_2\left(d-c\right)\right]}f(b,d),
\end{aligned}$$

$$
\begin{aligned}
& f(a+b-x,y) \\
\leq\ & \frac{\sin\left[\rho_1\left(x-a\right)\right]}{\sin\left[\rho_1\left(b-a\right)\right]}\frac{\sin\left[\rho_2\left(d-y\right)\right]}{\sin\left[\rho_2\left(d-c\right)\right]}f(a,c)+\frac{\sin\left[\rho_1\left(x-a\right)\right]}{\sin\left[\rho_1\left(b-a\right)\right]}\frac{\sin\left[\rho_2\left(y-c\right)\right]}{\sin\left[\rho_2\left(d-c\right)\right]}f(a,d) \\
& +\frac{\sin\left[\rho_1\left(b-x\right)\right]}{\sin\left[\rho_1\left(b-a\right)\right]}\frac{\sin\left[\rho_2\left(d-y\right)\right]}{\sin\left[\rho_2\left(d-c\right)\right]}f(b,c)+\frac{\sin\left[\rho_1\left(b-x\right)\right]}{\sin\left[\rho_1\left(b-a\right)\right]}\frac{\sin\left[\rho_2\left(y-c\right)\right]}{\sin\left[\rho_2\left(d-c\right)\right]}f(b,d),
\end{aligned}
\tag{2.7}
$$

and

$$
\begin{aligned}
& f(a+b-x,c+d-y) \\
\leq\ & \frac{\sin\left[\rho_1\left(x-a\right)\right]}{\sin\left[\rho_1\left(b-a\right)\right]}\frac{\sin\left[\rho_2\left(y-c\right)\right]}{\sin\left[\rho_2\left(d-c\right)\right]}f(a,c)+\frac{\sin\left[\rho_1\left(x-a\right)\right]}{\sin\left[\rho_1\left(b-a\right)\right]}\frac{\sin\left[\rho_2\left(d-y\right)\right]}{\sin\left[\rho_2\left(d-c\right)\right]}f(a,d) \\
& +\frac{\sin\left[\rho_1\left(b-x\right)\right]}{\sin\left[\rho_1\left(b-a\right)\right]}\frac{\sin\left[\rho_2\left(y-c\right)\right]}{\sin\left[\rho_2\left(d-c\right)\right]}f(b,c)+\frac{\sin\left[\rho_1\left(b-x\right)\right]}{\sin\left[\rho_1\left(b-a\right)\right]}\frac{\sin\left[\rho_2\left(d-y\right)\right]}{\sin\left[\rho_2\left(d-c\right)\right]}f(b,d)
\end{aligned}
\tag{2.8}
$$

for $(x,y)\in\Delta$. By summing the inequalities (2.5)-(2.8), we obtain

$$
\begin{aligned}
& f(x,y)+f(x,c+d-y)+f(a+b-x,y)+f(a+b-x,c+d-y) \\
\leq\ & \frac{\sin\left[\rho_1\left(b-x\right)\right]}{\sin\left[\rho_1\left(b-a\right)\right]}\frac{\sin\left[\rho_2\left(d-y\right)\right]}{\sin\left[\rho_2\left(d-c\right)\right]}f(a,c)+\frac{\sin\left[\rho_1\left(b-x\right)\right]}{\sin\left[\rho_1\left(b-a\right)\right]}\frac{\sin\left[\rho_2\left(y-c\right)\right]}{\sin\left[\rho_2\left(d-c\right)\right]}f(a,d) \\
& +\frac{\sin\left[\rho_1\left(x-a\right)\right]}{\sin\left[\rho_1\left(b-a\right)\right]}\frac{\sin\left[\rho_2\left(d-y\right)\right]}{\sin\left[\rho_2\left(d-c\right)\right]}f(b,c)+\frac{\sin\left[\rho_1\left(x-a\right)\right]}{\sin\left[\rho_1\left(b-a\right)\right]}\frac{\sin\left[\rho_2\left(y-c\right)\right]}{\sin\left[\rho_2\left(d-c\right)\right]}f(b,d) \\
& +\frac{\sin\left[\rho_1\left(b-x\right)\right]}{\sin\left[\rho_1\left(b-a\right)\right]}\frac{\sin\left[\rho_2\left(y-c\right)\right]}{\sin\left[\rho_2\left(d-c\right)\right]}f(a,c)+\frac{\sin\left[\rho_1\left(b-x\right)\right]}{\sin\left[\rho_1\left(b-a\right)\right]}\frac{\sin\left[\rho_2\left(d-y\right)\right]}{\sin\left[\rho_2\left(d-c\right)\right]}f(a,d) \\
& +\frac{\sin\left[\rho_1\left(x-a\right)\right]}{\sin\left[\rho_1\left(b-a\right)\right]}\frac{\sin\left[\rho_2\left(y-c\right)\right]}{\sin\left[\rho_2\left(d-c\right)\right]}f(b,c)+\frac{\sin\left[\rho_1\left(x-a\right)\right]}{\sin\left[\rho_1\left(b-a\right)\right]}\frac{\sin\left[\rho_2\left(d-y\right)\right]}{\sin\left[\rho_2\left(d-c\right)\right]}f(b,d) \\
& +\frac{\sin\left[\rho_1\left(x-a\right)\right]}{\sin\left[\rho_1\left(b-a\right)\right]}\frac{\sin\left[\rho_2\left(d-y\right)\right]}{\sin\left[\rho_2\left(d-c\right)\right]}f(a,c)+\frac{\sin\left[\rho_1\left(x-a\right)\right]}{\sin\left[\rho_1\left(b-a\right)\right]}\frac{\sin\left[\rho_2\left(y-c\right)\right]}{\sin\left[\rho_2\left(d-c\right)\right]}f(a,d)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\sin\left[\rho_1\left(b-x\right)\right]}{\sin\left[\rho_1\left(b-a\right)\right]}\frac{\sin\left[\rho_2\left(d-y\right)\right]}{\sin\left[\rho_2\left(d-c\right)\right]}f(b,c)+\frac{\sin\left[\rho_1\left(b-x\right)\right]}{\sin\left[\rho_1\left(b-a\right)\right]}\frac{\sin\left[\rho_2\left(y-c\right)\right]}{\sin\left[\rho_2\left(d-c\right)\right]}f(b,d)\\
&+\frac{\sin\left[\rho_1\left(x-a\right)\right]}{\sin\left[\rho_1\left(b-a\right)\right]}\frac{\sin\left[\rho_2\left(y-c\right)\right]}{\sin\left[\rho_2\left(d-c\right)\right]}f(a,c)+\frac{\sin\left[\rho_1\left(x-a\right)\right]}{\sin\left[\rho_1\left(b-a\right)\right]}\frac{\sin\left[\rho_2\left(d-y\right)\right]}{\sin\left[\rho_2\left(d-c\right)\right]}f(a,d)\\
&+\frac{\sin\left[\rho_1\left(b-x\right)\right]}{\sin\left[\rho_1\left(b-a\right)\right]}\frac{\sin\left[\rho_2\left(y-c\right)\right]}{\sin\left[\rho_2\left(d-c\right)\right]}f(b,c)+\frac{\sin\left[\rho_1\left(b-x\right)\right]}{\sin\left[\rho_1\left(b-a\right)\right]}\frac{\sin\left[\rho_2\left(d-y\right)\right]}{\sin\left[\rho_2\left(d-c\right)\right]}f(b,d)\\
=\;&\frac{\sin\left[\rho_1\left(b-x\right)\right]+\sin\left[\rho_1\left(x-a\right)\right]}{\sin\left[\rho_1\left(b-a\right)\right]}\frac{\sin\left[\rho_2\left(d-y\right)\right]+\sin\left[\rho_2\left(y-c\right)\right]}{\sin\left[\rho_2\left(d-c\right)\right]}\\
&\times\left[f(a,c)+f(a,d)+f(b,c)+f(b,d)\right].
\end{aligned}
$$

