On cohomologies and algebraic K-theory of Lie p-superalgebras

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Abstract

An enveloping associative superalgebra $\Lambda[L, \alpha, \beta]$ and its groups of cohomologies are defined and it is proved that there exists Frobenius multiplication of the Quillen algebraic K-functors of $\Lambda[L, \alpha, \beta]$. These results generalize corresponding results for Lie p-algebras which were proved by the author earlier.

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1 Introduction

The notion of Lie p-superalgebras was introduced independently by A. Mikhalev [5] and the author of this article [10]. Let L be a Lie p-superalgebra. In this work an enveloping associative superalgebra $\Lambda[L, \alpha, \beta]$ and its groups of cohomologies are defined and it is proved that there exists Frobenius multiplication of the Quillen algebraic K-functors of $\Lambda[L, \alpha, \beta]$. These results generalize corresponding results from [8] and [11] for Lie p-algebras.

All modules will be considered over a field k and $chark = p > 0$ in the case of Lie p-superalgebras. We let Hom and $\otimes$ denote corresponding functors over k.

A Lie superalgebra is a Z_2-graded module $L = L_{\bar{0}} \oplus L_{\bar{1}}$ with such multiplication that

$$[L_i, L_J] \subseteq L_{i+j}],\ [xy] = (-1)^{i+j}[yx],$$

$$(-1)^{ri}[x[yz]] + (-1)^{ij}[y[zx]] + (-1)^{jr}[z[xy]] = 0,$$

where $x \in L_i, y \in L_j, z \in L_r$.

Definition 1. A Lie p-superalgebra is a Lie superalgebra over a field of characteristic $p > 0$ if $L_{\bar{0}}$ is a Lie p-algebra with a p-mapping $x \to x^{[p]}$ and $[x^{[p]}y] = (adx)^p(y)$, where $x \in L_{\bar{0}}$ and $y \in L_{\bar{1}}$.

Let $x_1, x_2, ..., x_n \in L_{\bar{0}},\ y_1, y_2, ..., y_m \in L_{\bar{1}}$ be a basis of Lie superalgebra. Then elements $x_1^{i_1} \cdots x_n^{i_n} y_1^{\varepsilon_1} \cdots y_m^{\varepsilon_m},\ \varepsilon_j = 0 \vee 1$ are basis [13] of an universal enveloping associative algebra $k(L)$ of L.

If L is the Lie p-superalgebra and $I \subseteq k(L)$ is a homogeneous two sided ideal generated by elements $x^p - x^{[p]},\ x \in L_{\bar{0}}$, then it is easy to prove that elements

$$x_1^{i_1} \cdots x_n^{i_n} y_1^{\varepsilon_1} \cdots y_m^{\varepsilon_m},\ 0 \leq_j < p,\ \varepsilon_j = 0 \vee 1$$

generate the basis of a restricted universal associative enveloping algebra $k[L] = k(L)/I$ of the Lie p-superalgebra L.

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2 On cohomologies of Lie p-superalgebras.

Let $X = X_{\bar{0}} \oplus X_{\bar{1}}$ be a Z_2-graded module, $T(X)$ the tensor algebra over X and $T_L(X)$ the Lie p-superalgebra which arises from $T(X)$ in natural way by the additive commutator and the p-mapping. It is clear that the Lie p-sub-superalgebra of $T_L(X)$ generated by X is a free Lie p-superalgebra $F(X)$. A functor of the free Lie p-superalgebras F induces a cotriple [14] functor which we denote by the same symbol F. Consider a simplicial resolution of the Lie p-superalgebra L

$$... \overrightarrow{\rightarrow} F^3(L) \overrightarrow{\rightarrow} F^2(L) \rightarrow F(L) \rightarrow 0, \tag{1}$$

where $F^i(L) = F^i(L)_{\bar{0}} \oplus F^i(L)_{\bar{1}}$. Let L act on a commutative Lie p-superalgebra M with nontrivial p-mapping in general. Let $Der(L, M)$ be the commutative group of superderivations $d : L \rightarrow M$; this means that

$$\begin{gathered} d([xy]) = xd(y) - (-1)^{ij} yd(x), \quad x \in L_i, \ \ y \in L_j, \\ d(x^{[p]}) = x^{p-1}d(x) + d(x)^{[p]}, \quad x \in L_0, \ \ d(x) \in M_0, \\ d(x^{[p]}) = x^{p-1}d(x), \quad x \in L_0, \ \ d(x) \in M_1. \end{gathered} \tag{2}$$

