Monodromy matrices as universal set of quantum gates and dynamics of cold trapped ions

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Abstract

In this paper we describe a feasible construction of universal set of quantum gates using monodromy matrices of Fuchsian system. Fuchsian systems are considered as Schrödinger type equations and it is shown that such quantum systems are exactly solvable. We also show that dynamics of trapped cold ions may be described by a Fuchsian system which also describes the critical points of logarithmic potential associated with equilibrium positions of trapped ions in line geometry. Two different approaches to the inverse problem are also discussed.

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1 Introduction

In the classical theory of computing it is known that any Boolean function $f : \{0,1\}^n \to \{0,1\}^m$ can be expressed in disjunctive normal form using binary operators OR, AND and unary operator NOT. These operators are sufficient, i.e. they form a *basis* and are called *fundamental* or *universal gates* in classical computation.

Construction of a basis for quantum computing is one of the fundamental problems which was investigated in many papers; let us refer to the seminal works [1], [2], [3], [4]. By practical considerations it is convenient to construct a basis whose elements are of local nature, i.e. operators act on a small amount of neighborly qubits. The solution of the Schrödinger equation represents the evolution of the initial state (input). Here before treating the Schrödinger equation, the time evolution of the state (the unitary operators) is known as a discrete evolution and is determined by the unitary matrices. Our problem is to find out which type of the Schrödinger equation has evolutionary solutions the same as the evolutions given by the quantum gates.

One qubit gates are elements of $SU(2)$ and act on $\mathbb{C}^2$. For realization of quantum algorithms one needs all operators from $SU(2^n)$. This problem can be reduced to finding such elements M_i of $SU(4)$ which together with $SU(2) \otimes SU(2)$ generate the whole $SU(4)$. By definition a two qubit gate A is a *universal* for the quantum computation if A together with local unitary transformations (i.e transformation of one qubit states) generates all unitary transformations of the complex vector space of dimension 2^n to itself [1].

First we consider *geometric* or *Berry phase* [5], [6]. A phase is, for our purposes, not a state of matter but a complex number of unit modulus, an element of the group $U(1)$ or, more general, an element of matrix group $U(N)$. The phases are often obtained with cyclic evolution of a physical system. Cyclic variation of external parameters often leads to a net evolution involving a phase depending only on the geometry of the path traversed is parameter space. The natural mathematical context for geometric phases in the theory of $U(N)$ fibre bundles. In that context, one defines a

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phase, known as holonomy, that depends on the geometry of a loop, and is independent of coordinate choice.

Examples of geometric phases abound in many areas of physics: in optics, Pancharatnm's phase led to measurable interference effects; in molecular physics, the molecular Aharonov-Bohm effect was found to have a significant effect on molecular dynamics. The universal significance of these phases means that whenever an adiabatic approximation applies, we may expect to find a geometric phase [7]. The crucial role of adiabatic condition is to make sure that the cyclic variable of parameters leads to cyclic evolution. F.Wilczeck and A.Zee consider non-Abelian phases associated with adiabatic evolution of degenerate Hamiltonians (see [8]). Presence of the geometric phase in quantum systems and its properties have been used by P. Zanardi and M. Rasetti [4] to construct a new model for quantum computing, which they called the holonomic quantum computing. Subsequently quite a number of papers has been devoted to the holonomic quantum computing (see for example [8] and references therein). The model of holonomic quantum computing when the holonomy depends only homotopy class of integration path is known as monodromic quantum computing [9].

One of possible physical realizations of monodromic quantum computing is the quantum processor based on cold trapped ion technology. This is the main postulate of this paper and we give mathematical foundation for this hypothesis.

Earnshaw's theorem states that a collection of point charges cannot be maintained in a stable stationary equilibrium configuration solely by the electrostatic interaction of the charges [10]. The Earnshaw's theorem applies to classical inverse-square law forces (electric and gravitational) and also to the magnetic forces of permanent magnets, if the magnets are hard (the magnets do not vary in strength with external fields). Cold ion trap technologies (see [10], pp. 309-324) get around Earnshaw's theorem by using oscillating electric fields to trap charged particles, as the theorem strictly applies only to static fields.

Equilibria of point charges confined to a line segment with charged end-points have been studied from various points of view (see, e.g., [5], [11]). In particular, the case of equal charges interacting by logarithmic potential has been studied in big detail in a seminal paper of T.J. Stieltjes (see [5]). The results of Stieltjes served as a paradigm for a vast topic which is nowadays called electrostatic interpretation of configurations of points on the line.

Over in the last decades, the interest towards these topics increased in connection with the mathematical aspects of *linear ion traps* (Paul traps), [10], [12]. In this context it is natural to consider both the logarithmic potential and Coulomb potential.

In the last section, in the framework of Fuchsian equations we consider equilibria of point charges with respect to logarithmic potential and reduce equilibrium problem to Riemann-Hilbert monodromy problem. In the second section we describe monodromic model of quantum computation [9], [13] and in the third section we give a way of construction of universal set of quantum gates for exactly solvable models [6]. In section four we give a general procedure of reduction of some type Schrödinger equations to Fuchsian systems [14], [15].

2 Monodromic quantum computing

Let M be a smooth manifold, which in our case plays the role of parameter space. Let a family of Hamiltonians H_λ be given, depending continuously on parameters $\lambda \in M$, all of which have a set of n degenerate levels. By a simple renormalization of the energies we can suppose that these levels are at $E = 0$.

Theorem 2.1. (see [8]) If the parameters are slowly varied from an initial value λ_0 to some final value λ_1 over long time interval T, and the given space of degenerate levels does not cross other levels, then solutions of

$$H_{\lambda_0}\Psi(t) = 0 \tag{1}$$

are mapped onto solutions of

$$H_{\lambda_1}\Psi(t) = 0 \tag{2}$$

by solving the time-dependent Schrödinder equation

$$i\frac{\partial\Psi(t)}{\partial t} = H(\lambda(t))\Psi(t)$$

with the boundary conditions

$$\lambda(0) = \lambda_0, \lambda(T) = \lambda_1.$$

The essence of this theorem is in the fact that if one can find eigenvalues of the parameter-dependent Hamiltonian H_λ for a value λ_0 of the parameter, then it is also possible to find its eigenvalues for the value λ_1 of the parameter.