It can be easily seen that

$$
\frac{\sin\left[\rho_1\left(b-x\right)\right]+\sin\left[\rho_1\left(x-a\right)\right]}{\sin\left[\rho_1\left(b-a\right)\right]}=\frac{\cos\left[\rho_1\left(x-\frac{a+b}{2}\right)\right]}{\cos\left[\frac{\rho_1(b-a)}{2}\right]}
$$

and

$$
\frac{\sin\left[\rho_2\left(d-y\right)\right]+\sin\left[\rho_2\left(y-c\right)\right]}{\sin\left[\rho_2\left(d-c\right)\right]}=\frac{\cos\left[\rho_2\left(y-\frac{c+d}{2}\right)\right]}{\cos\left[\frac{\rho_2(d-c)}{2}\right]}.
$$

This completes the proof of second inequality in (2.3). Q.E.D.

3 Hermite-Hadamard type inequlities

In this section we establish some Hermite-Hadamard type inequalities for co-ordinated trigonometrically ρ-convex functions.

Theorem 3.1. If $f:\Delta\to\mathbb{R}$ is a co-ordinated trigonometrically ρ-convex functions on Δ with $0<b-a<\frac{\pi}{\rho_1}$ and $0<d-c<\frac{\pi}{\rho_2}$, then we have the following Hermite-Hadamard type inequalities

$$
\begin{aligned}
&\frac{4}{\rho_1\rho_2}\sin\left[\frac{\rho_1\left(b-a\right)}{2}\right]\sin\left[\frac{\rho_2\left(d-c\right)}{2}\right]f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\
\leq\;&\int_a^b\int_c^d f(x,y)dydx \qquad (3.1)\\
\leq\;&\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{\rho_1\rho_2}\tan\left[\frac{\rho_1\left(b-a\right)}{2}\right]\tan\left[\frac{\rho_2\left(d-c\right)}{2}\right].
\end{aligned}
$$

Proof. Integrating the inequality (2.3) with respect to (x,y) on Δ, we obtain

$$\begin{aligned} & f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\left(\int_a^b \cos\left[\rho_1\left(x-\frac{a+b}{2}\right)\right]dx\right)\left(\int_c^d \cos\left[\rho_2\left(y-\frac{c+d}{2}\right)\right]dy\right) \\ \leq & \frac{1}{4}\int_a^b\int_c^d \left[f(x,y)+f(x,c+d-y)+f(a+b-x,y)+f(a+b-x,c+d-y)\right]dydx \quad (3.2) \\ \leq & \left(\int_a^b \cos\left[\rho_1\left(x-\frac{a+b}{2}\right)\right]dx\right)\left(\int_c^d \cos\left[\rho_2\left(y-\frac{c+d}{2}\right)\right]dy\right) \\ & \times\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4\cos\left[\frac{\rho_1(b-a)}{2}\right]\cos\left[\frac{\rho_2(d-c)}{2}\right]}. \end{aligned}$$

Here we have

$$\int_a^b \cos\left[\rho_1\left(x-\frac{a+b}{2}\right)\right]dx=\frac{2}{\rho_1}\sin\left[\frac{\rho_1(b-a)}{2}\right] \quad (3.3)$$

and similarly

$$\int_c^d \cos\left[\rho_2\left(y-\frac{c+d}{2}\right)\right]dy=\frac{2}{\rho_2}\sin\left[\frac{\rho_2(d-c)}{2}\right]. \quad (3.4)$$

Moreover by change of variable, we get

$$\begin{aligned} \int_a^b\int_c^d f(x,c+d-y)dydx &= \int_a^b\int_c^d f(a+b-x,y)\,dydx \\ &= \int_a^b\int_c^d f(a+b-x,c+d-y)dydx \quad (3.5) \\ &= \int_a^b\int_c^d f(x,y)dydx. \end{aligned}$$

If we substitute the equalities (3.3)-(3.5) in (3.2), then we obtain the required result (3.1). Q.E.D.

Remark 3.2. For $\rho_1\to 0$ and $\rho_2\to 0$ we observe that

$$\lim_{\rho_1\to 0}\frac{2}{\rho_1}\sin\left[\frac{\rho_1(b-a)}{2}\right]=b-a,\quad \lim_{\rho_2\to 0}\frac{2}{\rho_2}\sin\left[\frac{\rho_2(d-c)}{2}\right]=d-c \quad (3.6)$$

and

$$\lim_{\rho_1\to 0}\frac{1}{\rho_1}\tan\left[\frac{\rho_1(b-a)}{2}\right]=\frac{b-a}{2},\quad \lim_{\rho_2\to 0}\frac{1}{\rho_2}\tan\left[\frac{\rho_2(d-c)}{2}\right]=\frac{d-c}{2}. \quad (3.7)$$

Thus, from Theorem 3.1 in the limit $\rho_1 \to 0$ and $\rho_2 \to 0$, we the inequalities

$$f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \leq \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)dydx \leq \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} \quad (3.8)$$

which proved by Dragomir in [16].