It is clear that

$$Der(L, M) = (D_0^0 \oplus D_1^1) \oplus (D_1^0 \oplus D_0^1), \quad D_j^i \subset Hom(L, M) \cap Der(L_i, M_j).$$

If we denote $D_{\bar{0}} = D_0^0 \oplus D_1^1$, $D_{\bar{1}} = D_1^0 \oplus D_0^1$, then the conditions (2) are satisfied in $D_{\bar{0}}$ and $D_{\bar{1}}$. Consider M as an $F^i(L)$-module by (1). Applying the functor $Der(L, M)$ to (1), it is easy to verify that we get the complex

$$0 \rightarrow C_0^0 \oplus C_1^0 \stackrel{\partial}{\rightarrow} C_0^1 \oplus C_1^1 \stackrel{\partial}{\rightarrow} C_0^2 \oplus C_1^2 \stackrel{\partial}{\rightarrow} ..., \tag{3}$$

where $C_0^i \oplus C_1^i = Der(F^i(L), M)$ and $\partial(C_j^i) \subset C_j^{i+1}, j = 0 \vee 1$. Therefore the complex (3) defines Z_2-graded F-cotriple cohomologies of L with coefficients in M:

$$H_F^n(L, M) = H_F^n(L, M)_{\bar{0}} \oplus H_F^n(L, M)_{\bar{1}}. \tag{4}$$

If L is a Lie p-algebra, the cotriple cohomologies $H_F^n(L, M)$ coincide to Pareigis cohomologies [6].

Let $Ext(L, M)$ denote equivalent classes of extensions of Lie p-superalgebras

$$0 \rightarrow M \stackrel{\psi}{\rightarrow} \Gamma \stackrel{\psi}{\rightarrow} L \rightarrow 0 \tag{5}$$

with the natural structure of commutative algebra.

Theorem 1. $H_F^1(F, M)_{\bar{0}} \simeq Ext(L, M)$.

Proof. Let $\sigma \in Hom(L, \Gamma), \varphi\sigma = id$. We may construct

$$\Phi : H_F^1(F, M)_{\bar{0}} \rightarrow Ext(L, M)$$

in the following way. Consider Γ as the direct sum of k-modules $\Gamma = L \oplus M$ and let $d \in Der(F^2(L), M)_{\bar{0}}$, i.e.

$$d = d_0 + d_1, d_i \in Hom(F^i(L), M).$$

Let us define

$$\begin{gathered}[(x_1,0),(x_2,0)]=([x_1,x_2],d_0(\overline{[x_1,x_2]}),\ x_1,x_2\in L_{\overline{0}};\\ [(x,0),(y,0)]=([x,y],d_1(\overline{[x,y]}),\ x\in L_{\overline{0}},\ y\in L_{\overline{1}};\\ [(y_1,0),(y_2,0)]=([y_1,y_2],d_0(\overline{[y_1,y_2]}),\ y_1,y_2\in L_{\overline{1}};\\ (x,0)^{[p]}=\left(x^{[p]},d\left(\overline{x^p}\right)\right),\ x\in L_{\overline{0}}\end{gathered}\tag{6}$$

($\overline{x}$ denotes the image of x by the natural embedding $X\to F(X)$). It is easy to prove that (6) defines the structure of Lie p-superalgebra on $L\oplus M$, since d is a cocycle. The theorem is proved as the corresponding theorem from [6] but we must take into account the structure of the Lie p-superalgebra on L.