Suppose that the Hamiltonian of the system depends on an external parameter τ, whose cyclic adiabatic change induces appearance of an additional phase factor, the evolution operator. This phase factor depends on the geometry of the parameter space. After the work of M.Berry (see [8]) it became clear that the operator $e^{-iH_\tau t}$ can be represented as a product of operators $e^{-iH^1_\tau}e^{-iH^2 t}$, where the first factor is exactly the holonomy operator or the geometric phase. This holonomy operator is defined on the total space of a bundle over the parameter space as base. As is known, the adequate mathematical formalism for description of the geometric phase is a bundle with connection. As was already noted, holonomic quantum computation is based on a smooth bundle $E \to X$ with connection Ω. Quantum gates for computing are obtained from the holonomy operator $\Gamma_\Omega(\gamma)$ acting on the fibre F of the given bundle, which is an n-dimensional vector bundle. When the connection is fixed within one gauge equivalence class, to construct the needed gate one has to choose the integration path γ in an appropriate way.

Let σ be a smooth curve in X with initial point x_0 and end point x_1. Let $v_0 \in F_{x_0}$ and assume that there as a global horizontal lifting of σ with initial point v_0. The endpoint $v_1 \in F_{x_0}$ of σ is called the parallel transport of v_0 along σ. It is obviously unique if it exists. If the parallel transport exists for all $v_0 \in F_{x_0}$ then the map $v_0 \to v_1$ defines a diffeomorphism $F_{x_0} \to F_{x_1}$ which is a linear isomorphism.

It is useful to have explicit forms for the parallel transport map. This is given by a construction known as time-ordered exponential integrals. Let F be a finite dimensional vector space and consider the following non-autonomous differential equation in F:

$$\frac{v(t)}{dt} = A(t)v(t) \tag{3}$$

where $A(t)$ is a linear operator which is a C^∞-function of t. By the existence, uniqueness and regularity theorems for solutions of ordinary differential equations, for any $v \in F$, there is a unique solution $v(t)$ with $v(a)$ given. In differential equation theory, one generally introduces what is known as the fundamental solution of (3), that is, an $End(F)$ valued function which satisfies

$$\frac{\partial}{\partial t}\Phi(t,a) = A(t)\Phi(t,a), \quad \Phi(a,a) = 1.$$

One now has

$$v(t) = \Phi(t, a)v(t).$$

Other representation of the $v(t)$ is:

$$v(t) = Te^{\int_a^t A(s)ds}v(a)$$

which is called the time-ordered exponential integral.

The effect of parallel transporting a group element over an oriented curve σ in U is to multiply it on the left by $Pexp(-\int_\sigma A)$.

This result shows that parallel transport is globally defined as the path-oriented exponential integral and exists for any compact oriented curve with end-points.

If a physical system or theory is described by the principal bundle $P \to M$, with structural group G, then let us say that we have *gauge theory*. The group G is called the gauge group. The choice of a trivialization of the principal bundle is called a *gauge*. Analog of classical elementary field in gauge theory is a connection on the principal bundle called *gauge field* or *vector potential*. *Gauge transformation (change of gauge*) is an automorphism of the principal bundle which covers the identity mapping of the base manifold. Maxwell's electro-magnetism and Yang-Mills theory are examples, correspondingly, of abelian and nonabelian gauge theories.

Theorem 2.2. [4] Any computation code can be realized by a connection Γ.

Let us now give a simplified quantum-mechanical interpretation of the above holonomy operator. Suppose $\{|\varphi_j\rangle\}$ is a set of eigenstates of the Hamiltonian H, i. e.

$$H|\varphi_j\rangle = E_j|\varphi_j\rangle.$$

From this, any state has form $|\psi(0)\rangle = \sum_j a_j|\varphi_j\rangle$ and the state $|\psi(t)\rangle$ satisfies the Shrödinger equation

$$-iH|\psi(t)\rangle = \frac{\partial}{\partial t}|\psi(t)\rangle,$$

from this $|\varphi(t)\rangle = e^{iHt}|\psi(0)$ and therefore

$$|\psi(t)\rangle = \sum_j a_j e^{iE_j t}|\varphi_j\rangle.$$

The term $e^{iE_j t}$ is called *dynamical phase*. In the case $H = H(t)$ we have

$$|\psi(t)\rangle = \sum_j a_j(t)Te^{-i\int_0^\tau H(\tau)d\tau}|\varphi_j(t)\rangle, \tag{4}$$

where $\{|\varphi_j(t)\rangle\}$ is a basis of eigenstates of $H(t)$ and T denotes the time ordering as above.

Suppose that the Hamiltonian H depends on a parameter $\lambda \in M$, where M is an n-dimensional manifold; such Hamiltonian is called parametric Hamiltonian. Using the parametric Hamiltonian identity (4) becomes:

$$|\varphi(t)\rangle = \sum_j a_j(t)Te^{-i\int_0^\tau H(\lambda(\tau))d\tau}|\varphi_j(\lambda(t))\rangle.$$

Using the adiabatic Theorem 2.1 with $|\psi(0) = |\varphi_j(0)\rangle$, we have that

$$|\psi(t) = e^{-i\gamma_j(t)} e^{-i\int_0^\tau E_j(\tau)d\tau} |\varphi_j(t)\rangle,$$

where $e^{-i\gamma_j}$ is obtained from adiabatic evolution.

If we substitute this into the Schödinger equation, we have

$$i\frac{\partial\gamma_j}{\partial t}|\varphi_j(t)\rangle = \frac{\partial}{\partial t}|\varphi_j(t)\rangle,$$

from this

$$\frac{\partial\gamma_j}{\partial t} = -i\langle\varphi_j(t)|\frac{\partial}{\partial t}|\varphi_j(t)\rangle.$$

If we integrate the above identity over a closed path in the parameter space M, we obtain the Berry's phase γ_j as holonomy:

$$\gamma_j = -i\int\langle\varphi_j(t)|\frac{\partial}{\partial t}|\varphi_j(t)\rangle dt = -i\int_0^\tau \langle\varphi_j(t)|\frac{\partial}{\partial t}|\varphi_j(t)\rangle dt,$$

where τ is defined in such a way that $H(\lambda(t)) = H(\lambda(0))$.

In general, let $E \to X$ be a vector bundle with connection ω and let γ be a loop in X at point x_0: $\gamma : [0,1] \to X, \gamma(0) = \gamma(1)$. For this γ, one has a *holonomy* operator $\Gamma_\omega(\gamma) = Pexp(-\int_\gamma \omega)$. It acts on the fibre F_{x_0} at x_0 of the vector bundle $E \to X$ as follows: $E \ni v \mapsto \Gamma_\omega(\gamma)v$. Suppose $\Gamma_\omega(\gamma) \in SU(n)$ and fibre of the bundle is isomorphic to $\mathbb{C}^2 \otimes ... \otimes \mathbb{C}^2$. In this case we speak of *gauge quantum computer*.

In gauge quantum computing the *encoding space of information* is the fibre F of the vector bundle $E \to X$ and *processing of information* is the holonomy operator defined by a connection ω of this bundle and acts on the fibre as $v \mapsto \Gamma_\omega(\gamma)v$.

