Almost tri-cubic functions with Lipschitz condition

Ismail Nikoufar

Department of Mathematics, Payame Noor University, P.O. Box 19395-3697 Tehran, Iran.
E-mail: nikoufar@pnu.ac.ir

Abstract

In this paper, we approximate a family of functions with a tri-symmetric left invariant mean property by tri-cubic functions in Lipschitz spaces.

2010 Mathematics Subject Classification. **39B82** 39B52
Keywords. tri-cubic functional equation, Lipschitz space, stability.

1 Introduction

The issue of stability of a functional equation has been motivated by a question raised in 1940 by Ulam and the first answer to Ulam's question was given by Hyers in [6]. Some generalizations of Hyers' Theorem have been considered in [22, 23]. Let G be an abelian group and $\mathcal{W}$ a real vector space. A function $\mathcal{C} : G^3 \longrightarrow \mathcal{W}$ is called tri-cubic if $\mathcal{C}$ satisfies the system of equations

$$2\mathcal{C}(x+y,z,w)+2\mathcal{C}(x-y,z,w)+12\mathcal{C}(x,z,w)=\mathcal{C}(2x+y,z,w)+\mathcal{C}(2x-y,z,w)$$

$$2\mathcal{C}(x,y+z,w)+2\mathcal{C}(x,y-z,w)+12\mathcal{C}(x,y,w)=\mathcal{C}(x,2y+z,w)+\mathcal{C}(x,2y-z,w)$$

$$2\mathcal{C}(x,y,z+w)+2\mathcal{C}(x,y,z-w)+12\mathcal{C}(x,y,z)=\mathcal{C}(x,y,2z+w)+\mathcal{C}(x,y,2z-w)$$

for all $x, y, z \in G$. We define the tri-cubic difference of a given function $\mathcal{A} : G^3 \longrightarrow \mathcal{W}$ as follows:

$$\begin{aligned} TC_{\mathcal{A}}(x,y,z,w) := {} & 2\mathcal{A}(x+y,z,w)+2\mathcal{A}(x-y,z,w)+12\mathcal{A}(x,z,w) \\ & -\mathcal{A}(2x+y,z,w)-\mathcal{A}(2x-y,z,w) \end{aligned}$$

for all $x, y, z, w \in G$.

Jun and Kim [9] introduced the following cubic functional equation

$$\mathcal{C}(2x+y)+\mathcal{C}(2x-y)=2\mathcal{C}(x+y)+2\mathcal{C}(x-y)+12\mathcal{C}(x). \tag{1.1}$$

They proved the general solution and the Hyers-Ulam-Rassias stability problem of (1.1) for mapping from a real vector space to a Banach space. Every solution of the cubic functional equation is said to be a cubic function.

Stability of cubic functional equations have been investigated by many authors in different spaces [19, 11]. For instance, in non-Archimedean spaces [3], random normed spaces [4], various normed spaces [10], β-normed left Banach modules on Banach algebras [5], and Fuzzy spaces [7].

Czerwik et al. [1] established the stability of the quadratic functional equations in Lipschitz spaces. The Lipschitz stability type problems for some functional equations were also studied by Tabor [20, 21]. In Lipschitz spaces we investigated the stability of cubic functional equations [2, 17] and the stability of qaudratic [12, 13, 15, 16, 18] and quartic [14] functional equations. The

Tbilisi Mathematical Journal 13(2) (2020), pp. 207–216.
Tbilisi Centre for Mathematical Sciences.
Received by the editors: 24 June 2017.
Accepted for publication: 07 March 2020.

generalized Hyers-Ulam stability of the bi-quadratic functional equations verified in quasi-normed spaces (cf. [8]).

The algebra of Lipschitz functions on a complete metric space plays an important role in noncommutative metric theory similar to that played by the algebra of continuous functions on a compact space in noncommutative topology.

In this paper, we prove a stability result for tri-cubic functional equations in Lipschitz spaces. We prove a family of functions which satisfies tri-symmetric left invariant mean property can be approximated by tri-cubic functions in Lipschitz spaces.

2 Lipschitz conditions for tri-quadratic functional equations

In this section, we prove a family of functions with tri-symmetric left invariant mean (TSLIM in brief) property can be approximated by tri-cubic functions in Lipschitz spaces.

