Z - transform for integrable Boehmians

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Abstract

Boehmians, being the specific class of the generalized functions (the theory of distributions), are exposed to applications within mathematics and beyond. This paper is involved into the investigation of the Z - transform for integrable Boehmians.

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1 Introduction

Among several methods of analysis of discrete systems applied, the transform techniques and operator methods have remained much significant. One of them is the Z - transform, which is extensively developed and applied to many areas of discrete system theory, details of which can be found in [3, 4, 12] .

As integral transform, such as Fourier, Laplace, etc. have played an important role in studying Lebesgue space, distribution spaces and Boehmian spaces. A constructive composition of the paper is that Section 2 contains notation, definition of Z - transform, and its properties. In Section 3, we introduce the construction of Boehmians and integrable Boehmians. In Section 4, we intend to establish Z - transform for integrable Boehmians. In order to draw a more constructive view of the matter, despite several fundamentals of Boehmian theory known and can be seen in references mentioned, we write some of them here because we want to make the paper a sharing of concepts.

2 Z - transform and properties

The sampled function [3, pp. 446-448] is defined by

$$\begin{aligned} f^*(t) &= f(t)\sum_{n=0}^{\infty}\delta(t-nT) \\ &= \sum_{n=0}^{\infty} f(nT)\delta(t-nT) \\ &= \sum_{n=-\infty}^{\infty} f(t_n)\delta(t-t_n) \quad , \end{aligned} \tag{2.1}$$

where t_n (n is an integer) is any particular value of t and $f(t)$ is a continuous function in any interval containing the point $t = t_n$. When $t_n = nT$, the series $\sum_{n=-\infty}^{\infty}\delta(t-nT)$ is called the impulse train and T is called the sampling period.

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The Laplace transform of the sampled function (2.1) [2, p. 449] is defined by

$$L\{f^*(t)\} = \overline{f}^*(s) = \sum_{n=0}^{\infty} f(nT)\exp(-nsT) \quad , \tag{2.2}$$

where s is the transform variable (which is a complex variable).

If we consider $z = \exp(sT)$, then

$$L\{f^*(t)\} = \mathcal{F}(z) = \sum_{n=0}^{\infty} f(nT)z^{-n} \quad . \tag{2.3}$$

Therefore, $\mathcal{F}(z)$ is called the Z - transform of the sequence $\{f(nT)\}$, where T is the sampling time. As T has no effect on the properties of the Z - transform, we may consider $T = 1$. The Z - transform of a sequence $\{f(n)\}$, in terms of function $\mathcal{F}(z)$ of a complex variable z , is defined by

$$Z\{f(n)\} = \mathcal{F}(z) = \sum_{n=0}^{\infty} f(n)\ z^{-n} \tag{2.4}$$

Thus, Z is the linear transformation and can be considered as an operator mapping sequence of scalars into functions of the complex variable z.

The inverse Z - transform is given by [3, p. 449]

$$Z^{-1}\{\mathcal{F}(z)\} = f(n) = \frac{1}{2\pi i}\oint_C \mathcal{F}(z)\ z^{n-1}dz \quad , \tag{2.5}$$

where C is the simple closed contour, enclosing the origin and lying outside the circle $|z| = R$.

The bilateral Z - transform is defined by

$$Z\{f(n)\} = \mathcal{F}(z) = \sum_{n=-\infty}^{\infty} f(n)\ z^{-n} \quad , \tag{2.6}$$

for all complex numbers z for which the series converges. Substituting $z = re^{i\theta}$ and for $r = 1$, the Z - transform (2.6) is written as

$$\mathcal{F}(e^{i\theta}) = \sum_{n=-\infty}^{\infty} f(n)\ e^{-in\theta} \tag{2.7}$$

i.e.

$$\mathcal{F}(\theta) = \sum_{n=-\infty}^{\infty} f(n)\ e^{-in\theta} = \boldsymbol{F}\{f(n)\} \quad , \tag{2.8}$$

where $\boldsymbol{F}$ is known as the Fourier transform of the sequence $\{f(n)\}_{-\infty}^{\infty}$.