Consider a trivial bundle $\mathbb{C}^n \times X \to X$. A fixed trivialization gives a one-to-one correspondence between connections ∇ on this bundle and matrix valued 1-forms $\omega \in \Lambda X \otimes \mathfrak{gl}(n, \mathbb{C})$ according to the rule

$$\nabla f = df - \omega f,$$

where ΛX are smooth 1-forms on manifold X, and $f : X \to \mathbb{C}^n$ is a smooth vector function. Connection ∇ is flat if and only if the 1-form is integrable, i.e.

$$d\omega + \omega \wedge \omega = 0.$$

We can define the *iterated integral* (see for example [8] and references therein) on the space of piecewise smooth paths in X, which we denote by PX:

$$\int_\gamma \omega_1 ... \omega_r : PX \to \mathbb{R}.$$

The holonomy (or parallel transport) $\Gamma_\omega : PX \to GL(n, \mathbb{C})$ of the connection 1-form ω on $C^n \times X \to X$ is given by

$$\Gamma_\omega(\gamma) = 1 + \int_\gamma \omega + \int_\gamma \omega\omega + \int_\gamma \omega\omega\omega + ...$$

If ω is integrable, the $\Gamma_\omega(\gamma)$ depends only on homotopy class γ relative to its end-points. Therefore holonomy operator Γ_ω induces the *monodromy* representation

$$\rho : \pi_1(X, x_0) \to GL(n, \mathbb{C}),$$

$$\gamma \mapsto \Gamma_\omega(\gamma).$$

In this case we speak of *monodromic quantum computer* or *monodromic model of quantum computation.*

3 Construction of an Universal Set of Gates for exactly solvable model

In this section, we present the way to obtain a universal set of gates by using the mentioned time-dependent Hamiltonians. Two-qubit gates described by 2×2 matrices together with the special matrices, whose application led to the interchange of the states, form an universal set of quantum gates. All of these matrices can be obtained from the dynamic evolution of quantum system, which will provide quantum program with needed unitary matrices at each fixed moment. It is known that three-parametric family is enough for approximation of all elements from $SU(2)$.

The time evolution of quantum system is governed by the Schrödinger equation

$$i\frac{d|\Psi(r,t)\rangle}{\partial t} = H(r,t)|\Psi(r,t)\rangle \tag{5}$$

with $\hbar = 1$ and T-periodic time-dependent Hamiltonian, $H(t) = H(t+T)$.

In terms of the evolution operator $\mathcal{U}(t) = \mathcal{U}(t,0)$, the solution $|\Psi(r,t)\rangle$ is

$$|\Psi(r,t)\rangle = \mathcal{U}(t)|\Psi(r,0)\rangle, \;\; \mathcal{U}(0) = 1. \tag{6}$$

Using the evolution operator and cyclic solutions $\Psi_1(t), \Psi_2(t)$ we obtain the evolution matrix

$$U(t) = \begin{pmatrix} u_{11}(t) & u_{12}(t) \\ u_{21}(t) & u_{22}(t) \end{pmatrix}. \tag{7}$$

Using the notations from [8] (section 2.3 p.30) let us introduce the matrix

$$U(\alpha,\beta,\lambda;t) = \begin{pmatrix} u_{11}(\alpha,\beta,\lambda;t) & u_{12}(\alpha,\beta,\lambda;t) \\ u_{21}(\alpha,\beta,\lambda;t) & u_{22}(\alpha,\beta,\lambda;t) \end{pmatrix},$$

where

$$\begin{aligned}
u_{11}(\alpha,\beta,\lambda;t) &= \cos(\alpha t)\exp(-i\lambda t)\cos(\beta) - \sin(\alpha t)\exp(i\lambda t)\sin(\beta),\\
u_{12}(\alpha,\beta,\lambda;t) &= -\cos(\alpha t)\exp(-i\lambda t)\sin(\beta) - \sin(\alpha t)\exp(i\lambda t)\cos(\beta),\\
u_{21}(\alpha,\beta,\lambda;t) &= \sin(\alpha t)\exp(-i\lambda t)\cos(\beta) + \cos(\alpha t)\exp(i\lambda t)\sin(\beta),\\
u_{22}(\alpha,\beta,\lambda;t) &= -\sin(\alpha t)\exp(-i\lambda t)\sin(\beta) + \cos(\alpha t)\exp(i\lambda t)\sin(\beta)
\end{aligned}$$

and rewrite the given evolution operator as a product of 3 unitary matrices

$$U(t) = \begin{pmatrix} \cos(\alpha t) & \sin(\alpha t) \\ \sin(\alpha t) & \cos(\alpha t) \end{pmatrix} \begin{pmatrix} \exp(-i\lambda t) & 0 \\ 0 & \exp(i\lambda t) \end{pmatrix} \begin{pmatrix} \cos(\beta) & -\sin(\beta) \\ \sin(\beta) & \cos(\beta) \end{pmatrix}. \tag{8}$$

Evidently, $U(\alpha,\beta,\lambda,t)$ generates all matrices from $SU(2)$. The action of the right matrix on the basis state $|0\rangle$ or $|1\rangle$ yields all coherent states, which by using the diagonal matrix from the given form of $U(t)$ can be rotated around y-axis in a controlled way.

For any α,β,γ, at $t=0$ from (8) we obtain the gate

$$U(\alpha,\beta,\gamma;t)=\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}. \tag{9}$$

To obtain the Hadamard gate

$$H=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} \tag{10}$$

we multiply $U(\alpha,\beta,\gamma;t)$ from the left on the gate

$$NOT=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}. \tag{11}$$

At $t=\frac{\pi}{2\alpha}$ and $\lambda=0$, we have $U(\alpha,\beta,0;\frac{\pi}{2\alpha})=NOT$ and therefore the Hadamard gate H is a result of sequence of gates

$$H=U(\alpha,\beta,0;\frac{\pi}{2\alpha})U(\alpha,\beta,\gamma;0).$$

The single qubit gate U can be obtained as a product of unitary matrices

$$U=U(\alpha_k,\beta_k,\gamma_k;t_k)U(\alpha_{k-1},\beta_{k-1},\gamma_{k-1};t_{k-1})...U(\alpha_0,\beta_0,\gamma_0;t_0=0).$$

Here, $\alpha_j,\beta_j,\gamma_j$, $j=0,...,k$ are any real parameters and the parameters of discrete time t_j, $j=0,...,k$ satisfy the conditions $0=t_0<t_1<...<t_k$.. Note, in our approach, the effect of geometric phase is naturally taking into account the construction of single qubit gates.