Suppose that L is a Lie superalgebra. The standard projective resolution of the graded Lie algebra $L=\oplus_{i=0}^{\infty}L_i$ was constructed in [8]. If $L_{\overline{0}}=\oplus_{i=0}^{\infty}L_{2i},\ L_{\overline{1}}=\oplus_{i=0}^{\infty}L_{2i+1}$, then $L=L_{\overline{0}}\oplus L_{\overline{1}}$ will be the superalgebra. Therefore mentioned resolution will be the standard projective resolution of Lie superalgebras. This resolution is

$$X(L)=k[L]\otimes E(L_{\overline{0}})\otimes\Gamma(L_{\overline{1}}),\tag{7}$$

where $E(L_{\overline{0}})$ is an exterior algebra, $\Gamma(L_{\overline{1}})$ is an algebra of divided powers [2] and the derivation is expressed by the formula (5.5) from [6]. If $char\,k=0$, then $\Gamma(L_{\overline{1}})\simeq S(L_{\overline{1}})$ and (7) coincides to the standard resolution from [1]. We shall denote corresponding cohomologies of the Lie superalgebra L with coefficients in the L-module M as

$$H^n(L,M)=H^n(L,M)_{\overline{0}}\oplus H^n(L,M)_{\overline{1}}.$$

Consider (5) as the extension of Lie superalgebras. Then it is easy to show that

$$[(x,0),(y,0)]=([xy],\alpha(x,y)),[\alpha]\in H^2(L,M)_{\overline{0}},$$

because

$$\begin{gathered}(-1)^{ri}\alpha(x,[yz])+(-1)^{ij}\alpha(y,[zx])+(-1)^{jr}\alpha(z,[xy])+\\ +(-1)^{ri}x\alpha(y,z)+(-1)^{ij}y\alpha(z,x)+(-1)^{jr}z\alpha(x,y)=0,\\ x\in L_i,\ y\in L_j,\ z\in L_r.\end{gathered}\tag{8}$$

It is clear that

$$\alpha_1:L_{\overline{0}}\otimes L_{\overline{0}}\to M_{\overline{0}},\ \alpha_2:L_{\overline{0}}\otimes L_{\overline{1}}\to M_{\overline{1}},\ \alpha_3:L_{\overline{1}}\otimes L_{\overline{1}}\to M_{\overline{0}}.$$

For other σ in (5) we get such α' that

$$\alpha'(x,y)-\alpha(x,y)=x\nu(y)-(-1)^{ij}y\nu(x)-\nu([xy])\tag{9}$$

where

$$x\in L_i,\ y\in L_j,\ \gamma\in Hom(L,M)_{\overline{0}}$$

i.e.

$$\nu=\nu_0+\nu_1,\ \nu_0:L_{\overline{0}}\to M_{\overline{0}},\ \nu_1:L_{\overline{1}}\to M_{\overline{1}}.$$

Since there exists the p-mapping in (5), we must have

$$(x,0)^{[p]} = (x^{[p]}, \beta(x)),\ x \in L_{\overline{0}},\ \beta : L_{\overline{0}} \to M_{\overline{0}}.$$

It is easy to verify that β satisfies the conditions

$$\beta(\lambda x) = \gamma^p \beta(x),\ x \in L_{\overline{0}},$$

$$\alpha(x^{[p]}, y) - y\beta(x) = \sum\nolimits_{i=0}^{p-1} x^{p-1-j} \alpha_j(x, (adx)^i(y)),\ j = 1,2;\ y \in L_{\overline{j}}, \tag{10}$$

$$\beta(x_1 + x_2) - \beta(x_1) - \beta(x_2) =$$

$$\sum_{i=1}^{p-1} \sum_{j=1}^{p-1} \frac{1}{t^{-1}(1)} x_{t(1)} ... x_{t(i-1)} \alpha_1(x_{t(i)}, [x_{t(i+1)}, [...[x_{t(p-1)}, x_2]...]]),$$

where $x_1, x_2 \in L_{\overline{0}}, t : \{1,2,...,p-1\} \to \{1,2\}, 1 \le |t^{-1}(1)| = card\{n | t(n) = 1\}$.