Properties of the Z - transform are given as :

1. *Linearity of Z - transform* : For all c_1 and c_2 constants, the property holds

$$Z\{c_1 f_1 + c_2 f_2\} = c_1 Z\{f_1(n)\} + c_2 Z\{f_2(n)\} \tag{2.9}$$

2. *Multiplication* : If $Z\{f(n)\} = F(z)$, then

$$Z\{nf(n)\} = -z \frac{d}{dz}\mathcal{F}(z) \tag{2.10}$$

and

$$Z\{n^k f(n)\} = (-1)^k \left(z\frac{d}{dz}\right)^k \mathcal{F}(z) \quad , k = 0, 1, 2, \ldots \tag{2.11}$$

3. *Convolution* : If $Z\{f(n)\} = \mathcal{F}(z)$ and $Z\{g(n)\} = G(z)$, then the Z - transform of the convolution $f(n) * g(n)$ is given by

$$Z\{f(n) * g(n)\} = Z\{f(n\}Z\{g(n)\} \quad . \tag{2.12}$$

4. *Parseval's Formula* : If $\mathcal{F}(z) = Z\{f(n)\}$ and $G(z) = Z\{g(n)\}$, then

$$\sum_{n=-\infty}^{\infty} f(n)\overline{g}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathcal{F}(e^{i\theta})\overline{G}(e^{i\theta})d\theta \tag{2.13}$$

In particular,

$$\sum_{n=-\infty}^{\infty} |f(n)|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|\mathcal{F}(e^{i\theta})\right|^2 d\theta \quad . \tag{2.14}$$

5. If $f(n) = \delta(n)$ and $Z\{f(n)\} = \mathcal{F}(z)$, then

$$Z\{\delta(n)\} = \mathcal{F}(z) = 1. \tag{2.15}$$

Equation (2.14) shows square integrable functions which involves the L_p spaces, where functions are bounded and continuous. The testing function space $\mathcal{D}$ and its dual $\mathcal{D}'$ are employed for the Z transform [6]. The distributional Z transform is defined by

$$F(z) = Z\{f(n)\} = \sum_{n=0}^{\infty} \left\langle f(n), z^{-n} \right\rangle \quad , \tag{2.16}$$

where $f(n) \in \mathcal{D}'$ and $z \in \mathcal{D}$.

3 Integrable Boehmians

Boehmians are classes of generalized functions whose construction is algebraic that is shown [8, 9, 10] as an extension of the notion of spaces of different types, are defined using different types of function spaces. The general construction of Boehmians is given in [7, 8]. Boehmians are termed as all objects by an abstract algebra of regular operators [2]. Its construction is as follows that makes the paper self contained.

Let $\mathcal{G}$ be a linear space and S be the sub - space of $\mathcal{G}$. Let $*$ be a binary operation from $\mathcal{G} \times S$ to $\mathcal{G}$ such that following conditions hold:

(i) if $\varphi, \psi \in S$, then $\varphi * \psi \in S$ and $\varphi * \psi = \psi * \varphi$
(ii) if $f \in \mathcal{G}, \varphi, \psi \in S$, then $(f * \varphi) * \psi = f * (\ \varphi * \psi)$
(iii) if $f, g \in \mathcal{G}, \varphi \in S$ and $\lambda \in R$, then $(f + g) * \varphi = (f* \varphi) + (g * \varphi)$ and $\lambda(f * \varphi) = (\lambda f)* \varphi$.
Let Δ be a family of sequence of elements from S such that
(i) if $\alpha, \beta \in \mathcal{G}, (\delta_n) \in \Delta$ and $(f * \delta_n) = (g * \delta_n)$ for all $(n = 1, 2, \ldots)$, then $f = g$.
(ii) if $(\varphi_n), (\delta_n) \in \Delta$, then $(\varphi_n * \delta_n) \in \Delta$.

Elements of Δ will be called delta sequence. The class A of pair of the sequences defined by $A = \{(\ f_n, \varphi_n) : (f_n) \subseteq \mathcal{G}^N, (\varphi_n) \in \Delta\}$ for element $(f_n, \varphi_n) \in A$, is called quotient of sequence, denoted by f_n/φ_n if

$$f_i * \varphi_j = f_j * \varphi_i, \qquad \text{for all } i, j \in N.$$

Two quotients of sequences f_n/φ_n and g_n/ψ_n are said to be equivalent if

$$f_n * \psi_n = g_n * \varphi_n, \qquad \text{if } n \in N. \tag{3.1}$$

The equivalence class containing f_n/φ_n is denoted by $[f_n/\varphi_n]$.These equivalence classes are called Boehmians and space of all Boehmians is denoted by $B = B(G, \Delta)$.