Indeed, for the periodic evolution matrix with the period $T=2\pi m$ it follows that the state changes sign when $\alpha_j t_j=2\pi m$ or $\gamma_j t_j=2\pi m$. By this way, we arrive at the binary representation of the numbers from 0 to 2^n-1 :

$$(H\otimes H\otimes\cdots\otimes H)|00\ldots0\rangle=$$

$$U(\alpha,\beta,0;\frac{\pi}{2\alpha}+2n\pi)U(\alpha,\beta,\gamma;2n\pi)\otimes\cdots\otimes U(\alpha,\beta,0;\frac{\pi}{2\alpha})U(\alpha,\beta,\gamma;0)|00\ldots0\rangle=$$

$$\frac{1}{\sqrt{2^n}}((|0\rangle+|1\rangle)\otimes(|0\rangle+|1\rangle)\otimes\cdots\otimes(|0\rangle+|1\rangle))=\frac{1}{\sqrt{2^n}}\sum_{x=0}^{2^n-1}|x\rangle.$$

It should be noted, for this presented sequence of gates in obtaining states $\frac{1}{\sqrt{2^n}}\sum_{x=0}^{2^n-1}|x\rangle$ are not unique.

Let us now turn to construction of two qubit gates. Note, the operator $SWAP:C^2\otimes C^2\to C^2\otimes C^2$, acting on tensor product of vectors by the rule $SWAP(u\otimes v)=v\otimes u$ and taking in the standard basis the form

$$SWAP=\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}$$

does not give the entangled operator, and the operator, obtained from the Burau representation $bur = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ (see [8], [3]) as well as the operator $CNOT$ is the entangled operator despite all these operators have the same standard Jordan's form.

The Hermitian operator of the form $A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ is of rank 2 and commutes with the operators of the form

$$bur = \begin{pmatrix} x_{11} & 0 & 0 & x_{14} \\ 0 & a & b & 0 \\ 0 & b & a & 0 \\ x_{41} & 0 & 0 & x_{44} \end{pmatrix}$$

and the operator $exp(-iAt)$ is an entangled one. Indeed,

$$exp(-iAt) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(t) & -i\sin(t) & 0 \\ 0 & -i\sin(t) & \cos(t) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

takes the form at $t = \pi/4$

$$exp(-iA\pi/4) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sqrt{2}/2 & -i\sqrt{2}/2 & 0 \\ 0 & -i\sqrt{2}/2 & \sqrt{2}/2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The universal $CNOT$ is constructed from $exp(-iA\pi/4)$ and one-qubit gates as follows

$$CNOT = (X_1 \otimes 1)(1 \otimes 1)(X_2 \otimes 1)(1 \otimes Y_2)\times$$

$$\times exp(-iA\pi/4)(X_3 \otimes 1 + 1 \otimes Y_3)exp(-iA\pi/4)(X_4 \otimes 1 + 1 \otimes Y_4)(1 \otimes Y_5)(X_5 \otimes 1),$$

where

$$X_1 = \sqrt{2}/2 \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, X_2 = \sqrt{2}/2 \begin{pmatrix} -i & 1 \\ i & 1 \end{pmatrix}, Y_2 = \sqrt{2}/2 \begin{pmatrix} 1 & i \\ i & -i \end{pmatrix},$$

$$X_3 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, Y_3 = 1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, X_4 = 1/2 \begin{pmatrix} 1-i & -1-i \\ -1-i & 1-i \end{pmatrix},$$

$$Y_4 = \sqrt{2}/2 \begin{pmatrix} 1 & -1 \\ -i & 1 \end{pmatrix}, X_5 = \sqrt{2}/2 \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, Y_5 = \sqrt{2}/2 \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix}.$$

Proposition 3.1. Let H be a two-dimensional diagonal time-independent Hamiltonian of the form

$$h = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}.$$

Then the evolution operator of the matrix Schrödinger equation (5) with the Hamiltonian $H = h \otimes 1 + 1 \otimes h + \varepsilon A$ is expressed as follows

$$U(t) = (e^{-iht} \otimes e^{-iht}) e^{-iAt},$$

where $\varepsilon \in \{0, 1\}$.

Indeed, from the above property of operator A, it follows that the matrix

$$h \otimes 1 + 1 \otimes h = \begin{pmatrix} 2a & 0 & 0 & 0 \\ 0 & a+b & 0 & 0 \\ 0 & 0 & a+b & 0 \\ 0 & 0 & 0 & 2b \end{pmatrix}$$

commutes with A and as consequence

$$U(t) = e^{-i(h \otimes 1 + 1 \otimes h + A)t} = (e^{-iht} \otimes e^{-iht}) e^{-iAt}. \tag{12}$$

The Pauli matrix $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ satisfies the condition of this proposition. We substitute $h = \sigma_3$ into (12) and get the entangled operator

$$U(t) = \begin{pmatrix} e^{it} & 0 & 0 & 0 \\ 0 & \cos(t) & -i\sin(t) & 0 \\ 0 & -i\sin(t) & \cos(t) & 0 \\ 0 & 0 & 0 & e^{-it} \end{pmatrix}.$$

It is interesting to note, even in the case when the initial Hamiltonian does not give the entangled operator, after the suitable choice of the gauge transformation one can get Hamiltonian giving entanglement. In addition, the geometric phase is obtained from the cyclic evolution of states.

4 Schrödinger Equations as Fuchsian Systems and exactly solvable models

It is known that certain physically interesting Schrödinger equations can be rewritten as Fuchsian systems [14], [15], [16]. For us, especially interesting are the cases that arise for Fuchsian systems which are of an Okubo type [8], [14].

The system of differential equation

$$(z\mathbb{I} - B)\dot{X} = AX, \tag{13}$$

where X is a complex-valued (n-column) vector function of complex variable z, dot denotes differentiation over z and

$$B = s_1 \mathbb{I}_{n_1} \oplus ... \oplus s_p \mathbb{I}_{n_p},\ s_i \in \mathbb{C}; s_i \neq s_j,\ \text{when } i \neq j; n_1 + ... + n_p = n$$

and $A \in End(n, \mathbb{C})$, is called a *system of Okubo normal form.*

Denoted by Δ the partition $n_1, ..., n_p$ of n. For the fixed partition Δ, we present the matrix A by the blocks $A = (A_{ij})_{1 \neq i, j \neq p}$.

Eigenvalues of A_{ii} are denoted by Δ_i, $i = 1, ..., p$. The set of eigenvalues of A is denoted by Δ_∞. It is known that

$$\sum_{i=1}^{n} \sum_{\lambda \in \Delta_i} \lambda = \sum_{\rho \in \Delta_\infty} \rho. \tag{14}$$

The system (13) is Fuchsian and has singular points at points $s_1, ..., s_p, \infty$. The set of characteristic exponents at s_i is Δ_i, $i = 1, ..., p$ and the set of characteristic exponents at ∞ is Δ_∞. In this definition the identity means a Fuchs's relation. Suppose that the subgroup G_Δ of $GL_n(\mathbb{C})$ associated with partition Δ has the form $G_\Delta = GL_{n_1}(\mathbb{C}) \oplus ... \oplus GL_{n_p}(\mathbb{C})$. We shall first describe one multidimensional situation that arises from an exactly solvable model of quantum dynamics (see [16] and references therein).