Theorem 3.3. If $f : \Delta \to \mathbb{R}$ is a co-ordinated trigonometrically ρ-convex functions on Δ with $0 < b-a < \frac{\pi}{\rho_1}$ and $0 < d-c < \frac{\pi}{\rho_2}$, then we have the following Hermite-Hadamard type inequalities

$$\begin{aligned} & f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\ \leq & \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y) \sec\left[\rho_1\left(x-\frac{a+b}{2}\right)\right] \sec\left[\rho_2\left(y-\frac{c+d}{2}\right)\right] dydx \quad (3.9) \\ \leq & \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} \sec\left[\frac{\rho_1(b-a)}{2}\right] \sec\left[\frac{\rho_2(d-c)}{2}\right]. \end{aligned}$$

Proof. If we divide the inequality (2.3) by $\cos\left[\rho_1\left(x-\frac{a+b}{2}\right)\right] \cos\left[\rho_2\left(y-\frac{c+d}{2}\right)\right]$, we obtain

$$\begin{aligned} & f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\ \leq & \frac{[f(x,y)+f(x,c+d-y)+f(a+b-x,y)+f(a+b-x,c+d-y)]}{4} \quad (3.10) \\ & \times \sec\left[\rho_1\left(x-\frac{a+b}{2}\right)\right] \sec\left[\rho_2\left(y-\frac{c+d}{2}\right)\right] \\ \leq & \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} \sec\left[\frac{\rho_1(b-a)}{2}\right] \sec\left[\frac{\rho_2(d-c)}{2}\right]. \end{aligned}$$

Integrating the inequalities (3.10) on Δ, we have

$$\begin{aligned} & f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\ \leq & \frac{1}{4(b-a)(d-c)} \int_a^b \int_c^d [f(x,y)+f(x,c+d-y)+f(a+b-x,y)+f(a+b-x,c+d-y)] \quad (3.11) \\ & \times \sec\left[\rho_1\left(x-\frac{a+b}{2}\right)\right] \sec\left[\rho_2\left(y-\frac{c+d}{2}\right)\right] dydx \\ \leq & \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} \sec\left[\frac{\rho_1(b-a)}{2}\right] \sec\left[\frac{\rho_2(d-c)}{2}\right]. \end{aligned}$$

By the change of variable $v = c + d - y$, we have

$$\begin{aligned}
&\int_a^b \int_c^d f(x, c+d-y) \sec\left[\rho_1\left(x - \frac{a+b}{2}\right)\right] \sec\left[\rho_2\left(y - \frac{c+d}{2}\right)\right] dydx \\
&= \int_a^b \int_c^d f(x, v) \sec\left[\rho_1\left(x - \frac{a+b}{2}\right)\right] \sec\left[\rho_2\left(\frac{c+d}{2} - v\right)\right] dvdx, \qquad (3.12) \\
&= \int_a^b \int_c^d f(x, v) \sec\left[\rho_1\left(x - \frac{a+b}{2}\right)\right] \sec\left[\rho_2\left(v - \frac{c+d}{2}\right)\right] dvdx.
\end{aligned}$$

Similarly using the change of variable we obtain the following equalities.

$$\begin{aligned}
&\int_a^b \int_c^d f(a+b-x, y) \sec\left[\rho_1\left(x - \frac{a+b}{2}\right)\right] \sec\left[\rho_2\left(y - \frac{c+d}{2}\right)\right] dydx \\
&= \int_a^b \int_c^d f(u, y) \sec\left[\rho_1\left(u - \frac{a+b}{2}\right)\right] \sec\left[\rho_2\left(y - \frac{c+d}{2}\right)\right] dydu,
\end{aligned} \qquad (3.13)$$

and

$$\begin{aligned}
&\int_a^b \int_c^d f(a+b-x, c+d-y) \sec\left[\rho_1\left(x - \frac{a+b}{2}\right)\right] \sec\left[\rho_2\left(y - \frac{c+d}{2}\right)\right] dydx \\
&= \int_a^b \int_c^d f(u, v) \sec\left[\rho_1\left(u - \frac{a+b}{2}\right)\right] \sec\left[\rho_2\left(v - \frac{c+d}{2}\right)\right] dvdu.
\end{aligned} \qquad (3.14)$$

By substituting the equalities (3.12)-(3.14) in (3.11), we establish the desired result (3.9). Q.E.D.

Remark 3.4. If we choose $\rho_1 = \rho_2 = 0$ in Theorem 3.3, the inequalities (3.9) reduce to the inequalities (3.8).

Theorem 3.5. If $f : \Delta \to \mathbb{R}$ is a co-ordinated trigonometrically ρ-convex functions on Δ with

$0 < b-a < \frac{\pi}{\rho_1}$ and $0 < d-c < \frac{\pi}{\rho_2}$, then we have the following Hermite-Hadamard type inequalities,

$$\begin{aligned}
&\frac{1}{4} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right)\left[(d-c)+\frac{1}{2\rho_2}\sin\left[\rho_2(d-c)\right]\right]\left[(b-a)+\frac{1}{2\rho_1}\sin\left[\rho_1(b-a)\right]\right] \\
\leq\ & \int_a^b\int_c^d f(x,y)\cos\left[\rho_1\left(x-\frac{a+b}{2}\right)\right]\cos\left[\rho_2\left(y-\frac{c+d}{2}\right)\right]dydx \qquad (3.15)\\
\leq\ & \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{16}\left[(d-c)+\frac{1}{2\rho_2}\sin\left[\rho_2(d-c)\right]\right] \\
&\times\left[(b-a)+\frac{1}{2\rho_1}\sin\left[\rho_1(b-a)\right]\right]\sec\left[\frac{\rho_1(b-a)}{2}\right]\sec\left[\frac{\rho_2(d-c)}{2}\right].
\end{aligned}$$

Proof. If we multiply the inequality (2.3) by $\cos\left[\rho_1\left(x-\frac{a+b}{2}\right)\right]\cos\left[\rho_2\left(y-\frac{c+d}{2}\right)\right]$, we obtain

$$\begin{aligned}
&\cos^2\left[\rho_1\left(x-\frac{a+b}{2}\right)\right]\cos^2\left[\rho_2\left(y-\frac{c+d}{2}\right)\right] f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\
\leq\ & \frac{1}{4}\left[f(x,y)+f(x,c+d-y)+f(a+b-x,y)+f(a+b-x,c+d-y)\right. \qquad (3.16)\\
&\left.\times\cos\left[\rho_1\left(x-\frac{a+b}{2}\right)\right]\cos\left[\rho_2\left(y-\frac{c+d}{2}\right)\right]\right] \\
\leq\ & \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\frac{\cos^2\left[\rho_1\left(x-\frac{a+b}{2}\right)\right]}{\cos\left[\frac{\rho_1(b-a)}{2}\right]}\frac{\cos^2\left[\rho_2\left(y-\frac{c+d}{2}\right)\right]}{\cos\left[\frac{\rho_2(d-c)}{2}\right]}.
\end{aligned}$$