A family S of subsets of $\mathcal{W}$ is called linearly invariant if $A + \alpha B \in S$ for $A, B \in S$, $\alpha \in \mathbb{R}$ and $x + A \in S$ for $A \in S$, $x \in \mathcal{W}$. For example, the family of all closed balls with center at zero is a linearly invariant family in a normed vector space. We denote this family by $CB(\mathcal{W})$. Let $\mathcal{L}(\mathcal{W})$ be a linearly invariant family of subsets of $\mathcal{W}$. By $\mathcal{B}(G, \mathcal{L}(\mathcal{W}))$ we denote the family of all functions $\mathcal{A} : G \longrightarrow \mathcal{W}$ such that $\mathtt{Im}\ \mathcal{A} \subset B$ for some $B \in \mathcal{L}(\mathcal{W})$.

Definition 2.1. A function $\mathcal{A} : G^3 \longrightarrow \mathcal{W}$ is called tri-symmetric if

$$\mathcal{A}(x, y, z) = \mathcal{A}(y, z, x), \quad \mathcal{A}(y, z, x) = \mathcal{A}(z, x, y),$$

$$\mathcal{A}(z, x, y) = \mathcal{A}(z, y, x), \quad \mathcal{A}(z, y, x) = \mathcal{A}(x, z, y),$$

$$\mathcal{A}(x, z, y) = \mathcal{A}(y, x, z)$$

for all $x, y, z \in G$.

Definition 2.2. We say that $\mathcal{A} : G^3 \longrightarrow \mathcal{W}$ is a tri-odd function if

$$\mathcal{A}(-x, y, z) = -\mathcal{A}(x, y, z), \quad \mathcal{A}(x, -y, z) = -\mathcal{A}(x, y, z),$$

$$\mathcal{A}(x, y, -z) = -\mathcal{A}(x, y, z)$$

for all $(x, y, z) \in G^3$, that is, $\mathcal{A}$ is odd in each variable.

Definition 2.3. We say that $\mathcal{B}(G, \mathcal{L}(\mathcal{W}))$ admits a tri-symmetric left invariant mean (briefly TSLIM), if the family $\mathcal{L}(\mathcal{W})$ is linearly invariant and there exists a linear operator $\Pi : \mathcal{B}(G, \mathcal{L}(\mathcal{W})) \longrightarrow \mathcal{W}$ such that

(i) if $\mathcal{A}_{x,y,z} \in \mathcal{B}(G, \mathcal{L}(\mathcal{W}))$ and $x, y, z \in G$, then

$$\Pi[\mathcal{A}_{x,y,z}] = \Pi[\mathcal{A}_{y,z,x}], \quad \Pi[\mathcal{A}_{y,z,x}] = \Pi[\mathcal{A}_{z,x,y}],$$

$$\Pi[\mathcal{A}_{z,x,y}] = \Pi[\mathcal{A}_{z,y,x}], \quad \Pi[\mathcal{A}_{z,y,x}] = \Pi[\mathcal{A}_{x,z,y}],$$

$$\Pi[\mathcal{A}_{x,z,y}] = \Pi[\mathcal{A}_{y,x,z}],$$

(ii) if $\mathcal{A} \in \mathcal{B}(G, \mathcal{L}(\mathcal{W}))$ and $\mathtt{Im}\ \mathcal{A} \subset B$ for some $B \in \mathcal{L}(\mathcal{W})$, then $\Pi[\mathcal{A}] \in B$,

(iii) if $\mathcal{A} \in \mathcal{B}(G, \mathcal{L}(\mathcal{W}))$ and $a \in G$, then $\Pi[\mathcal{A}^a] = \Pi[\mathcal{A}]$, where $\mathcal{A}^a(x) = \mathcal{A}(x + a)$.

Definition 2.4. Let $\Delta : G^3 \times G^3 \longrightarrow \mathcal{L}(\mathcal{W})$ be a set-valued function such that

$$\begin{aligned}&\Delta((x + a, y + b, z + c), (u + a, v + b, w + c))\\ &\quad = \Delta((a + x, b + y, c + z), (a + u, b + v, c + w)) = \Delta((x, y, z), (u, v, w))\end{aligned}$$

for all $(a, b, c), (x, y, z), (u, v, w) \in G^3$. A function $\mathcal{A} : G^3 \longrightarrow \mathcal{W}$ is said to be Δ-Lipschitz if

$$\mathcal{A}(x, y, z) - \mathcal{A}(u, v, w) \in \Delta((x, y, z), (u, v, w))$$

for all $(x, y, z), (u, v, w) \in G^3$.

Theorem 2.5. Let G be an Abelian group and let $\mathcal{W}$ be a vector space. Assume that the family $\mathcal{B}(G, \mathcal{L}(\mathcal{W}))$ admits TSLIM. If $\mathcal{A} : G^3 \longrightarrow \mathcal{W}$ is a tri-odd function and $TC_{\mathcal{A}}(\cdot, t, \cdot, \cdot) : G^3 \longrightarrow \mathcal{W}$ is Δ-Lipschitz for every $t \in G$, then there exists a tri-cubic function $\mathcal{C} : G^3 \longrightarrow \mathcal{W}$ such that $\mathcal{A} - \mathcal{C}$ is $\frac{1}{12}\Delta$-Lipschitz.