Consider a Schrödinger equation of the form

$$i\frac{\partial \Psi(t)}{\partial t} = H(t)\Psi(t) \tag{15}$$

with time-dependent Hamiltonian $H(t) = (H_{ij}(t))$, $i, j = 1, ..., N$, where

$$H_{11} = \varepsilon(t), H_{12} = V_2, H_{13} = V_3, ..., H_{1N} = V_N,$$

$$H_{21} = V_2, H_{31} = V_3, ..., H_{N1} = V_N, \text{ and } H_{ij} = 0 \text{ otherwise},$$

$\Psi(t) = (\psi_1(t), ..., \psi_N(t))$ is a wave function, $V_j, j = 2, \ldots, n$ are constants, and the time-dependent part ε has the form $\varepsilon(t) = E_1 \tanh(t/T)$ with constant E_1 and T.

Theorem 4.1. Equation (15) is reducible to an N-dimensional Fuchsian system of Okubo type

$$(zI_N - B)\frac{d\Phi(z)}{dz} = A\Phi(z) \tag{16}$$

with $B = \operatorname{diag}(i, -i, \ldots, -i)$.

Remark 4.1. From this theorem follows that we can construct Schrödinger equations with prescribed qualitative properties of solutions (see, e.g., [15]).

In two-dimensional case (i.e., for $N = 2$) this result can be generalized by permitting Hamiltonians of more general form. Namely, consider a Schrödinger equation with two-component phase function

$$i\frac{\partial f(t)}{\partial t} = H(t)f(t), \tag{17}$$

where $f(t) = (f_1(t), f_2(t))$, and time-dependent Hamiltonian $H(t)$ has the form

$$H(t) = \begin{pmatrix} \varepsilon(t) & V(t) \\ V(t) & -\varepsilon(t) \end{pmatrix}, \tag{18}$$

where

$$\varepsilon(t) = \frac{E_0 T + E_1 T y}{1 + y^2}\frac{dy}{dt},\ V(t) = \frac{V_0 T}{\sqrt{1 + y^2}}\frac{dy}{dt}$$

with some monotonically increasing differentiable function $y : \mathbb{R} \to \mathbb{R}$ satisfying $y(t) \to \pm\infty$ as $t \to \pm\infty$.

Theorem 4.2. (see [14]) Equation (17) is reducible to a system of two hypergeometric equations of the form

$$z(z-1)\frac{d^2g}{dz^2}+(\gamma-(1+\alpha+\beta)z)\frac{dg}{dz}-\alpha\beta g(z)=0 \tag{19}$$

with appropriate constants α,β,γ.

Thus, Schrödinger equations with potentials such as above can be written in the Okubo form with explicitly known coefficient matrices, which enables one to compute the monodromy and asymptotics of solutions [16].

As was pointed out earlier, the role of local and entangling operators in monodromic quantum computing is played by monodromy matrices of connection with logarithmic singularities acting on the fiber of a holomorphic vector bundle as on the space of qubits.

We consider *Fuchsian differential equations on Riemann sphere as completely integrable systems*. These systems are also known in mathematical physics as the *Calogero-Moser-Sutherland* systems. For example, in particular cases, the trigonometric Calogero-Moser-Sutherland system describes the dynamics of a quantum mechanical system consisting of n particles moving on the real line under the influence of a pairwise interaction potential which is proportional to the inverse square by the hyperbolic sine *sh* distance of the particles. It is also known, that the trigonometric Calogero-Moser-Sutherland system is obtained as a limit for the elliptic Calogero-Moser-Sutherland system.

Also, one has the classification of integrable models of quantum mechanics which are invariant under the action of a Weyl group with certain assumptions. For the case of B_N ($N\geq 3$), the generic model coincides with the BC_N *Inozemtsev* model. The eigenvalue problem for the Hamiltonian of the BC_1 (one particle) Inozemtsev model can be transformed to the *Heun* equation:

$$\frac{d^2u}{dt^2}+\left(\frac{\gamma}{t}+\frac{\delta}{t-1}+\frac{\varepsilon}{t-d}\right)\frac{du}{dt}+\frac{\alpha\beta t-q}{t(t-1)(t-d)}\,u=0. \tag{20}$$

Here $d\in\mathbb{C}$, is a parameter ($d\neq 0,1$), and $\alpha,\beta,\gamma,\delta,\varepsilon\in\mathbb{C}$ are exponent-related parameters. The Riemann P-symbol is

$$P\left\{\begin{array}{cccc} 0 & 1 & d & \infty \\ 0 & 0 & 0 & \alpha \\ 1-\gamma & 1-\delta & 1-\varepsilon & \beta \end{array}\;;t\right\}.$$

This does not uniquely specify the equation and its solutions, since it omits the accessory parameter $q\in\mathbb{C}$. The exponents are constrained by

$$\alpha+\beta-\gamma-\delta-\varepsilon+1=0. \tag{21}$$

This is a special case of Fuchs's relation, according to which the sum of the $2n$ characteristic exponents of any second-order Fuchsian equation on $\mathbb{CP}^1$ with n singular points must equal $n-2$.

The canonical form of Fuchsian system of the form (21) with exponents $(0,1/2),(0,1/2),(0,1/2)$ and $(l/2,-(l+1)/2)$ is the Lamé equation

$$y^{''}(x)+\frac{1}{2}\left(\frac{1}{x}+\frac{1}{x-1}+\frac{1}{x-s}\right)y^{'}(x)-\frac{l(l+1)x+4q}{4x(x-1)(x-s)}y(x)=0. \tag{22}$$

The Weierstrass form of Lamé equation is

$$\frac{d^2\psi}{du^2}-\left[\ell(\ell+1)\wp(u)+B\right]\psi=0, \tag{23}$$

which is a Fuchsian equation on torus $\mathbb{T}$ with exactly one singular point (at $(\wp, \wp') = (\infty, \infty)$) and a single accessory parameter, B.

Remark 4.2. It is known that the *Lamé* operator

$$L_m = \frac{d^2}{dx^2} - m(m+1)\wp - B,$$

is a nontrivial example of an operator of algebraic type (i.e., its eigenvalue problem is explicitly solvable in quadratures.)

More exactly, $L_{m,B}$ is of algebraic type if and only if m is a nonnegative integer. The equation

$$\frac{d^2}{dx^2}u - (m(m+1)\wp + B)u = 0,$$

is a Schrödinger equation with an elliptic potential extended to the complex domain.