Integrating the inequalities (3.16) on Δ, we have

$$\begin{aligned}
&\int_a^b\int_c^d \cos^2\left[\rho_1\left(x-\frac{a+b}{2}\right)\right]\cos^2\left[\rho_2\left(y-\frac{c+d}{2}\right)\right] f\left(\frac{a+b}{2}, \frac{c+d}{2}\right)dydx \\
\leq\ & \int_a^b\int_c^d \frac{1}{4}\left[f(x,y)+f(x,c+d-y)+f(a+b-x,y)+f(a+b-x,c+d-y)\right] \qquad (3.17)\\
&\times\cos\left[\rho_1\left(x-\frac{a+b}{2}\right)\right]\cos\left[\rho_2\left(y-\frac{c+d}{2}\right)\right]dydx \\
\leq\ & \int_a^b\int_c^d \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4\cos\left[\frac{\rho_1(b-a)}{2}\right]\cos\left[\frac{\rho_2(d-c)}{2}\right]}\cos^2\left[\rho_1\left(x-\frac{a+b}{2}\right)\right]\cos^2\left[\rho_2\left(y-\frac{c+d}{2}\right)\right]dydx.
\end{aligned}$$

That is

$$\begin{aligned}
& f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\left(\int_a^b \cos^2\left[\rho_1\left(x-\frac{a+b}{2}\right)\right]dx\right)\left(\int_c^d \cos^2\left[\rho_2\left(y-\frac{c+d}{2}\right)\right]dy\right) \\
\leq & \int_a^b\int_c^d \frac{1}{4}\left[f(x,y)+f(x,c+d-y)+f(a+b-x,y)+f(a+b-x,c+d-y)\right] \\
& \times\cos\left[\rho_1\left(x-\frac{a+b}{2}\right)\right]\cos\left[\rho_2\left(y-\frac{c+d}{2}\right)\right]dydx \qquad (3.18)\\
\leq & \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4\cos\left[\frac{\rho_1(b-a)}{2}\right]\cos\left[\frac{\rho_2(d-c)}{2}\right]} \\
& \times\left(\int_a^b \cos^2\left[\rho_1\left(x-\frac{a+b}{2}\right)\right]dx\right)\left(\int_c^d \cos^2\left[\rho_2\left(y-\frac{c+d}{2}\right)\right]dy\right).
\end{aligned}$$

A simple calculation we have

$$\begin{aligned}
& \int_a^b \cos^2\left[\rho_1\left(x-\frac{a+b}{2}\right)\right]dx \\
= & \frac{1}{2}\left[(b-a)+\int_a^b \cos\left[2\rho_1\left(x-\frac{a+b}{2}\right)\right]dx\right] \\
= & \frac{1}{2}\left[(b-a)+\frac{1}{2\rho_1}\sin\left[2\rho_1\left(x-\frac{a+b}{2}\right)\right]\Big|_a^b\right] \qquad (3.19)\\
= & \frac{1}{2}\left[(b-a)+\frac{1}{2\rho_1}\sin\left[2\rho_1\left(b-\frac{a+b}{2}\right)\right]-\frac{1}{2\rho_1}\sin\left[2\rho_1\left(a-\frac{a+b}{2}\right)\right]\right] \\
= & \frac{1}{2}\left[(b-a)+\frac{1}{\rho_1}\sin\left[\rho_1(b-a)\right]\right]
\end{aligned}$$

and similarly

$$\int_c^d \cos^2\left[\rho_2\left(y-\frac{c+d}{2}\right)\right]dy=\frac{1}{2}\left[(d-c)+\frac{1}{\rho_2}\sin\left[\rho_2(d-c)\right]\right]. \qquad (3.20)$$

By change of variable $v = c + d - y$, then w have

$$\begin{aligned} & \int_a^b \int_c^d f(x, c+d-y) \cos\left[\rho_1\left(x - \frac{a+b}{2}\right)\right] \cos\left[\rho_2\left(y - \frac{c+d}{2}\right)\right] dydx \\ = & \int_a^b \int_c^d f(x,v) \cos\left[\rho_1\left(x - \frac{a+b}{2}\right)\right] \cos\left[\rho_2\left(v - \frac{c+d}{2}\right)\right] dvdx. \end{aligned} \tag{3.21}$$

Similarly, using change of variable we have

$$\begin{aligned} \leq & \int_a^b \int_c^d f(a+b-x, y) \cos\left[\rho_1\left(x - \frac{a+b}{2}\right)\right] \cos\left[\rho_2\left(y - \frac{c+d}{2}\right)\right] dydx \\ = & \int_a^b \int_c^d f(u,y) \cos\left[\rho_1\left(u - \frac{a+b}{2}\right)\right] \cos\left[\rho_2\left(y - \frac{c+d}{2}\right)\right] dydu, \end{aligned} \tag{3.22}$$

and

$$\begin{aligned} \leq & \int_a^b \int_c^d f(a+b-x, c+d-y) \cos\left[\rho_1\left(x - \frac{a+b}{2}\right)\right] \cos\left[\rho_2\left(y - \frac{c+d}{2}\right)\right] dydx \\ = & \int_a^b \int_c^d f(u,v) \cos\left[\rho_1\left(u - \frac{a+b}{2}\right)\right] \cos\left[\rho_2\left(v - \frac{c+d}{2}\right)\right] dvdu. \end{aligned} \tag{3.23}$$

By substituting the equalities (3.19)-(3.23) in (3.18), we establish the desired result (3.15) Q.E.D.

Remark 3.6. For $\rho_1 \to 0$ and $\rho_2 \to 0$ we observe that

$$\lim_{\rho_1 \to 0} \frac{1}{2\rho_1} \sin\left[\rho_1 (b-a)\right] = \frac{b-a}{2}, \quad \lim_{\rho_2 \to 0} \frac{1}{2\rho_2} \sin\left[\rho_2 (d-c)\right] = \frac{d-c}{2}.$$

Thus, from Theorem 3.5 in the limit $\rho_1 \to 0$ and $\rho_2 \to 0$, we have the following Hermite-Hadamard type inequalities

$$f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \leq \frac{16}{9\,(d-c)\,(b-a)} \int_a^b \int_c^d f(x,y) dydx \leq \frac{f(a,c) + f(a,d) + f(b,c) + f(b,d)}{4}.$$

Now we give the following important inequality for co-ordinated trigonometrically ρ-convex functions.