For other σ in (5) we get such different β' that

$$\beta'(x) - \beta(x) = \nu(x)^{[p]} - x^{p-1}\nu(x) - \nu(x^{[p]}). \tag{11}$$

Corollary. If we denote by $S(L,M)$ equivalent classes of pairs (α, β) respect to conditions (9) and (11); then

$$H_F^2(L,M)_{\overline{0}} \simeq Ext(L,M) \simeq S(L,M).$$

3 Crossed enveloping associative algebras of Lie p-superalgebras

Let A be an associative superalgebra over k and M be a Z_2-graded A-module. Let us define on the k-module

$$T(A,M) = A \oplus (A \otimes M \otimes A) \oplus (A \otimes M \otimes A \otimes M \otimes A) \oplus ...$$

the multiplication as

$$(...m_1 \otimes a_1)(m_2 \otimes a_2...) = ...m_1 \otimes a_1 m_2 \otimes a_2....$$

Let us construct the maps

$$T(A,M),\ a \to a;\quad M \to T(A,M),\ m \to 1 \otimes m \otimes 1.$$

The triple $(T(A,M), i_1, i_2)$ has the natural properties of universality.

Let L be a Lie superalgebra, and Λ a commutative associative algebra on which L acts by superderivations.

Definition 1. Let $[\alpha] \in H^2(L, \Lambda^+)_{\overline{0}}$, where Λ is the underground space of Λ^+. Let I be the two sided ideal of $T(\Lambda, L)$, generated by elements

$$1 \otimes x \otimes \lambda - (-1)^{ri} \lambda \otimes x \otimes 1 - x\lambda, 1 \otimes x \otimes 1 \otimes y \otimes 1 - (-1)^{ij} 1 \otimes y \otimes 1 \otimes x \otimes 1 - 1 \otimes [xy] \otimes 1,$$

where $\lambda \in \Delta_r,\ x \in L_i,\ y \in L_j$. Then the triple $(\Lambda(L,\alpha), i_1, i_2)$, where $\Lambda(L,\alpha) = T(\Lambda, L)/I$ and i_1, i_2 are induced by embeddings $L, \Lambda \to T(\Lambda, L)$ is called an α-crossed universal enveloping associative algebra of the Lie superalgebra L.

Let L be a Lie p-superalgebra.

Definition 2. Let $[(\alpha,\beta)] \in S(L,\Lambda_p^+)$, where Λ is the underground space of Λ_p^+. Let I be the homogenous ideal of $\Lambda(L,\alpha)$ generated by elements $x^p - x^{[p]} - \beta(x)$. Then the triple $(\Lambda[L,\alpha,\beta], i_1, i_2)$, where $\Lambda[L,\alpha,\beta] = \Lambda(L,\alpha)/I$ and i_1, i_2 are induced by the natural embeddings $L, \Lambda \to \Lambda(L,\alpha)$ is called an (α,β)-crossed universal enveloping associative algebra of the Lie p-superalgebra L.

Remark. From Proposition 2 below follows that Definitions 1 and 2 are well defined.

The associative superalgebras $\Lambda(L,\alpha)$ and $(\Lambda[L,\alpha,\beta], i_1, i_2)$ are equipped with the same natural universal properties as the similar constructions in the non-graded cases [8], [11].