In the theory of holonomic quantum computing [4], it is established that necessary condition for the controllability of the correspondence quantum system is the irreducibility of the connection, in which case the holonomy group of connection coincides with $SU(2)$.

The following theorem imposes on the hypergeometric equation those conditions which ensure construction of local gates (see also [17]).

Theorem 4.3. The physical system described by Schrödinger equation (17) with Hamiltonian (18) defines a universal set of gates if and only if it satisfies the following conditions:

$$1) iT(-E_1 + \sqrt{E_1^2 + V_0^2}) \notin \mathbb{Z};\ \ 2) iT(-E_1 - \sqrt{E_1^2 + V_0^2}) \notin \mathbb{Z};$$

$$3) \frac{1}{2} + E_0T + iT\sqrt{E_1^2 + V_0^2} \notin \mathbb{Z};\ \ 4) \frac{1}{2} + E_0T - 2iE_1T + iT\sqrt{E_1^2 + V_0^2} \notin \mathbb{Z};$$

$$5) \frac{1}{2} - E_0T + iT\sqrt{E_1^2 + V_0^2} \notin \mathbb{Z};\ \ 6) \frac{1}{2} - E_0T - iT\sqrt{E_1^2 + V_0^2} \notin \mathbb{Z};$$

$\lhd$ It is known, that equation (19) defines a two-dimensional holomorphic vector bundle on $\mathbb{CP}^1$ with holomorphic connection having logarithmic singularities [9], [15]. The fiber of this bundle is considered as the space of qubits. The monodromy matrices of equation (19) act on the fibre in a natural way. It is also known that the associated connection gives a universal set of gates if and only if its monodromy representation is irreducible [4]. For equation (19) irreducibility condition means, that in Riemann P-symbol

$$P\begin{pmatrix} 0 & 1 & \infty \\ 0 & 0 & \alpha \\ 1-\gamma & \gamma-\alpha-\beta & \beta \end{pmatrix} \tag{24}$$

the constants α, β, γ satisfy the conditions

$$\alpha, \beta, \gamma - \alpha, \gamma - \beta, 1 - \gamma + \alpha, 1 - \gamma + \beta \notin \mathbb{Z}.$$

From the definition, the universally set of gates with irreducibility of the monodromy representation in hypergeometric equation it follows, that the representation associated from equation (19) irreducible if and only if the system is completely controllable. From this condition of irreducibility it follows, that the constants from (18) can be chosen to satisfy the conditions 1), 2), 3), 4), 5) and 6) in the theorem.▷

If $\alpha = 1$, $q = 1$, or $\alpha = 0$, $q = 0$ then the equation (20) reduces to the hypergeometric equation.

Theorem 4.4. The Hamiltonian of the BC_1 Inozemtsev model

$$H = -\frac{d^2}{dx^2} + \sum_{i=0}^{3} l_i(l_i+1)\wp(x+\omega_i) \tag{25}$$

for appropriate values of parameters (the conditions for the parameters are given in the proof) defines a completely controllable quantum system.

◁ From (25) one obtains Fuchsian equation (20) with 4 singular points and therefore obtains the Heun equation with singular points $(0, 1, t, \infty)$. Denote by ∇ the connection of the two-dimensional vector bundle induced from the Heun equation. Let the eigenvalues for residues of ∇ be:

$$Res_0\nabla = (\lambda_0, -\lambda_0), Res_1\nabla = (\lambda_1, -\lambda_1),$$

$$Res_t\nabla = (\lambda_t, -\lambda_t), Res_\infty\nabla = (\lambda_\infty, -\lambda_\infty).$$

It is known, that if $\sum \varepsilon_i\lambda_i \notin \mathbb{Z}$, where $(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) \in (\mathbb{Z}/2\mathbb{Z})^4$, then the monodromy representation of ∇ is irreducible and the two-dimensional vector bundle is stable. From this follows the controllability of the Hamiltonian (25).▷

Remark 4.3. The sixth Painlevé equation P_{VI} is the nonlinear differential equation of order two on $\mathbb{CP}^1$ with at most four regular singularities and without movable branch points $0, 1, t, \infty$:

$$\frac{d^2x}{dt^2} = \frac{1}{2}(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{x-t})(\frac{dx}{dt})^2 - (\frac{1}{t} + \frac{1}{t-1} + \frac{1}{x-t})\frac{dx}{dt} +$$
$$\frac{x(x-1)(x-t)}{t^2(t-1)^2}(\alpha - \beta\frac{t}{x^2} + \gamma\frac{t-1}{(x-1)^2} + (\frac{1}{2} - \delta)\frac{t(t-1)}{(x-t)^2}).$$

It expresses the isomonodromic condition of the Heun equation. The parameters $\alpha, \beta, \gamma, \delta$ represent the eigenvalues $\delta_0, \delta_1, \delta_t, \delta_\infty$ of the residues of logarithmic connection Δ as before in the following way:

$$\alpha = \frac{1}{2}\lambda_\infty^2,\ \beta = \frac{1}{2}\lambda_0^2,\ \gamma = \frac{1}{2}\lambda_1^2,\ \delta = \frac{1}{2}\delta_t^2.$$

The Painlevé P_{VI} is a Hamilton type equation

$$\frac{dx}{dt} = \frac{\partial H}{\partial p},\ \frac{dp}{dt} = -\frac{\partial H}{\partial x}$$

with the Hamiltonian

$$H = \frac{1}{t(t-1)}[x(x-1)(x-t)p^2 - \{\lambda_0(x-1)(x-t) +$$

$$\lambda_1 x(x-t) + (\lambda_t - 1)x(x-1)\}p + \frac{1}{4}((\lambda_0 + \lambda_1 + \lambda_t - 1)^2 - \lambda_\infty^2)(x-t)].$$

Remark 4.4. A second order differential equation on the Riemann sphere with coefficients in the field of rational functions induces a two-dimensional vector bundle on $\mathbb{CP}^1$. In Fuchsian case, with $n \geq 3$ singular points, the equation will be determined up to isomorphism by location of singular points a difference of characteristic exponents, and $n-3$ accessory parameters which have more a global than local significance.

Remark 4.5. The equation (19) is the two-dimensional case of (13) in which

$$A = \begin{pmatrix} 1-\gamma & 1 \\ (\alpha-\gamma+1)(\gamma-\beta-1) & \gamma-\alpha-\beta-1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

It is known that, for every hypergeometric equation (19), there exists a Fuchsian system with the same singular points and the some monodromy. This system has the form:

$$\frac{dF(z)}{dz} = \begin{pmatrix} 0 & 0 \\ -\alpha\beta & \gamma \end{pmatrix} \frac{F(z)}{z} + \begin{pmatrix} 0 & 1 \\ 0 & \gamma-(\alpha+\beta) \end{pmatrix} \frac{F(z)}{z-1}$$

where $F(z)$ is a 2-dimensional vector function.