Proof. Since $TC_{\mathcal{A}}(\cdot, t, \cdot, \cdot) : G^3 \longrightarrow \mathcal{W}$ is Δ-Lipschitz for every $t \in G$,

$$\frac{1}{12}TC_{\mathcal{A}}(x, t, y, z) - \frac{1}{12}TC_{\mathcal{A}}(u, t, v, w) \in \frac{1}{12}\Delta((x, y, z), (u, v, w)) \tag{2.1}$$

for all $(x, y, z), (u, v, w) \in G^3$. So,

$$\mathtt{Im}(\frac{1}{12}TC_{\mathcal{A}}(x, \cdot, y, z) - \frac{1}{12}TC_{\mathcal{A}}(u, \cdot, v, w)) \subseteq \frac{1}{12}\Delta((x, y, z), (u, v, w)). \tag{2.2}$$

For every $a \in G$ we define $\psi_a : G^3 \longrightarrow \mathcal{W}$ by

$$\psi_a(\cdot, y, z) := \frac{1}{12}\mathcal{A}(2a + \cdot, y, z) + \frac{1}{12}\mathcal{A}(2a - \cdot, y, z) - \frac{1}{6}\mathcal{A}(a + \cdot, y, z) - \frac{1}{6}\mathcal{A}(a - \cdot, y, z).$$

Fix $a \in G^3$. Since $\mathcal{A}$ is odd on its first variable, we have

$$\begin{aligned}\psi_a(\cdot, y, z) &= \frac{1}{12}\mathcal{A}(2a + \cdot, y, z) + \frac{1}{12}\mathcal{A}(2a - \cdot, y, z)\\ &\quad - \frac{1}{6}\mathcal{A}(a + \cdot, y, z) - \frac{1}{6}\mathcal{A}(a - \cdot, y, z)\\ &= \mathcal{A}(0, y, z) + \frac{1}{12}\mathcal{A}(2a + \cdot, y, z) + \frac{1}{12}\mathcal{A}(2a - \cdot, y, z)\\ &\quad - \frac{1}{6}\mathcal{A}(a + \cdot, y, z) - \frac{1}{6}\mathcal{A}(a - \cdot, y, z) - \mathcal{A}(a, y, z) + \mathcal{A}(a, y, z) - \mathcal{A}(0, y, z)\\ &= \frac{1}{12}TC_{\mathcal{A}}(0, \cdot, y, z) - \frac{1}{12}TC_{\mathcal{A}}(a, \cdot, y, z) + \mathcal{A}(a, y, z) - \mathcal{A}(0, y, z).\end{aligned}$$

Let $S := \frac{1}{12}\Delta((0, y, z), (a, y, z)) + \mathcal{A}(a, y, z) - \mathcal{A}(0, y, z)$. By (2.1) we observe that $\mathtt{Im}\ \psi_a(\cdot, y, z) \subseteq S$. The family $\mathcal{B}(G, \mathcal{L}(\mathcal{W}))$ admits TSLIM, so there exists a linear operator $\Pi : \mathcal{B}(G, \mathcal{L}(\mathcal{W})) \longrightarrow \mathcal{W}$ such that

(i) We have

$$\begin{aligned}&\Pi[\psi_x(\cdot,y,z)]=\Pi[\psi_y(\cdot,z,x)],\quad \Pi[\psi_y(\cdot,z,x)]=\Pi[\psi_z(\cdot,x,y)],\\&\Pi[\psi_z(\cdot,x,y)]=\Pi[\psi_y(\cdot,x,z)],\quad \Pi[\psi_y(\cdot,x,z)]=\Pi[\psi_x(\cdot,z,y)],\\&\Pi[\psi_x(\cdot,z,y)]=\Pi[\psi_z(\cdot,y,x)]\end{aligned}$$

for every $x,y,z\in G$,

(ii) $\Pi[\psi_x(\cdot,y,z)]\in A$ for some $A\in\mathcal{L}(\mathcal{W})$ and every $x,y,z\in G$,

(iii) if $u\in G$ and $\psi_x^u(\cdot,y,z):G\longrightarrow\mathcal{W}$ defined by $\psi_x^u(\cdot,y,z):=\psi_x(\cdot+u,y,z)$ for every $x,y,z\in G$, then $\psi_x^u(\cdot,y,z)\in\mathcal{B}(G,\mathcal{L}(\mathcal{W}))$ and $\Pi[\psi_x^u(\cdot,y,z)]=\Pi[\psi_x(\cdot,y,z)]$.