Theorem 3.7. Suppose that $f : \Delta \to \mathbb{R}$ is a co-ordinated trigonometrically ρ-convex functions on Δ. Then for $0 < b-a < \frac{\pi}{\rho_1}$ and $0 < d-c < \frac{\pi}{\rho_2}$ and for all $(x,y) \in \Delta$, we have the following inequality

$$\begin{aligned}
&\frac{4}{\rho_1\rho_2}\sin\left[\frac{\rho_1(b-a)}{2}\right]\sin\left[\frac{\rho_2(d-c)}{2}\right] f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\
\leq\ &\frac{1}{\rho_1}\sin\left[\frac{\rho_1(b-a)}{2}\right]\int_c^d f\left(\frac{a+b}{2},y\right)dy+\frac{1}{\rho_2}\sin\left[\frac{\rho_2(d-c)}{2}\right]\int_a^b f\left(x,\frac{c+d}{2}\right)dx\\
\leq\ &\int_a^b\int_c^d f(x,y)\,dydx\\
\leq\ &\frac{1}{2}\left[\frac{1}{\rho_2}\tan\left[\frac{\rho_2(d-c)}{2}\right]\int_a^b [f(x,c)+f(x,d)]\,dx\right.\\
&\left.+\frac{1}{\rho_1}\tan\left[\frac{\rho_1(b-a)}{2}\right]\int_c^d [f(a,y)+f(b,y)]\,dy\right]\\
\leq\ &\tan\left[\frac{\rho_1(b-a)}{2}\right]\tan\left[\frac{\rho_2(d-c)}{2}\right]\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{\rho_1\rho_2}.
\end{aligned} \tag{3.24}$$

Proof. Since $f : \Delta = [a,b]\times[c,d] \to \mathbb{R}$ is trigonometrically ρ-convex on the co-ordinates on Δ, it follows that the mapping $g_x : [c,d] \to \mathbb{R}$, $g_x(y) = f(x,y)$ is trigonometrically ρ-convex on $[c,d]$ for all $x \in [a,b]$. By Hermite-Hadamard inequality (1.6) for trigonometrically ρ-convex functions (see: [15]), we get:

$$\frac{2}{\rho_2}g_x\left(\frac{c+d}{2}\right)\sin\left[\frac{\rho_2(d-c)}{2}\right]\leq\int_c^d g_x(y)dy\leq\frac{g_x(c)+g_x(d)}{\rho_2}\tan\left[\frac{\rho_2(d-c)}{2}\right].$$

That is,

$$\begin{aligned}
\frac{2}{\rho_2}f\left(x,\frac{c+d}{2}\right)\sin\left[\frac{\rho_2(d-c)}{2}\right] &\leq \int_c^d f(x,y)\,dy\\
&\leq \frac{f(x,c)+f(x,d)}{\rho_2}\tan\left[\frac{\rho_2(d-c)}{2}\right].
\end{aligned} \tag{3.25}$$

Integrating the inequality (3.25) with respect to x on $[a,b]$, we have

$$\begin{aligned}\frac{2}{\rho_2}\sin\left[\frac{\rho_2(d-c)}{2}\right]\int_a^b f\left(x,\frac{c+d}{2}\right)dx &\leq \int_a^b\int_c^d f(x,y)\,dydx \\ &\leq \frac{1}{\rho_2}\tan\left[\frac{\rho_2(d-c)}{2}\right]\int_a^b [f(x,c)+f(x,d)]\,dx.\end{aligned} \tag{3.26}$$

By a similar argument applied for mapping $g_y:[a,b]\to\mathbb{R}$, $g_y:=f(x,y)$, we get

$$\begin{aligned}\frac{2}{\rho_1}\sin\left[\frac{\rho_1(b-a)}{2}\right]\int_c^d f\left(\frac{a+b}{2},y\right)dy &\leq \int_a^b\int_c^d f(x,y)\,dydx \\ &\leq \frac{1}{\rho_1}\tan\left[\frac{\rho_1(b-a)}{2}\right]\int_c^d [f(a,y)+f(b,y)]\,dy.\end{aligned} \tag{3.27}$$

Summing the inequalities (3.26) and (3.27), we obtain

$$\begin{aligned}&\frac{2}{\rho_1}\sin\left[\frac{\rho_1(b-a)}{2}\right]\int_c^d f\left(\frac{a+b}{2},y\right)dy+\frac{2}{\rho_2}\sin\left[\frac{\rho_2(d-c)}{2}\right]\int_a^b [f(x,c)+f(x,d)]\,dx \\ \leq\ & 2\int_a^b\int_c^d f(x,y)\,dydx \\ \leq\ & \frac{2}{\rho_2}\tan\left[\frac{\rho_2(d-c)}{2}\right]\int_a^b [f(x,c)+f(x,d)]\,dx+\frac{2}{\rho_1}\tan\left[\frac{\rho_1(b-a)}{2}\right]\int_c^d [f(a,y)+f(b,y)]\,dy.\end{aligned} \tag{3.28}$$

If we divide (3.26) by 2, then we prove the second and third inequalities in (3.24).
By first inequality in (1.6), we have

$$\begin{aligned}&\frac{2}{\rho_1\rho_2}\sin\left[\frac{\rho_1(b-a)}{2}\right]\sin\left[\frac{\rho_2(d-c)}{2}\right]f\left(\frac{a+b}{2},\frac{c+d}{2}\right) \\ \leq\ & \frac{1}{\rho_2}\sin\left[\frac{\rho_2(d-c)}{2}\right]\int_a^b f\left(x,\frac{c+d}{2}\right)dx\end{aligned} \tag{3.29}$$

and

$$\frac{2}{\rho_1\rho_2}\sin\left[\frac{\rho_1(b-a)}{2}\right]\sin\left[\frac{\rho_2(d-c)}{2}\right]f\left(\frac{a+b}{2},\frac{c+d}{2}\right) \tag{3.30}$$

$$\leq \frac{1}{\rho_1}\sin\left[\frac{\rho_1(b-a)}{2}\right]\int_c^d f\left(\frac{a+b}{2},y\right)dy.$$

If we add the inequalities (3.29) and (3.30), then we have

$$\frac{4}{\rho_1\rho_2}\sin\left[\frac{\rho_1(b-a)}{2}\right]\sin\left[\frac{\rho_2(d-c)}{2}\right]f\left(\frac{a+b}{2},\frac{c+d}{2}\right) \tag{3.31}$$

$$\leq \frac{1}{\rho_1}\sin\left[\frac{\rho_1(b-a)}{2}\right]\int_c^d f\left(\frac{a+b}{2},y\right)dy+\frac{1}{\rho_2}\sin\left[\frac{\rho_2(d-c)}{2}\right]\int_a^b f\left(x,\frac{c+d}{2}\right)dx$$

which gives the first inequality in (3.24).