5 Mathematics of linear ion traps and Fuchsian equations

It is known that ion traps are good candidates to produce quantum computational processes because they are one of the proposals that fulfill all the criteria. The ion traps that run with static and dynamic electric fields are usually called *Paul* or *radio frequency traps.*

The motion of charged particles using time varying electric fields is described by the Mathieu's equation:

$$w^{''}(\tau) + (a - 2q\cos(2\tau))w(\tau) = 0.$$

By the change of variable $z = sin^2\tau$ we obtain the algebraic form of the Mathieu's equation:

$$z(1-z)w^{''} + \frac{1}{2}(1-2z)w^{'} + \frac{1}{4}(a-2q(1-2z))w = 0.$$

This equation is a confluence Heun's equation and has regular singularity at 0 and 1, both with exponents 0, $\frac{1}{2}$ and irregular singular point at ∞.

Remark 5.1. Other algebraic form of Mathieu's equation is

$$(1-z^2)w^{''} - zw^{'} + (a+2q-4qz^2)w = 0$$

which is obtained by substitution $z = cos\tau$.

Ion traps can produce logical operations through the use of quantum gates, as was firstly proposed by Cirac and Zoller [7] (see also [10], [20]). Equilibrium positions of ions in a linear trap were investigated in [12].

Let us consider the following model [12]. Suppose a chain of N ions in a trap is given. The ions are assumed to be strongly bound in the y and z directions but weakly bound in an harmonic potential in the x direction. The position of the m-th ion, where the ions are numbered from left to right, will be denoted $x_m(t)$. The motion of each ion will be influenced by an overall harmonic potential due to the trap electrodes and by the Coulomb force exerted by all of the other ions. We

will assume that the binding potential in the y and z directions is sufficiently strong so that motion along these axes can be neglected. Hence the energy of the ion chain is given by the following expression:

$$V = \sum_{j=1}^{N} \frac{1}{2} M\nu^2 x_m(t)^2 + \sum_{i,j=1, i\neq j}^{N} \frac{Z^2 e^2}{8\pi\varepsilon_0} \frac{1}{|x_i(t) - x_j(t)|}, \tag{26}$$

where M is the mass of each ion, e is the electron charge, Z is the degree of ionization of the ions, ε_0 is the permitivity of free space, and ν is the trap frequency, which characterizes the strength of the trapping potential in the axial direction.

Assume that the ions are sufficiently cold so that the position of the m-th ion can be approximated by the formula

$$x_m(t) \approx x_m^{(0)} + q_m(t), \tag{27}$$

where $x_m^{(0)}$ is the equilibrium position of the ion, and $q_m(t)$ is a small displacement. The equilibrium positions will be determined by the following equation:

$$\left[\frac{\partial V}{\partial x_m}\right]_{x_m = x_m^{(0)}} = 0 \tag{28}$$

and the dimensionless equilibrium position $u_m = \frac{x_m^{(0)}}{l}$. Then (28) may be rewritten as the following system of N algebraic equations for the values of u_m :

$$u_m - \sum_{m=1}^{N-1} \frac{1}{(x_m - x_n)^2} + \sum_{n=m+1}^{N} \frac{1}{(x_m - x_n)^2} = 0, \quad m = 1, ..., N. \tag{29}$$

The described model is related to the so-called direct problem of electrostatics for Coulomb potential (see [11], [19]): given a compact conductor X and collection of m positive numbers $Q = (q_i)$, find all equilibrium configurations of charges q_i in X and determine their types as critical points of Coulomb potential $E_Q|X_m$, where X_m is the space of m different points in X.

We also consider inverse problem [19]: *given a finite configuration $P = (x_1, ..., x_m)$ of points in a fixed subset X, does there exist a collection of nonzero real numbers $Q = (q_1, ..., q_m)$, interpreted as values of point charges, such that if they are placed at the points x_j then the given configuration is a critical point of Coulomb energy E_Q restricted to X_m?*

Let N be the number of point charges considered. Then the resultant force on q_i in position x_i is given by

$$F_m = \frac{q_m t_1}{x_m^2} + \sum_{j=1}^{m-1} \frac{q_m q_j}{(x_m - x_j)^2} - \sum_{j=m+1}^{N} \frac{q_m q_j}{(x_j - x_m)^2} - \frac{q_m t_2}{(L - x_m)^2}, \quad m = 1, ..., N.$$

The relations

$$F_1 = 0, F_2 = 0, ..., F_N = 0,$$

give a system of non-homogeneous linear equations

$$MQ = G, \tag{30}$$

for unknowns $q_1, ..., q_N$, where

$$G = \left(\frac{t_2}{(L-x_1)^2} - \frac{t_1}{x_1^2}, \frac{t_2}{(L-x_2)^2} - \frac{t_1}{x_2^2}, ..., \frac{t_2}{(L-x_N)^2} - \frac{t_1}{x_N^2} \right)^T,$$

$M = (m_{ij})_{i,j=1}^N$ is an antisymmetric matrix and $m_{ij} = (-1)^{\tau_{ij}}(x_j - x_i)^{-2}$, $i \neq j$, $\tau_{ij} = 0$, if $i > j$; and $\tau_{ij} = 1$, if $i < j$.

Below we discuss some aspects of the inverse problem for logarithmic potential.

The equilibrium points of n free unit charges in interval $(-1, 1) \subset \mathbb{R}^1$ in the field generated by two fixed charges $\frac{\alpha+1}{2}$ at 1 and $\frac{\beta+1}{2}$ at -1, where the charges repel each other according to the law of logarithmic potential, are zeros of classical *Jacobi polynomials* $\left(P_n^{\alpha,\beta}(x)\right)_{n=0}^{\infty}$, orthogonal on $[-1, 1]$ with respect to weight function $(1-x)^\alpha(1+x)^\beta$. This means that the energy of the field has a local minimum at the zeros of the Jacobi polynomial of degree n. This deep result belongs to Stieltjes (see [18]). Szegö proved that the energy has a unique minimum, thus establishing the stability of the equilibrium [18].

Theorem 5.1. [18] Let $\alpha > -1$ and $\beta > -1$. Then the differential equation

$$(1-x^2)y'' + (\beta - \alpha - (\alpha + \beta + 2)x)y' + \gamma y = 0, \tag{31}$$

where γ is a parameter, has a non-identically vanishing polynomial solution if and only if γ has the form $n(n + \alpha + \beta + 1)$, $n = 0, 1, ...$ Moreover, if $\gamma = n(n + \alpha + \beta + 1)$, the polynomials $cP_n^{\alpha,\beta}(x)$, where c is any constant, are solutions of this equation, and there are no other polynomial solutions.