Define the function $\mathcal{C}:G^3\longrightarrow\mathcal{W}$ by $\mathcal{C}(x,y,z):=\Pi[\psi_x(\cdot,y,z)]$. Recall that $\mathcal{B}(G,S(\mathcal{W}))$ contains constant functions. Property (ii) of Π shows that for a constant function $K:G^3\longrightarrow\mathcal{W}$, $\Pi[K]=K$. We proceed the proof by showing that $\mathcal{A}-\mathcal{C}$ is $\frac{1}{12}\Delta$-Lipschitz. For any $(x,y,z)\in G^3$ we consider the constant function $K_{x,y,z}:G\longrightarrow\mathcal{W}$ by $K_{x,y,z}(\cdot):=\mathcal{A}(x,y,z)$. We find that

$$\begin{aligned}&(\mathcal{A}(x,y,z)-\mathcal{C}(x,y,z))-(\mathcal{A}(u,v,w)-\mathcal{C}(u,v,w))\\&\quad=(\Pi[K_{x,y,z}(\cdot)]-\Pi[\psi_x(\cdot,y,z)])-(\Pi[K_{u,v,w}(\cdot)]-\Pi[\psi_u(\cdot,v,w)])\\&\quad=\Pi[K_{x,y,z}(\cdot)-\psi_x(\cdot,y,z)]-\Pi[K_{u,v,w}(\cdot)-\psi_u(\cdot,v,w)]\\&\quad=\Pi[\frac{1}{12}TC_{\mathcal{A}}(x,\cdot,y,z)-\frac{1}{12}TC_{\mathcal{A}}(u,\cdot,v,w)]\end{aligned}$$

for all $(x,y,z),(u,v,w)\in G^3$. In the light of property (ii) of Π we have

$$\Pi[\frac{1}{12}TC_{\mathcal{A}}(x,\cdot,y,z)-\frac{1}{12}TC_{\mathcal{A}}(u,\cdot,v,w)]\in\frac{1}{12}\Delta((x,y,z),(u,v,w))$$

for all $(x,y,z),(u,v,w)\in G^3$. This shows that

$$(\mathcal{A}(x,y,z)-\mathcal{C}(x,y,z))-(\mathcal{A}(u,v,w)-\mathcal{C}(u,v,w))\in\frac{1}{12}\Delta((x,y,z),(u,v,w))$$

for all $(x,y,z),(u,v,w)\in G^3$, i.e., $\mathcal{A}-\mathcal{C}$ is a $\frac{1}{12}\Delta$-Lipschitz function. As a consequence of property (iii) of Π, we see that

$$\begin{aligned}&2\mathcal{C}(y+z,u,v)+2\mathcal{C}(y-z,u,v)+12\mathcal{C}(y,u,v)\\&=2\Pi[\psi_{y+z}(\cdot,u,v)]+2\Pi[\psi_{y-z}(\cdot,u,v)]+12\Pi[\psi_y(\cdot,u,v)]\\&=\Pi[\psi_{y+z}^{2y}(\cdot,u,v)]+\Pi[\psi_{y+z}^{-2y}(\cdot,u,v)]\\&+\Pi[\psi_{y-z}^{2y}(\cdot,u,v)]+\Pi[\psi_{y-z}^{-2y}(\cdot,u,v)]\\&+2\Pi[\psi_y^{y+z}(\cdot,u,v)]+2\Pi[\psi_y^{y-z}(\cdot,u,v)]+2\Pi[\psi_y^{-y-z}(\cdot,u,v)]\\&+2\Pi[\psi_y^{-y+z}(\cdot,u,v)]+2\Pi[\psi_y^{z}(\cdot,u,v)]+2\Pi[\psi_y^{-z}(\cdot,u,v)].\end{aligned}\tag{2.3}$$

In view of property (ii) of Π, the following equalities hold:

$$\begin{aligned}\Pi[\psi_{y+z}^{2y}(\cdot,u,v)]=\Pi[&\frac{1}{12}\mathcal{A}(4y+2z+\cdot,u,v)+\frac{1}{12}\mathcal{A}(2z-\cdot,u,v)\\&-\frac{1}{6}\mathcal{A}(3y+\cdot+z,u,v)-\frac{1}{6}\mathcal{A}(z-\cdot-y,u,v)],\end{aligned}\tag{2.4}$$

$$\Pi[\psi_{y+z}^{-2y}(\cdot,u,v)] = \Pi[\frac{1}{12}\mathcal{A}(2z+\cdot,u,v) + \frac{1}{12}\mathcal{A}(4y+2z-\cdot,u,v) - \frac{1}{6}\mathcal{A}(\cdot-y+z,u,v) - \frac{1}{6}\mathcal{A}(3y+z-\cdot,u,v)]. \tag{2.5}$$