The operations of addition, scalar multiplication and differentiation is satisfied by Boehmian spaces.

In B, two types of convergence, $\delta-$ convergence and Δ - convergence, are defined as follows:

(1) A sequence of Boehmians (x_n) in B is said to be δ - convergent to a Boehmian x in B, denoted by $x_n \xrightarrow{\delta} x$, if there exists a delta sequence (δ_n) such that

$$(x_n * \delta_n), (x, \delta_n) \in \mathcal{G},\ n \in N$$

and

$$(x_n * \delta_k) \to (x * \delta_k) \text{ as } n \to \infty \ \text{ in } \mathcal{G}, \forall k \in N.$$

(2) A sequence of Boehmians (x_n) in B is said to be Δ - convergent to a Boehmian x in B, denoted by $x_n \xrightarrow{\Delta} \delta x$, if there exists a $(\delta_n) \in \Delta$ such that

$$\begin{aligned} (x_n - x) * \delta_n &\in \mathcal{G}, \quad \forall n \in N \\ (x_n - x) * \delta_n &\to 0 \text{ as } n \to \infty \text{ in } \mathcal{G}. \end{aligned}$$

We consider $\mathcal{G}$ to be the set of all Lebesgue integrable functions (locally integrable function) on R denoted as $L_p(p = 1, 2)$ and identify two such functions whenever they are equal almost equal everywhere with respect to the usual Lebesgue measure on R.

Also consider $S = \mathcal{D}(R)$ (the distribution space) and Δ to be class of sequence from $\mathcal{D}$, we define corresponding Boehmian space $B = B(G, \Delta) = B(L_p, \mathcal{D}, *, \Delta)$ as the space of locally integrable Boehmian, where $p = 1$ and denoted as B_{L_1} and if $B = B(L_2, \mathcal{D}, *, \Delta) = B_{L_2}$, then elements of B_{L_2} are called square integrable Boehmians.

$\mathcal{D}'(R)$ can be imbedded continuously in the above mentioned Boehmian space in the sense that the mapping $\mathcal{D}' \to B$, given by $x \to \left[\frac{x*\delta_n}{\delta_n}\right]$, defines a one to one function in such a way that $x_m \underset{\to}{\delta} x$ in $\mathcal{D}' \Rightarrow x_m \underset{\to}{\delta} x$ in B.The Boehmian space B_{L_1} as constructed by considering $\mathcal{G}$=$L_1(R)$ and the class Δ, consisting of sequence of continuous real function $\delta_n \in L_1$ which satisfies the conditions

(i) $\int_{\mathbb{R}} \delta_n(x)dx = 1 \ \ , \forall n \in N$

(ii) $||\delta_n|| < M$, for some $M \in R$ and all $n \in N$, and

(iii) $\lim\limits_{n\to\infty} \int_{|x|>\epsilon} |\delta_n(x)|\, dx = 0$, for each $\epsilon > 0$

and $*$ is the usual convolution defined by

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x - y)g(y)dm(y)$$

except for the use of ordinary Lebesgue measure.

A function $f \in L_p(p = 1, 2)$ that is locally integrable function on R, can be identified with the Boehmian $[f * \delta_n/\delta_n]$, where (δ_n) is a delta sequence. Thus, L_1 is viewed as a subspace of B_{L_1}.If $F = [f_n/\varphi_n]$, then $F * \delta_n = f_n$ and $F * \delta_n \in L_1, \forall\ n \in N$.

The properties of Boehmian space of locally integrable space (Lebesgue spaces) satisfy addition, multiplication, convolution and δ and Δ - convergences; for details see [8, 9]

4 Z - transform of integrable Boehmian

While referring to [1, 5, 11], it can be realized how the integrable Boehmian have association with integral transforms such as Fourier, Mellin, and several other. The Z - transform is yet to be ventured into distributions and Boehmian space. In this section we endeavour the approach of the Z - transform to the integrable Boehmians. Mathematics in this context, is shown here.

Lemma 1. *If* $[f_\mathsf{p}/\delta_\mathsf{p}] \in B_{L_1}$, *then the sequence*

$$Z\{f_\mathsf{p}(n)\} = \mathcal{F}_\mathsf{p}(z) = \sum_{n=0}^{\infty} f_\mathsf{p}(n)\ z^{-n} \tag{4.1}$$

converges uniformly on each compact set in $\mathbb{R}$.