It is well known that $k(L)$ and $k[L]$ are cocommutative Hopf algebras with antipodes [13]. An associative superalgebra A is a comodule superalgebra [5] over Hopf superalgebra H, if A is the comodule over H with such a coaction $\chi : A \to A \otimes H$ that χ is a morphism of associative superalgebras. Let us denote $\chi(a) = \sum a_{(i_1)} \otimes h_{(i_2)}$, $\Delta(h) = \sum h_{(i_1)} \otimes h_{(i_2)}$, $a_{(i)} \in A_i$, $h_{(i)} \in H_i$ (Δ is a coproduct in H). If M is a supermodule over A, N is a supermodule over H, then $M \odot N$ will be denote the A-supermodule with the action $a(m \odot n) = \sum (-1)^{i_2 r} a_{(i_1)} m \otimes h_{(i_2)} n$, $m \in M_r$. If H has an antipode S, then $Hom(N,M)$ is the left A-supermodule by the action $(af)(n) = \sum (-1)^{jr} a_{(i_1)} f(S(h_{(i_2)})n)$, where $f \in Hom(N,M)_r$, i.e. $f(N_i) \subset M_{i+r}$. Finally the action of A on $A \otimes M$ by the formula $a(b \otimes m) = ab \otimes m$ will be denoted $\langle A \otimes M \rangle$.

Proposition 1. a) $\Lambda(L,\alpha)$ is the comodule superalgebra over $k(L)$;

b) $\Lambda[L,\alpha,\beta]$ is the comodule superalgebra over $k[L]$.

Proof. Proposition follows from properties of universality of the superalgebras $\Lambda(L,\alpha)$ and $\Lambda[L,\alpha,\beta]$.

Proposition 2. a) $L(L,\alpha)$ and $L(L,\alpha')$ are isomorphic if and only if $\alpha \sim \alpha'$ in $H^2(L,\Lambda^+)_{\bar{0}}$.

b) $\Lambda[L,\alpha,\beta]$ and $\Lambda[L,\alpha',\beta']$ are isomorphic if and only if $(\alpha,\beta) \sim (\alpha',\beta')$.

Proof. Proposition follows from the conditions (9) and (11).

Corollary. a) The set of equivalence classes of superalgebras $\Lambda(L,\alpha)$ is isomorphic to $H^2(L,\Lambda^+)_{\bar{0}}$ as a commutative algebra.

b) The set of equivalence classes of superalgebras $\Lambda[L,\alpha,\beta]$ is isomorphic to $H^2(L,\Lambda_p^+)_{\bar{0}}$ as a commutative algebra.

Theorem 2 (Poincare-Birkhoff-Witt theorem). Let $I \cup J$ be well-ordered set where $(i \in I, j \in J) \Rightarrow i < j$ and $I \bigcap J = \varnothing$. Let $\{x_i\}_{i\in I}$ be a basis of $L_{\bar{0}}$ and $\{y_j\}_{j\in J}$ a basis of $L_{\bar{1}}$.

a) The elements $\bar{x}_{i_1}^{r_1} \ldots \bar{x}_{i_n}^{r_n} \bar{y}_{j_1}^{\varepsilon_1} \ldots \bar{y}_{j_m}^{\varepsilon_m}$, $\varepsilon_j = 0 \vee 1$ are Λ-basis of $\Lambda(L,\alpha)$.

b) The elements $\bar{x}_{i_1}^{r_1} \ldots \bar{x}_{i_n}^{r_n} \bar{y}_{j_1}^{\varepsilon_1} \ldots \bar{y}_{j_m}^{\varepsilon_m}$, $0 \le r_s < p$, $\varepsilon_j = 0 \vee 1$ are Λ-basis of $\Lambda[L,\alpha,\beta]$ (as we already have noted, $i_1(x) = \bar{x}$, $i_2(y) = \bar{y}$).

Proof. Theorem 2 was proved in [10] similarly as in [8], [11].

Lemma. a) In $k(L)$ and $k[L]$ the following conditions are satisfied:

$$(\forall n \in Z) \sum (-1)^{ni_2} h_{(i_1)} \varepsilon(h_{(i_2)}) = \sum (-1)^{ni_1} \varepsilon(h_{(i_1)}) h_{(i_2)} = h, \tag{12}$$

where $h \in k(L)$ (or $h \in k[L]$) and ε is the counit of $k(L)$ (or $k[L]$).

b) In $L(\Lambda,\alpha)$ and $L[\Lambda,\alpha,\beta]$ the following condition is satisfied:

$$(\forall n \in Z) \sum (-1)^{ni_2} a_{(i_1)} \varepsilon(h_{(i_2)}) = a, \tag{13}$$

where $a \in L(\Lambda,\alpha)$ (or $a \in L[\Lambda,\alpha,\beta]$).