We also have

$$\Pi[\psi_{y-z}^{2y}(\cdot,u,v)] = \Pi[\frac{1}{12}\mathcal{A}(4y-2z+\cdot,u,v) + \frac{1}{12}\mathcal{A}(-2z-\cdot,u,v) - \frac{1}{6}\mathcal{A}(3y+\cdot-z,u,v) - \frac{1}{6}\mathcal{A}(-z-\cdot-y,u,v)], \tag{2.6}$$

$$\Pi[\psi_{y-z}^{-2y}(\cdot,u,v)] = \Pi[\frac{1}{12}\mathcal{A}(-2z+\cdot,u,v) + \frac{1}{12}\mathcal{A}(4y-2z-\cdot,u,v) - \frac{1}{6}\mathcal{A}(-z+\cdot-y,u,v) - \frac{1}{6}\mathcal{A}(3y-z-\cdot,u,v)]. \tag{2.7}$$

A straightforward calculation shows that

$$2\Pi[\psi_{y}^{y+z}(\cdot,u,v)] = \Pi[\frac{1}{6}\mathcal{A}(3y+\cdot+z,u,v) + \frac{1}{6}\mathcal{A}(y-\cdot-z,u,v) - \frac{1}{3}\mathcal{A}(2y+\cdot+z,u,v) - \frac{1}{3}\mathcal{A}(-\cdot-z,u,v)], \tag{2.8}$$

$$2\Pi[\psi_{y}^{y-z}(\cdot,u,v)] = \Pi[\frac{1}{6}\mathcal{A}(3y+\cdot-z,u,v) + \frac{1}{6}\mathcal{A}(y-\cdot+z,u,v) - \frac{1}{3}\mathcal{A}(2y+\cdot-z,u,v) - \frac{1}{3}\mathcal{A}(-\cdot+z,u,v)], \tag{2.9}$$

$$2\Pi[\psi_{y}^{-y-z}(\cdot,u,v)] = \Pi[\frac{1}{6}\mathcal{A}(\cdot+y-z,u,v) + \frac{1}{6}\mathcal{A}(3y-\cdot+z,u,v) - \frac{1}{3}\mathcal{A}(\cdot-z,u,v) - \frac{1}{3}\mathcal{A}(2y-\cdot+z,u,v)]. \tag{2.10}$$

We also have

$$2\Pi[\psi_{y}^{-y+z}(\cdot,u,v)] = \Pi[\frac{1}{6}\mathcal{A}(\cdot+y+z,u,v) + \frac{1}{6}\mathcal{A}(3y-\cdot-z,u,v) - \frac{1}{3}\mathcal{A}(\cdot+z,u,v) - \frac{1}{3}\mathcal{A}(2y-\cdot-z,u,v)], \tag{2.11}$$

$$2\Pi[\psi_{y}^{z}(\cdot,u,v)] = \Pi[\frac{1}{6}\mathcal{A}(2y+\cdot+z,u,v) + \frac{1}{6}\mathcal{A}(2y-\cdot-z,u,v) - \frac{1}{3}\mathcal{A}(\cdot+y+z,u,v) - \frac{1}{3}\mathcal{A}(y-\cdot-z,u,v)], \tag{2.12}$$

$$2\Pi[\psi_y^{-z}(\cdot,u,v)] = \Pi[\frac{1}{6}\mathcal{A}(2y+\cdot-z,u,v)+\frac{1}{6}\mathcal{A}(2y-\cdot+z,u,v) - \frac{1}{3}\mathcal{A}(\cdot+y-z,u,v)-\frac{1}{3}\mathcal{A}(y-\cdot+z,u,v)]. \tag{2.13}$$

Summing (2.4)–(2.13) and using (2.3) and oddness of $\mathcal{A}$ on its first variable, we detect that

$$\begin{aligned}
&2\mathcal{C}(y+z,u,v)+2\mathcal{C}(y-z,u,v)+12\mathcal{C}(y,u,v)\\
&\quad=\Pi[\frac{1}{12}\mathcal{A}(4y+2z+\cdot,u,v)+\frac{1}{12}\mathcal{A}(4y+2z-\cdot,u,v)\\
&\quad-\frac{1}{6}\mathcal{A}(2y+z+\cdot,u,v)-\frac{1}{6}\mathcal{A}(2y+z-\cdot,u,v)]\\
&\quad+\Pi[\frac{1}{12}\mathcal{A}(4y-2z+\cdot,u,v)+\frac{1}{12}\mathcal{A}(4y-2z-\cdot,u,v)\\
&\quad-\frac{1}{6}\mathcal{A}(2y-z+\cdot,u,v)-\frac{1}{6}\mathcal{A}(2y-z-\cdot,u,v)]\\
&\quad=\Pi[\psi_{2y+z}(\cdot,u,v)]+\Pi[\psi_{2y-z}(\cdot,u,v)]\\
&\quad=\mathcal{C}(2y+z,u,v)+\mathcal{C}(2y-z,u,v).
\end{aligned}$$