Finally, by the second inequality in (1.6), we can write

$$\int_a^b f(x,c)\,dx \leq \frac{f(a,c)+f(b,c)}{\rho_1}\tan\left[\frac{\rho_1(b-a)}{2}\right], \tag{3.32}$$

$$\int_a^b f(x,d)\,dx \leq \frac{f(a,d)+f(b,d)}{\rho_1}\tan\left[\frac{\rho_1(b-a)}{2}\right], \tag{3.33}$$

$$\int_c^d f(a,y)\,dy \leq \frac{f(a,c)+f(a,d)}{\rho_2}\tan\left[\frac{\rho_2(d-c)}{2}\right], \tag{3.34}$$

and

$$\int_c^d f(b,y)\,dy \leq \frac{f(b,c)+f(b,d)}{\rho_2}\tan\left[\frac{\rho_2(d-c)}{2}\right]. \tag{3.35}$$

If we substitute the inequalities (3.32)-(3.35) in (3.28), then we have

$$\begin{aligned}
&\frac{1}{2}\left[\frac{1}{\rho_2}\tan\left[\frac{\rho_2\left(d-c\right)}{2}\right]\int_a^b\left[f\left(x,c\right)+f\left(x,d\right)\right]dx\right.\\
&\left.+\frac{1}{\rho_1}\tan\left[\frac{\rho_1\left(b-a\right)}{2}\right]\int_c^d\left[f\left(a,y\right)+f\left(b,y\right)\right]dy\right]\\
\leq\ &\frac{1}{2}\left[\frac{1}{\rho_2}\tan\left[\frac{\rho_2\left(d-c\right)}{2}\right]\tan\left[\frac{\rho_1\left(b-a\right)}{2}\right]\frac{f\left(a,c\right)+f\left(a,d\right)+f\left(b,c\right)+f\left(b,d\right)}{\rho_1\rho_2}\right.\\
&\left.+\frac{1}{\rho_1}\tan\left[\frac{\rho_1\left(b-a\right)}{2}\right]\tan\left[\frac{\rho_2\left(d-c\right)}{2}\right]\frac{f\left(a,c\right)+f\left(a,d\right)+f\left(b,c\right)+f\left(b,d\right)}{\rho_1\rho_2}\right]\\
=\ &\tan\left[\frac{\rho_1\left(b-a\right)}{2}\right]\tan\left[\frac{\rho_2\left(d-c\right)}{2}\right]\frac{f\left(a,c\right)+f\left(a,d\right)+f\left(b,c\right)+f\left(b,d\right)}{\rho_1\rho_2}.
\end{aligned}$$

This gives last inequality in (3.24). The proof is completely completed. Q.E.D.

Remark 3.8. For $\rho_1\to 0$ and $\rho_2\to 0$, by using the equalities (3.6) and (3.7) in Theorem 3.7, then the inequalities (3.24) reduce to the inequalities (1.3) obtained by Dragomir in [16].

Theorem 3.9. Suppose that $f:\Delta=[a.b]\times[c.d]\to\mathbb{R}$ is a co-ordinated trigonometrically ρ-convex functions on Δ. Then for $0<b-a<\frac{\pi}{\rho_1}$ and $0<d-c<\frac{\pi}{\rho_2}$ and for all $(x,y)\in\Delta$, we have the following inequality,

$$\begin{aligned}
&f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\
\leq\ &\frac{1}{2}\left[\frac{1}{d-c}\int_a^b f\left(\frac{a+b}{2},y\right)\sec\left[\rho_2\left(y-\frac{c+d}{2}\right)\right]dy\right.\\
&\left.+\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)\sec\left[\rho_1\left(x-\frac{a+b}{2}\right)\right]dx\right]\\
\leq\ &\frac{1}{\left(b-a\right)\left(d-c\right)}\int_a^b\int_c^d f(x,y)\sec\left[\rho_1\left(x-\frac{a+b}{2}\right)\right]\sec\left[\rho_2\left(y-\frac{c+d}{2}\right)\right]dydx \qquad (3.36)\\
\leq\ &\frac{1}{4}\sec\left[\frac{\rho_1\left(b-a\right)}{2}\right]\int_c^d\left[f\left(a,y\right)+(b,y)\right]\left[\rho_2\left(y-\frac{c+d}{2}\right)\right]dy
\end{aligned}$$

$$+\frac{1}{4}\sec\left[\frac{\rho_2(d-c)}{2}\right]\int_a^b[f(x,c)+(x,d)]\sec\left[\rho_1\left(x-\frac{a+b}{2}\right)\right]dx$$

$$\leq \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\sec\left[\frac{\rho_1(b-a)}{2}\right]\sec\left[\frac{\rho_2(d-c)}{2}\right].$$

Proof. Since $f:\Delta\to\mathbb{R}$ is completed trigonometrically $\rho-$ convex, by the inequality (1.7), we have

$$f\left(\frac{a+b}{2},y\right)\leq\frac{1}{b-a}\int_a^b f(x,y)\sec\left[\rho_1\left(x-\frac{a+b}{2}\right)\right]dx\leq\frac{f(a,y)+f(b,y)}{2}\sec\left[\frac{\rho_1(b-a)}{2}\right] \tag{3.37}$$

and

$$f\left(x,\frac{c+d}{2}\right)\leq\frac{1}{d-c}\int_c^d f(x,y)\sec\left[\rho_2\left(y-\frac{c+d}{2}\right)\right]dy\leq\frac{f(x,c)+f(x,d)}{2}\sec\left[\frac{\rho_2(d-c)}{2}\right] \tag{3.38}$$

multiplying the inequalities (3.37) and (3.38) by $\sec\left[\rho_2\left(y-\frac{c+d}{2}\right)\right]$ and $\left[\rho_1\left(x-\frac{a+b}{2}\right)\right]$ respectively, and integrating the resultant inequalities on $[c,d]$ and $[a,b]$ respectively, we obtain

$$\begin{aligned}
&\int_c^d f\left(\frac{a+b}{2},y\right)\sec\left[\rho_2\left(y-\frac{c+d}{2}\right)\right]dy\\
&\leq\frac{1}{b-a}\int_a^b\int_c^d f(x,y)\sec\left[\rho_1\left(x-\frac{a+b}{2}\right)\right]\sec\left[\rho_2\left(y-\frac{c+d}{2}\right)\right]dydx \qquad (3.39)\\
&\leq\frac{1}{2}\sec\left[\frac{\rho_1(b-a)}{2}\right]\int_c^d[f(a,y)+f(b,y)]\left[\rho_2\left(y-\frac{c+d}{2}\right)\right]dy
\end{aligned}$$

and

$$\begin{aligned}
&\int_a^b f\left(x,\frac{c+d}{2}\right)\sec\left[\rho_1\left(x-\frac{a+b}{2}\right)\right]dx\\
&\leq\frac{1}{d-c}\int_a^b\int_c^d f(x,y)\sec\left[\rho_2\left(y-\frac{c+d}{2}\right)\right]\sec\left[\rho_1\left(x-\frac{a+b}{2}\right)\right]dydx \qquad (3.40)\\
&\leq\frac{1}{2}\sec\left[\frac{\rho_2(d-c)}{2}\right]\int_a^b[f(x,c)+f(x,d)]\sec\rho_1\left(x-\frac{a+b}{2}\right)dx.
\end{aligned}$$