Proof. The case $n = 0$ follows from the definition of coalgebra; the general case is also true, because $\varepsilon(h) = 0$ for all homogenous elements $h \notin k$.

Proposition 3. Let H be a cocommutative Hopf superalgebra equipped by an antipode, A - a H-comodule superalgebra satisfying the conditions (13), $A' \subset A$ - a comodule sub-superalgebra and N - a supermodule over H. Then there exists the natural isomorphism of A-supermodules

$$f : \langle A \otimes_{A'} (M \odot N) \rangle \to \langle A \otimes_{A'} M \rangle \odot N.$$

Proof. Define

$$f : (a \otimes (m \otimes n)) = \sum (-1)^{i_2 r} (a_{(i_1)} \otimes m) \otimes h_{(i_2)} n, \ m \in M_r, \ n \in N_t.$$

It is easy to verify that f is well defined. Let $b \in A'$. If $\chi(b) = \sum b_{(j_1)} \otimes g_{(j_2)}$, then $\Delta(ab) = \sum (-1)^{i_2 j_1} a_{(i_1)} b_{(j_1)} \otimes h_{(i_2)} g_{(j_2)}$; therefore

$$f(ab \otimes (m \otimes n)) = \sum (-1)^{i_2 j_1 + (i_2 + j_2) r} (a_{(i_1)} b_{(j_1)} \otimes m) \otimes h_{(i_2)} g_{(j_2)} n,$$

$$f(a \otimes b(m \otimes n)) = \sum f(a \otimes (-1)^{j_2 r} (b_{(j_1)} m \otimes g_{(j_2)} n) =$$

$$= \sum (-1)^{j_2 r + i_2 (j_1 + r)} (a_{(j_1)} \otimes b_{(j_1)} m) \otimes h_{(j_{12})} g_{(j_{12})} n.$$

It is clear that f is the morphism of A-supermodules. Indeed, if $b \in A$ then

$$f(a \langle b \otimes (m \otimes n) \rangle) = f(\langle ab \otimes (m \otimes n) \rangle) =$$

$$= \sum (-1)^{(i_2 + j_2) r + i_2 j_1} (a_{(i_1)} b_{(j_1)} \otimes m) \otimes h_{(i_2)} g_{(j_2)} n = a f(b \otimes (m \otimes n)).$$

Let $n \in N_t$. Let the mapping $\varphi_n : \langle A \otimes_{A'} M' \rangle \to \langle A \otimes_{A'} (M' \otimes N) \rangle$ be

$$\varphi_n(a \otimes m) = \sum (-1)^{i_2 r} \left\langle a_{(i_1)} \otimes \left(m \otimes S(h_{(2)}) n \right) \right\rangle.$$

It is easy to show that φ_n is also well defined:

$$\varphi_n(ab \otimes m) = \sum (-1)^{i_2 j_1 + (i_2 + j_2) r} a_{(i_1)} b_{(j_1)} \otimes \left(m \otimes S(h_{(i_2)} g_{(j_2)}) n \right)$$

$$= \sum (-1)^{i_2 (j_1 + j_2) + (i_2 + j_3) r + j_2 r} \left(a_{(i_1)} \otimes \left(b_{(j_1)} m \otimes g_{(j_2)} S(h_{(i_2)} g_{(j_3)}) n \right) \right)$$

$$= \sum (-1)^{i_2 (j_1 + j_2) + (i_2 + j_3) r + j_2 r + i_2 j_3} \left(a_{(i_1)} \otimes \left(b_{(j_1)} m \otimes g_{(j_2)} S(g_{(j_3)}) S(h_{(i_2)}) n \right) \right)$$

$$= \sum (-1)^{i_2 j + (j - j_1) r + j_1 r} \left(a_{(i_1)} \otimes \left(b_{(j_1)} m \varepsilon(g_{(j_2)}) \otimes S(h_{(i_2)}) n \right) \right)$$

$$= \sum (-1)^{i_2 (j + r)} \left(a_{(i_1)} \otimes \left(bm \otimes S(h_{(i_2)}) n \right) \right) = \varphi_n(a \otimes bm).$$