This implies that $\mathcal{C}$ is cubic on its first variable. It follows from property (i) of Π that $\mathcal{C}$ is tri-symmetric. So, $\mathcal{C}$ is cubic on its second and third variables and consequently $\mathcal{C}$ is tri-cubic. Q.E.D.

Theorem 2.6. Under the hypotheses of Theorem 2.5, if $\mathtt{Im}\ TC_{\mathcal{A}} \subset A$ for some $A \in \mathcal{L}(\mathcal{W})$, then $\mathtt{Im}\ (\mathcal{A}-\mathcal{C}) \subset \frac{1}{12}A$.

Proof. We have

$$\mathtt{Im}(\frac{1}{12}TC_{\mathcal{A}}(x,\cdot,y,z)) \subset \mathtt{Im}(\frac{1}{12}TC_{\mathcal{A}}) \subset \frac{1}{12}A$$

and so $\frac{1}{12}TC_{\mathcal{A}}(x,\cdot,y,z) \in \mathcal{B}(G^3,S(\mathcal{W}))$ for all $(x,y,z)\in G^3$. Related to the property (iii) of Π in Theorem 2.5 we mention that

$$(\mathcal{A}-\mathcal{C})(x,y,z) = \Pi[\frac{1}{12}TC_{\mathcal{A}}(x,\cdot,y,z)] \in \frac{1}{12}A$$

for all $(x,y,z)\in G^3$. Therefore, $\mathtt{Im}\ (\mathcal{A}-\mathcal{C}) \subset \frac{1}{12}A$. Q.E.D.

Definition 2.7. Let (G^3,ρ) be a metric group and $\mathcal{W}$ a normed space. The function $m_{\mathcal{A}}:\mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a module of continuity of $\mathcal{A}: G^3 \longrightarrow \mathcal{W}$ if $\rho((x,y,z),(u,v,w)) \leq \delta$ implies $||\mathcal{A}(x,y,z) - \mathcal{A}(u,v,w)|| \leq m_{\mathcal{A}}(\delta)$ for every $\delta>0$ and $(x,y,z),(u,v,w)\in G^3$.

Definition 2.8. A function $\mathcal{A}: G^3 \longrightarrow \mathcal{W}$ is called a Lipschitz function of order $\alpha>0$ if there exists a constant $L>0$ such that

$$||\mathcal{A}(x,y,z)-\mathcal{A}(u,v,w)|| \leq L\rho((x,y,z),(u,v,w))^{\alpha} \tag{2.14}$$

for every $(x,y,z),(u,v,w)\in G^3$.

Definition 2.9. Consider an Abelian group $(G^3,+)$ with a metric ρ invariant under translation, i.e., satisfying the condition

$$\begin{aligned}&\rho((x+a,y+b,z+c),(u+a,v+b,w+c))\\&\quad=\rho((a+x,b+y,c+z),(a+u,b+v,c+w))=\rho((x,y,z),(u,v,w))\end{aligned}$$

for all $(a,b,c),(x,y,z),(u,v,w)\in G^3$. A metric D on $G\times G^3$ is called a metric pair if it is invariant under translation and the following condition holds

$$\begin{aligned}D((a,x,y,z),(a,u,v,w))&=D((x,a,y,z),(u,a,v,w))\\&=\rho((x,y,z),(u,v,w))\end{aligned}$$

for all $a\in G,(x,y,z),(u,v,w)\in G^3$.