By dividing the inequalities (3.39) and (3.40) by $(b-a)$ and $(d-c)$ respectively, then by adding the resultant inequalitites, we obtain

$$\begin{aligned}
&\frac{1}{2}\left[\frac{1}{d-c}\int_a^b f\left(\frac{a+b}{2},y\right)\sec\left[\rho_2\left(y-\frac{c+d}{2}\right)\right]dy+\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)\sec\left[\rho_1\left(x-\frac{a+b}{2}\right)\right]dx\right]\\
\leq\ &\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\sec\left[\rho_1\left(x-\frac{a+b}{2}\right)\right]\sec\left[\rho_2\left(y-\frac{c+d}{2}\right)\right]dydx \qquad (3.41)\\
\leq\ &\frac{1}{4}\sec\left[\frac{\rho_1(b-a)}{2}\right]\int_c^d\left[f(a,y)+(b,y)\right]\left[\rho_2\left(y-\frac{c+d}{2}\right)\right]dy\\
&+\frac{1}{4}\sec\left[\frac{\rho_2(d-c)}{2}\right]\int_a^b\left[f(x,c)+(x,d)\right]\sec\left[\rho_1\left(x-\frac{a+b}{2}\right)\right]dx
\end{aligned}$$

the second and third inequalities in (3.36).

From the first inequality in (1.7), we have

$$f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\leq\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)\sec\left[\rho_1\left(x-\frac{a+b}{2}\right)\right]dx \qquad (3.42)$$

and

$$f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\leq\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)\sec\left[\rho_2\left(y-\frac{c+d}{2}\right)\right]dy \qquad (3.43)$$

by summing the inequalities (3.42) and (3.43), we establish first inequality in (3.36).

By the using second inequality in (1.7), we get

$$\frac{1}{d-c}\int_c^d f(a,y)\sec\left[\rho_2\left(y-\frac{c+d}{2}\right)\right]dy\leq\frac{f(a,c)+f(a,d)}{2}\sec\left[\frac{\rho_2(d-c)}{2}\right] \qquad (3.44)$$

$$\frac{1}{d-c}\int_c^d f(b,y)\sec\left[\rho_2\left(y-\frac{c+d}{2}\right)\right]dy\leq\frac{f(b,c)+f(b,d)}{2}\sec\left[\frac{\rho_2(d-c)}{2}\right] \qquad (3.45)$$

$$\frac{1}{b-a}\int_a^b f(x,c)\sec\left[\rho_1\left(x-\frac{a+b}{2}\right)\right]dx\leq\frac{f(a,c)+f(a,d)}{2}\sec\left[\frac{\rho_1(b-a)}{2}\right] \qquad (3.46)$$

and

$$\frac{1}{b-a}\int_a^b f(x,d)\sec\left[\rho_1\left(x-\frac{a+b}{2}\right)\right]dx\leq\frac{f(a,d)+f(b,d)}{2}\sec\left[\frac{\rho_1(b-a)}{2}\right] \tag{3.47}$$

If we substitute the inequalities (3.44)-(3.47) in (3.41) then we obtain the last inequality in (3.36).The proof is completed. Q.E.D.

Remark 3.10. By choosing $\rho_1=0$ and $\rho_2=0$ Theorem 3.9, then the inequalities (3.36) reduce to the inequalities (1.3) obtained by Dragomir in [16].

4 A refinement of Hermite-Hadamard type inequlities

By applying Theorem 1.6, we establish following Hermite Hadamard inequalities which refine the first and second inequality in (3.24).

Theorem 4.1. Suppose that $f:\Delta=[a.b]\times[c.d]\to\mathbb{R}$ is a co-ordinated trigonometrically ρ-convex functions on Δ. Then for $0<b-a<\frac{\pi}{\rho_1}$ and $0<d-c<\frac{\pi}{\rho_2}$, we have the following inequality

$$\begin{aligned}
&\frac{4}{\rho_1\rho_2}\sin\left[\frac{\rho_1(b-a)}{2}\right]\sin\left[\frac{\rho_2(d-c)}{2}\right]f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\
\leq\ &\Lambda_1(t,s)\\
\leq\ &\frac{1}{\rho_1}\sin\left[\frac{\rho_1(b-a)}{2}\right]\int_c^d f\left(\frac{a+b}{2},y\right)dy+\frac{1}{\rho_2}\sin\left[\frac{\rho_2(d-c)}{2}\right]\int_a^b f\left(x,\frac{c+d}{2}\right)dx \qquad (4.1)\\
\leq\ &\Lambda_2(t,s)\\
\leq\ &\int_a^b\int_c^d f(x,y)\,dydx
\end{aligned}$$

where $\Lambda_1(t,s)$ and $\Lambda_2(t,s)$ are defined by

$$\begin{aligned}
\Lambda_1(t,s) \ =\ &\frac{1}{2\rho_1}\sin\left[\frac{\rho_1(b-a)}{2}\right]\int_c^d\left(\cos\left[\frac{\rho_2(1-s)(d-y)}{2}\right]+\cos\left[\frac{\rho_2(1-s)(y-c)}{2}\right]\right)\\
&\times f\left(\frac{a+b}{2},sy+(1-s)\frac{c+d}{2}\right)\Big]dy\\
&+\frac{1}{2\rho_2}\sin\left[\frac{\rho_2(d-c)}{2}\right]\int_a^b\left(\cos\left[\frac{\rho_1(1-t)(b-x)}{2}\right]+\cos\left[\frac{\rho_1(1-t)(x-a)}{2}\right]\right)\\
&\times f\left(tx+(1-t)\frac{a+b}{2},\frac{c+d}{2}\right)\Big]dx
\end{aligned}$$

and

$$\begin{aligned}
&\Lambda_2(t,s)\\
&= \frac{1}{4}\int_a^b\int_c^d \left(\cos\left[\frac{\rho_1\,(1-t)\,(b-x)}{2}\right]+\cos\left[\frac{\rho_1\,(1-t)\,(x-a)}{2}\right]\right) f\left(tx+(1-t)\,\frac{a+b}{2},y\right)dydx\\
&+\frac{1}{4}\int_a^b\int_c^d \left(\cos\left[\frac{\rho_2\,(1-s)\,(d-y)}{2}\right]+\cos\left[\frac{\rho_1\,(1-s)\,(y-c)}{2}\right]\right) f\left(x,sy+(1-s)\,\frac{c+d}{2}\right)dydx.
\end{aligned}$$