If we define

$$\varphi : \langle A \otimes_{A'} M' \rangle \odot N \to \langle A \otimes_{A'} (M' \odot N) \rangle,$$

$$\varphi : \langle a \otimes m \rangle \otimes n \to \varphi_n (a \otimes m),$$

then it is easy to verify that $f\varphi = id$, $\varphi f = id$.

Proposition 4. Let L be a Lie p-superalgebra, $L' \subseteq L$ a Lie p-sub-superalgebra, N' - a $k[L']$-supermodule, M - a supermodule over $\Lambda[L, \alpha, \beta]$. Then there exists the natural isomorphism of left $\Lambda[L, \alpha, \beta]$-modules

$$\left\langle \Lambda[L, \alpha, \beta] \otimes_{\Lambda[L', \alpha, \beta]} (N' \otimes M) \right\rangle \simeq \left\langle k[L] \otimes_{k[L']} N' \right\rangle \odot M.$$

Proof. Proposition is proved in [10] similarly to the corresponding propositions from [8], [11].

Corollary. In the conditions of Proposition 3 there exists the natural isomorphism of left $\Lambda[L, \alpha, \beta]$-modules

$$\langle \Lambda[L, \alpha, \beta] \otimes_{\Lambda} M \rangle \simeq k[L] \odot M.$$

Proof. It is sufficient to assume in Proposition 4 that $L' = 0, \quad N' = k$.

Proposition 5. Let H be a cocommutative Hopf superalgebra equipped by an antipode, A a comodule H-superalgebra, which satisfies (13) and $M_1, M_2 \in A - \mathrm{mod}$, $N \in H - \mathrm{mod}$. Then there exists the natural isomorphism of k-modules

$$Hom_A (M_1 \odot N, M_2) \simeq Hom_A (M_1, Hom (N, M_2)).$$

Proof. It is proved as corresponding propositions from [9], [12] taking into account the existence of the superstructure.

Corollary. If M_1 is a projective A-supermodule, then $M_1 \odot N$ is the projective supermodule over A.

4 The functors $K_n(\Lambda[L, \alpha, \beta])$ as Frobenius modules

Let A be an associative superalgebra, $M_{\mathrm{sup}}(A)$ - a category of finitely generated A-supermodules, $P_{\mathrm{sup}}(A)$ - a category of finitely generated projective A-supermodules. Let us denote

$$G_0(A) = K_0 (M_{\mathrm{sup}}(A)), \ K_i(A) = K_i(P_{\mathrm{sup}}(A)), \ i = 0, 1, 2, ...,$$

where K_0 is the Grothendieck group and $K_i(P_{\mathrm{sup}}(A))$ are Quillen's algebraic K-functors [7] of the category $P_{\mathrm{sup}}(A)$.

Let L be a Lie p-superalgebra and $Sub(L)$ the category of Lie p-sub-superalgebras of L, which morphisms are monomorphisms. Consider the following functors from $Sub(L)$ to category of commutative groups Ab :

$$G_0(k[-]), \ K_i(\Lambda[-, \alpha, \beta]) : Sub(L) \to Ab, \ i = 0, 1, 2, ...$$

Theorem 3. Let L be a finite dimensional Lie p-superalgebra; then

a) $G_0(k[-])$ is a Frobenius functor [3] on the category $Sub(L)$.

b) $K_i(\Lambda[-, \alpha, \beta])$ are a Frobenius modules [3] over $G_0(k[-])$ on the category $Sub(L)$.

Proof. This theorem is proved (using Propositions 2, 3, 4 and general properties of Quillen algebraic K-functors [7]) similarly to Theorem 1.1 from [9], [12] which states that Quillen algebraic K-functors of Sweedler extensions of Hopf algebras are Frobenius modules.

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