For a metric group (G^3,ρ), a normed space $\mathcal{W}$, and $\alpha\in(0,1]$, let $Lip_\alpha(G^3,\mathcal{W})$ to be the Lipschitz space consisting of all bounded Lipschitz functions of order $\alpha>0$ with the norm

$$||\mathcal{A}||_\alpha:=||\mathcal{A}||_{\sup}+P_\alpha(\mathcal{A}),$$

where $||.||_{\sup}$ is the supremum norm and

$$\begin{aligned}P_\alpha(\mathcal{A})=\sup\Big\{&\frac{||\mathcal{A}(x,y,z)-\mathcal{A}(u,v,w)||}{\rho((x,y,z),(u,v,w))^\alpha}:\\&(x,y,z),(u,v,w)\in G^3,(x,y,z)\neq(u,v,w)\Big\}.\end{aligned}$$

Theorem 2.10. Let $(G^3,+,\rho,D)$ be a metric pair, $\mathcal{W}$ a normed space such that $\mathcal{B}(G^3,CB(\mathcal{W}))$ admits TSLIM, and $\mathcal{A}:G^3\longrightarrow\mathcal{W}$ a tri-odd function. If $TC_\mathcal{A}\in Lip_\alpha(G\times G^3,\mathcal{W})$, then there exists a tri-cubic function $\mathcal{C}$ such that

$$P_\alpha(\mathcal{A}-\mathcal{C})\leq\frac{1}{12}P_\alpha(TC_\mathcal{A}),$$

$$||\mathcal{A}-\mathcal{C}||_\alpha\leq\frac{1}{12}||TC_\mathcal{A}||_\alpha.$$

Proof. Assume that $m_{TC_\mathcal{A}}:\mathbb{R}^+\to\mathbb{R}^+$ is the module of continuity of $TC_\mathcal{A}$ with the metric pair D. Define the set-valued function $\Delta:G^3\times G^3\longrightarrow CB(\mathcal{W})$ by

$$\Delta((x,y,z),(u,v,w)):=\inf_{\rho((x,y,z),(u,v,w))\leq\delta}m_{TC_\mathcal{A}}(\delta)B(0,1),$$

where $B(0,1)$ is the closed unit ball with center at zero. The following inequality shows that $TC_\mathcal{A}(\cdot,t,\cdot,\cdot)$ is Δ-Lipschitz:

$$\begin{aligned}||TC_\mathcal{A}(x,t,y,z)-TC_\mathcal{A}(u,t,v,w)||&\leq\inf_{D((x,t,y,z),(u,t,v,w))\leq\delta}m_{TC_\mathcal{A}}(\delta)\\&=\inf_{\rho((x,y,z),(u,v,w))\leq\delta}m_{TC_\mathcal{A}}(\delta)\end{aligned}$$

for all $t \in G, (x,y,z), (u,v,w) \in G^3$ and consequently by Theorem 2.5 there exists a tri-cubic function $\mathcal{C}$ such that $\mathcal{A}-\mathcal{C}$ is $\frac{1}{12}\Delta$-Lipschitz. Thus,

$$||(\mathcal{A}-\mathcal{C})(x,y,z)-(\mathcal{A}-\mathcal{C})(u,v,w)|| \leq \inf_{\rho((x,y,z),(u,v,w))\leq\delta} \frac{1}{12} m_{TC_{\mathcal{A}}}(\delta),$$

which entails that $m_{\mathcal{A}-\mathcal{C}} = \frac{1}{12} m_{TC_{\mathcal{A}}}$. Moreover, $||TC_{\mathcal{A}}||_{\sup} < \infty$ and $\mathtt{Im}\ TC_{\mathcal{A}} \subset ||TC_{\mathcal{A}}||_{\sup} B(0,1)$. It follows from Theorem 2.6 that

$$||\mathcal{A}-\mathcal{C}||_{\sup} \leq \frac{1}{12}||TC_{\mathcal{A}}||_{\sup}. \tag{2.15}$$

Define the function $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ by $\omega(t) := P_\alpha(TC_{\mathcal{A}})t^\alpha$. Since $TC_{\mathcal{A}} \in Lip_\alpha(G \times G^3, \mathcal{W})$, we have

$$||TC_{\mathcal{A}}(x,t,y,z) - TC_{\mathcal{A}}(u,t,v,w)|| \leq \omega(D((x,t,y,z),(u,t,v,w)),$$

which means that ω is the module of continuity of the function $TC_{\mathcal{A}}$ and consequently $m_{\mathcal{A}-\mathcal{C}} = \frac{1}{12}\omega$. Then,

$$\begin{aligned} ||(\mathcal{A}-\mathcal{C})(x,y,z)-(\mathcal{A}-\mathcal{C})(u,v,w)|| &\leq \frac{1}{12}\omega(\rho((x,y,z),(u,v,w))) \\ &= \frac{1}{12}P_\alpha(TC_{\mathcal{A}})\rho((x,y,z),(u,v,w))^\alpha. \end{aligned}$$

This inequality implies that $\mathcal{A}-\mathcal{C}$ is a Lipschitz function of order α and $P_\alpha(\mathcal{A}-\mathcal{C}) \leq \frac{1}{12}P_\alpha(TC_{\mathcal{A}})$. From the inequality (2.15) it follows that

$$\begin{aligned} ||\mathcal{A}-\mathcal{C}||_\alpha &= ||\mathcal{A}-\mathcal{C}||_{\sup} + P_\alpha(\mathcal{A}-\mathcal{C}) \\ &\leq \frac{1}{12}||TC_{\mathcal{A}}||_{\sup} + \frac{1}{12}P_\alpha(TC_{\mathcal{A}}) \\ &= \frac{1}{12}||TC_{\mathcal{A}}||_\alpha. \end{aligned}$$

Q.E.D.