Remark 5.2. The *Jacobi polynomials* from theorem 5.1 can be given explicitly by expression

$$P_n^{(\alpha,\beta)}(x) = \frac{1}{2}\sum_{j=0}^{n} \frac{(n+\alpha)!}{(n-j)!}\frac{(n+\beta)!}{j!}(x-1)^j(x+1)^{n-j}.$$

From the Rodrigues formula

$$P_n^{(\alpha,\beta)}(x) = \frac{1}{2^n n!}(x-1)^{-\alpha}(x+1)^{-\beta}\left(\frac{d}{dx}\right)^n [(x-1)^{n+\alpha}(x+1)^{n+\beta}]$$

follows, that $P_n^{(\alpha,\beta)}(x)$ are analytic functions of the parameters $\alpha, \beta \in \mathbb{C}$ and that $\deg P_n^{(\alpha,\beta)}(x) \leq n$.

We now describe a more detailed *electrostatic model* of m unit point charges on line.

Let two positive fixed charges of mass $\frac{\beta+1}{2}$ and $\frac{\alpha+1}{2}$ at -1 and $+1$, respectively and allow m positive unit charges $X = \{x_1, ..., x_m\}$ to move freely in $(-1, 1)$. The total energy $E(X)$ of this system if the interaction obeys the logarhitmic potential law is equal to

$$E(X) = E_{int} + E_{ext},$$

where

$$E_{int} = -\sum_{1 \leq k \leq j \leq n} \ln|x_k - x_j|,$$

and

$$E_{ext} = \sum_{k=1}^{n} \varphi(x_k)$$

with external field $\varphi(x)$ created by the fixed charges:

$$\varphi(x) = -\frac{\beta+1}{2} \ln|x+1| - \frac{\alpha+1}{2} \ln|x-1|. \tag{32}$$

Theorem 5.1 means that there exists a unique configuration $X^* = \{x_1^*, ..., x_m^*\}$ providing the global minimum of $E(X)$ in $[-1,1]^m$, corresponding to the unique equilibrium position for given free charges, and the points x_j^* are the zeros of the polynomial $P_n^{(\alpha,\beta)}$.

The critical points of the energy functional $E(X)$ as the function of x_j are the solutions of the equation

$$\frac{\partial}{\partial x_k} E(X) = 0.$$

Suppose X^* is a critical configuration, then

$$\frac{\partial}{\partial x_k} E_{int}(X)|_{X=X^*} + \varphi^{'}(x_k) = 0\ . \tag{33}$$

Suppose $y(x) = (x-x_1^*)(x-x_2^*)...(x-x_n^*)$ is a monic polynomial with zeros at x_k^*'s, then

$$\frac{\partial}{\partial x_k} E_{int}(X)|_{X=X^*} = -\sum_{1\le j\le n, j\neq k} \frac{1}{x_k^* - x_j^*} = -\frac{1}{2}\frac{y^{''}(x_k^*)}{y^{'}(x_k^*)}$$

and

$$\varphi^{'}(x) = -\frac{\beta+1}{2(x+1)} - \frac{\alpha+1}{2(x-1)}.$$

From (33) we obtain

$$y^{''}(x) + \left(\frac{\beta+1}{x+1} + \frac{\alpha+1}{x-1}\right) y^{'} = 0 \tag{34}$$

for all $x \in X^*$.

From the equation (34) we obtain that the polynomial

$$(1-x^2)y^{''}(x) + (x(\alpha+\beta+2) + (\alpha-\beta))y^{'}$$

of degree n is equal to zero at the zeros of polynomial $y(x)$ and therefore equal to $\texttt{const} \times y(x)$. Denote by γ this constant, we obtain a second order differential equation (31) (see [18]).

In this context the inverse problem of electrostatics is the following problem:

Inverse Problem: *For fixed position of m points $x_1, ..., x_m$ in interval $(-1,1)$ find α and β, such that total logarithmic energy $E(X)$ will be minimal.*

The equation (31) has only one polynomial solution ([18], p.159). Second linear independent solution of this equation has branch points at -1, 1 and ∞. Therefore we have the monodromy representation

$$\rho : \pi_1(\mathbb{CP}^1 \setminus \{-1, 1, \infty\}) \to GL(2, \mathbb{C})$$

induced from (31) and the monodromy matrices depend on the α, β. It follows that the parameters α and β may be interpreted as elements of monodromy matrices and the inverse problem reduces to the following question:

For a given monodromy representation ρ, construct the differential equation of type (31) with given monodromy ρ.

Let us consider a more general problem.

Given $m+1$ positive charges q_j at p_j, $p_0 < ... < p_m$. Find all possible equilibrium locations x_k of n free unit charges.

For the analysis of above problem let us consider the differential equation

$$P(x)y^{''}(x) + 2Q(x)y^{'}(x) + C(x)y(x) = 0, \tag{35}$$

where $P(x) = (x-p_0)...(x-p_m)$ is a polynomial with simple zeros at the points p_j, $Q(x)$ and $C(x)$ are polynomials of degree m and $m-1$, respectively, and

$$\frac{Q(x)}{P(x)} = \sum_{j=0}^{m} \frac{q_j}{x-p_j}.$$

The equation (35) is called a *generalized Lamé equation* in algebraic form.

Theorem 5.2. [8] For given $P(x)$ and $Q(x)$, there exist exactly $\frac{(n+m-1)!}{n!(p-1)!}$ polynomials $C(x)$, such that, for each of them, there exists a polynomial solution $y(x)$ of degree n with only real zeros of the generalized Lamé equation (35).

The polynomial $C(x)$ is called a *Van Vleck polynomial* and the corresponding polynomial solution $y(x)$ of (35) is called a *Stieltjes polynomial*.

The zeros of Stieltjes polynomial are the solutions of the above posed problem and these points are critical points of the following logarithmic potential

$$E(x_1, ...x_n) = -q_1 \sum_{j=0}^{n} \lg|p_1 - x_j| - ... - q_m \sum_{j=0}^{n} \lg|p_m - x_j| -$$

$$\sum_{1 \le i < j \le n} \lg|x_j - x_i|.$$

In this context the inverse problem takes the form:

For given points $p_1, ...p_m$ and given monodromy representation

$$\rho : \pi_1(\mathbb{CP}^1 \setminus \{p_1, ..., p_m\} \to GL(2, \mathbb{C}) \tag{36}$$

construct a generalized Lame equation

$$y^{''}(x) + \sum_{j=0}^{m} \frac{q_j}{x-p_j} y^{'}(x) + \frac{C(x)}{\prod_{j=0}^{m}(x-p_j)} y(x) = 0, \tag{37}$$

where $C(x)$ is a Van Vleck polynomial such that the monodromy representation of (36) coincides (37).

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