Proof. If (δ_p) is a delta sequence, then $Z\{f_\mathsf{p}(n)\}$ converges uniformly on each compact set to a constant function 1. Therefore, for each compact set K, $Z(\delta_k) > 0$ for almost all $k \in K$, and

$$\mathcal{F}_\mathsf{p}(z) = Z\{f_\mathsf{p}\} = Z\{f_\mathsf{p}\})\frac{Z(\delta_k)}{Z(\delta_k)}$$

that is,

$$= \frac{Z(f_{\mathsf{p}} * \delta_k)}{Z(\delta_k)} = \frac{Z(f_k * \delta_{\mathsf{p}})}{Z(\delta_k)}$$
$$= \frac{Z(f_k\}Z\{\delta_{\mathsf{p}}\}}{Z(\delta_k)} = \frac{Z(f_k)}{Z(\delta_k)} Z\{\delta_{\mathsf{p}}\} \quad \text{on } K.$$

In view of the above lemma, the Z - transform of an integrable Boehmian $F = [f_{\mathsf{p}}/\delta_{\mathsf{p}}]$ can be defined as the limit of $Z\{f_{\mathsf{p}}\}$ in the space of continuous functions on $\mathbb{R}$.

Thus, the Z- transform of an integrable Boehmian is a continuous function.

Similarly, we can define that Z - transform of $F = [f_{\mathsf{p}}/\varphi_{\mathsf{p}}] \in B_{L_2}$ then $[Zf_{\mathsf{p}}/Z\varphi_{\mathsf{p}}] \in B_{L_2}$.
Since $F = [f_{\mathsf{p}}/\varphi_{\mathsf{p}}] = [g_{\mathsf{q}}/\psi_{\mathsf{q}}]$, from (12), $(Zf_{\mathsf{p}})(Z\psi_{\mathsf{q}}) = Z(f_{\mathsf{p}} * \psi_q) = Z(g_{\mathsf{q}} * \varphi_{\mathsf{p}}) = Z(g_{\mathsf{q}})Z(\varphi_{\mathsf{p}})$.

Theorem 1. Let F and $G \in B_{L_1}$. Then

(1) $Z(F+G) = Z(F) + Z(G)$ and $Z(\lambda F) = \lambda Z(F)$ (for any complex λ)
(2) $Z(F * G) = Z(F)Z(G)$.

Proof. Relations (1) and (2) follow directly from the corresponding properties of Z transform in locally integrable function L_1. The properties (2.9) and (2.12) are defined in the Section 1, which justify the completion of the proof. .

Theorem 2. If $F_{\mathsf{p}} \underset{\Rightarrow}{\Delta} F$ in B_{L_1} then $ZF_{\mathsf{p}} \underset{\Rightarrow}{\Delta} ZF$ in B_{L_1}.

Proof. To prove the theorem, it suffices to show that it is true for $F = 0$. So let $F_{\mathsf{p}} \underset{\Rightarrow}{\Delta} 0$ in B_{L_1} and let $\{\delta_{\mathsf{p}}\} \in \Delta$ be such that $f_{\mathsf{p}} = F_{\mathsf{p}} * \delta_{\mathsf{p}} \in L_1$ for all p and $f_{\mathsf{p}} \to 0$ in L_1. Since $Z\{\delta_k\} \in \Delta$,we have

$$Z(F_{\mathsf{p}})Z(\delta_{\mathsf{p}}) = Z(F_{\mathsf{p}} * \delta_{\mathsf{p}}) = Z(f_{\mathsf{p}})$$

which implies that by (2.13) and (2.14),

$$\|Z(F_{\mathsf{p}})Z(\delta_{\mathsf{p}})\|_{L^p} = \|Zf_{\mathsf{p}}\|_{L^p} = \|f_{\mathsf{p}}\|_{L^p} \to 0$$

as $p \to \infty$ and in $L^p, p = 1, 2$.

Thus, $ZF_{\mathsf{p}} \underset{\Rightarrow}{\Delta} 0$ in B_{L_1}.

The above results proved, is for B_{L_1} (integrable) Boehmian space. Similarly, the square integrable Boehmian space B_{L_2} can be proved for the Z - transform as the Z - transform satisfies the Lebesgue and the distribution spaces as proved in [6]

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