Proof. Since $f:\Delta=[a.b]\times[c.d]\to\mathbb{R}$ is co-ordinated trigonometrically ρ-convex on Δ, it follows that the mapping $g_x:[c,d]\to\mathbb{R}$, $g_x(y)=f(x,y)$ is trigonometrically ρ-convex on $[c,d]$ for all $x\in[a,b]$. By the inequality (1.8), we have

$$\begin{aligned}
&\frac{2}{\rho}g_x\left(\frac{c+d}{2}\right)\sin\left[\frac{\rho_2(d-c)}{2}\right]\\
\leq\ &\frac{1}{2}\left(\cos\left[\frac{\rho_2(1-s)\,(d-y)}{2}\right]+\cos\left[\frac{\rho_2(1-s)\,(y-c)}{2}\right]\right)g_x\left(sy+(1-s)\,\frac{c+d}{2}\right)dy \quad (4.2)\\
\leq\ &\int_a^b g_x(y)dy,
\end{aligned}$$

i.e,

$$\begin{aligned}
&\frac{2}{\rho_2}f\left(x,\frac{c+d}{2}\right)\sin\left[\frac{\rho_2(d-c)}{2}\right]\\
\leq\ &\frac{1}{2}\int_c^d\left(\cos\left[\frac{\rho_2(1-s)\,(d-y)}{2}\right]+\cos\left[\frac{\rho_2(1-s)\,(y-c)}{2}\right]\right)f\left(x,sy+(1-s)\,\frac{c+d}{2}\right)dy \quad (4.3)\\
\leq\ &\int_c^d f(x,y)dy.
\end{aligned}$$

Integrating the inequality (4.3) on $[a,b]$, we have

$$\begin{aligned}
&\frac{2}{\rho_2}\sin\left[\frac{\rho_2(d-c)}{2}\right]\int_a^b f\left(x,\frac{c+d}{2}\right)dx \\
&\leq \frac{1}{2}\int_a^b\int_c^d\left(\cos\left[\frac{\rho_2(1-s)\,(d-y)}{2}\right]+\cos\left[\frac{\rho_2(1-s)\,(y-c)}{2}\right]\right)f\left(x,sy+(1-s)\,\frac{c+d}{2}\right)dydx \quad (4.4)\\
&\leq \int_a^b\int_c^d f(x,y)dydx.
\end{aligned}$$

By similar argument applied for the mapping $g_y:[a,b]\to\mathbb{R}$, $g_y(x)=f(x,y)$ is trigonometrically ρ-convex on $[a,b]$ for all $y\in[c,d]$, we have

$$\begin{aligned}
&\frac{2}{\rho_1}\sin\left[\frac{\rho_1(b-a)}{2}\right]\int_c^d f\left(\frac{a+b}{2},y\right)dx \\
&\leq \frac{1}{2}\int_a^b\int_c^d\left(\cos\left[\frac{\rho_1(1-t)\,(b-x)}{2}\right]+\cos\left[\frac{\rho_1(1-t)\,(x-a)}{2}\right]\right)f\left(xt+(1-t)\,\frac{a+b}{2},y\right)dydx \quad (4.5)\\
&\leq \int_a^b\int_c^d f(x,y)dydx.
\end{aligned}$$

If we add the inequalities (4.4) and (4.5), then we obtain the following inequality

$$\begin{aligned}
&\frac{2}{\rho_1}\sin\left[\frac{\rho_1\,(b-a)}{2}\right]\int_c^d f\left(\frac{a+b}{2},y\right)dy+\frac{2}{\rho_2}\sin\left[\frac{\rho_2\,(d-c)}{2}\right]\int_a^b f\left(x,\frac{c+d}{2}\right)dx \\
&\leq 2\Lambda_2(t,s) \quad (4.6)\\
&\leq 2\int_a^b\int_c^d f\,(x,y)\,dydx.
\end{aligned}$$

If we divide the inequality (4.6) by 2 , then proof of third and fourth inequalities in (4.1) is completed.

Similarly, by the inequality (1.8), we have

$$\begin{aligned}
&\frac{2}{\rho_1}\sin\left[\frac{\rho_1(b-a)}{2}\right] f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\
\leq\ &\frac{1}{2}\int_a^b\left(\cos\left[\frac{\rho_1(1-t)(b-x)}{2}\right]+\cos\left[\frac{\rho_1(1-t)(x-a)}{2}\right]\right) f\left(tx+(1-t)\frac{a+b}{2},\frac{c+d}{2}\right)dx \quad (4.7)\\
\leq\ &\int_a^b f\left(x,\frac{c+d}{2}\right)
\end{aligned}$$

and

$$\begin{aligned}
&\frac{2}{\rho_2}\sin\left[\frac{\rho_2(d-c)}{2}\right] f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\
\leq\ &\frac{1}{2}\int_c^d\left(\cos\left[\frac{\rho_2(1-s)(d-y)}{2}\right]+\cos\left[\frac{\rho_2(1-s)(y-c)}{2}\right]\right) f\left(\frac{a+b}{2},sy+(1-s)\frac{c+d}{2}\right)dx \quad (4.8)\\
\leq\ &\int_c^d f\left(\frac{a+b}{2},y\right)dy.
\end{aligned}$$

If we multiply the inequalities (4.7) by $\frac{1}{\rho_2}\sin\left(\frac{\rho_2(d-c)}{2}\right)$ and the inequalities (4.8) by $\frac{1}{\rho_1}\sin\left(\frac{\rho_1(b-a)}{2}\right)$, then by summing the resultant inequalities, we obtain the first and second inequality in (4.1).

Therefore the proof is completed. Q.E.D.

Remark 4.2. For $\rho_1 \to 0$ and $\rho_2 \to 0$, by using the equalities (3.6) and (3.7) in Theorem 4.1, then the inequalities reduce to inequalities obtained by Ali et al. in [4, Theorem 2.1].

5 Conclusion

This paper introduced the co-ordinated trigonometrically ρ-convex functions. Using this concept, we presented some Hermite-Hadamard type inequalities which generalize some inequalities given earlier works. The next step in the research direction proposed here is to establish the weighted versions of the inequalities obtained in this paper.

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