References

[1] S. Czerwik and K. Dlutek, *Stability of the quadratic functional equation in Lipschitz spaces,* J. Math. Anal. Appl. 293 (2004) 79–88.

[2] A. Ebadian, N. Ghobadipour, I. Nikoufar, and M. E. Gordji, *Approximation of the Cubic Functional Equations in Lipschitz Spaces,* Analysis in Theory and Applications, **30(4)** (2014), 354–362.

[3] M. E. Gordji and M. B. Savadkouhi, *Stability of Cubic and Quartic Functional Equations in Non-Archimedean Spaces,* Acta Appl. Math. **110** (2010), 1321-1329

[4] M. E. Gordji and M. B. Savadkouhi, *Stability of Mixed Type Cubic and Quartic Functional Equations in Random Normed Spaces,* J. Ineq. Appl., (2009), Article ID 527462, 9 pages, doi:10.1155/2009/527462

[5] M. E. Gordji, H. Khodaei, and A. Najati, *Approximation of Cubic mappings with n-variables in β-normed left Banach modules on Banach algebras,* Bull. Korean Math. Soc. **48(5)** (2011), 1063–1078

[6] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Natl. Acad. Sci. U.S.A. **27** (1941), 222–224.

[7] A. K. Mirmostafaee and M. S. Moslehian, *Fuzzy approximately cubic mappings,* Info. Sci. **178** (2008) 3791-3798

[8] W.-G. Park and J.-H. Bae, *Approximate behavior of bi-quadratic mappings in quasinormed spaces,* J. Ineq. Appl., (2010), 8 pages, doi:10.1155/2010/472721.

[9] K.-W. Jun and H.-M. Kim, *The generalized Hyers-Ulam-Rassias stability of a cubic functional equation,* J. Math. Anal. Appl., **274** (2002), 867-878

[10] H. A. Kenary, H. Rezaei, S. Talebzadeh, and S. Jin Lee, *Stabilities of Cubic Mappings in Various Normed Spaces: Direct and Fixed Point Methods,* J. Appl. Math. (2012), Article ID 546819, 28 pages, doi:10.1155/2012/546819

[11] A. Najati, *Hyers-Ulam-Rassias stability of a cubic functional equation,* Bull. Korean Math. Soc., **44(4)** (2007), 825–840.

[12] I. Nikoufar, *Quartic functional equations in Lipschitz spaces,* Rend. Circ. Mat. Palermo, **64(2)** (2015), 171–176.

[13] I. Nikoufar, *Lipschitz approximation of the n-quadratic functional equations,* Mathematica (Cluj), **57(1-2)**, (2015), 67–74.

[14] I. Nikoufar, *Erratum to: Quartic functional equations in Lipschitz spaces,* Rend. Circ. Mat. Palermo, **65(2)** (2016), 345–350.

[15] I. Nikoufar, *Lipschitz criteria for bi-quadratic functional equations,* Commun. Korean Math. Soc., **31(4)**, (2016) 819-825.

[16] I. Nikoufar, *Approximate tri-quadratic functional equations via Lipschitz conditions,* Math. Bohem., **142(4)** (2017), 337–344.

[17] I. Nikoufar, *Behavior of bi-cubic functions in Lipschitz spaces*, Lobachevskii J. Math., **39** (2018), 803–808.

[18] I. Nikoufar, *Stability of multi–quadratic functions in Lipschitz spaces*, Iran J. Sci. Technol. Trans. Sci. **43** (2019), 621–625.

[19] R. Saadati, S. M. Vaezpour, and C. Park, *The stability of the cubic functional equation in various spaces*, Math. Commun. **16** (2011), 131–145.

[20] J. Tabor, *Lipschitz stability of the Cauchy and Jensen equations*, Results Math. **32** (1997) 133–144.

[21] J. Tabor, *Superstability of the Cauchy, Jensen and Isometry Equations*, Results Math. **35** (1999) 355–379.

[22] Th. M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), 297–300.

[23] J. M. Rassias, *On approximation of approximately linear mappings by linear mappings*, J. Funct. Anal. **46** (1982), 126–130.