Symmretric functions for second-order recurrence sequences

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Abstract

In this paper, we introduce new symmetric endomorphism operators on appropriate infinite product series. The main results show that after direct calculations, the proposed operators are qualified to obtain new generating functions for second-order recurrence sequences.

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1 Introduction and notations

Some sequences of number have been studied over several years. From these sequences, the Fibonacci numbers F_n are the terms of the sequence $\{0, 1, 1, 2, 3, 5, ...\}$ wherein each term is the sum of the two preceding terms, beginning with the values $F_0 = 0$, $F_1 = 1$. Similarly Jacobsthal numbers J_n and Jacobsthal-Lucas numbers j_n are also defined by the following recurrences

$$J_{n+2} = J_{n+1} + 2\ J_n, \quad J_0 = 0, \quad J_1 = 1,\ n \geq 0,$$

and

$$j_{n+2} = j_{n+1} + 2\ j_n, \quad j_0 = 2, \quad j_1 = 1,\ n \geq 0,$$

respectively ([7]) . Recently, many authors are dedicated to Fibonacci sequence, we have for example, the works of Falcón et *al.*, in ([8]) , Marques in ([10]) and Shattuck in ([11]).

In this contribution, we shall define a new useful operator denoted by $\delta^k_{b_1 b_2}\delta^k_{a_1 a_2}$ for which we can formulate, extend and prove new results based on our previous ones ([1, 3, 4]). In order to determine generating functions for Jacobsthal numbers and Jacobsthal-Lucas numbers, we combine between our indicated past techniques and these presented polishing approaches.

Definition 1.1. ([6]) Let A and C be any sets of given variables, then we give $S_n(A - C)$ by the following form:

$$\frac{\prod\limits_{c\in C}(1 - cz)}{\prod\limits_{a\in A}(1 - az)} = \sum_{n=0}^{\infty} S_n(A - C)\, z^n, \tag{1.1}$$

with the condition $S_n(A - C) = 0$ for $n < 0$.

Equation (1.1) can be rewritten in the following form:

$$\sum_{n=0}^{\infty} S_n(A - C)\, z^n = \left(\sum_{n=0}^{\infty} S_n(A)\, z^n\right) \times \left(\sum_{n=0}^{\infty} S_n(-C)\, z^n\right),$$

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where

$$S_n(A-C)=\sum_{i=0}^{n}S_i(A)\,S_{n-i}(-C).$$

Remark 1.2. Taking $A=\{0\}$ in (1.1) gives

$$\sum_{n=0}^{\infty}S_n(-C)\,z^n=\prod_{c\in C}(1-cz). \tag{1.2}$$

Definition 1.3. ([5]) Given a function f on $\mathbb{R}^n$, the divided difference operator is defined as follows:

$$\partial_{x_ix_{i+1}}f(x_1,...,x_i,x_{i+1},...,x_n)=\frac{f(x_1,...,x_i,x_{i+1},...,x_n)-f(x_1,...,x_{i-1},x_{i+1},x_i,...,x_n)}{x_i-x_{i+1}}.$$

Definition 1.4. Let n be positive integer and $A=\{a_1,a_2\}$ are set of given variables, then, the n -th symmetric function $S_n(a_1+a_2)$ is defined by

$$S_n(A)=S_n(a_1+a_2)=\frac{a_1^{n+1}-a_2^{n+1}}{a_1-a_2},$$

with

$$\begin{aligned}S_0(A) &= S_0(a_1+a_2)=1,\\ S_1(A) &= S_1(a_1+a_2)=a_1+a_2,\\ S_2(A) &= S_2(a_1+a_2)=a_1^2+a_1a_2+a_2^2,\\ &\vdots\end{aligned}$$

Definition 1.5. ([6]) The symmetrizing operator $\delta^k_{a_1a_2}$ is defined by

$$\delta^k_{a_1a_2}f(a_1)=\frac{a_1^kf(a_1)-a_2^kf(a_2)}{a_1-a_2},\ \text{for all } k\in\mathbb{N}_0. \tag{1.3}$$

If $f(a_1)=a_1$, the operator (1.3) gives us

$$\delta^k_{a_1a_2}f(a_1)=\frac{a_1^{k+1}-a_2^{k+1}}{a_1-a_2}=S_k(a_1+a_2).$$

Definition 1.6. For $k\in\mathbb{N}_0$, the symmetrizing operator $\delta^k_{b_1b_2}\delta^k_{a_1a_2}$ is defined by

$$\delta^k_{b_1b_2}\delta^k_{a_1a_2}f(a_1,b_1)=\frac{a_1^kb_1^kf(a_1b_1)-a_1^kb_2^kf(a_1b_2)-a_2^kb_1^kf(a_2b_1)+a_2^kb_2^kf(a_2b_2)}{(a_1-a_2)(b_1-b_2)}.$$

2 Main results

In this part, we are now in a position to provide theorem 1 and theorem 2. At the mean time we derive the new generating functions of the products of some known numbers.

Theorem 2.1. Let A and B be two alphabets, respectively, $\{a_1, a_2\}$ and $\{b_1, b_2\}$, then we have

$$\sum_{n=0}^{\infty} S_n(a_1+a_2)\, S_{n+k-1}(b_1+b_2)\, z^n = \frac{b_1^k - b_2^k - (a_1+a_2)\left(b_1^k b_2 - b_2^k b_1\right) z - a_1 a_2 \left(b_2^k b_1^2 - b_1^k b_2^2\right) z^2}{(b_1-b_2)\prod\limits_{a\in A}(1-ab_1 z)\prod\limits_{a\in A}(1-ab_2 z)},$$

for all $k \in \mathbb{N}_0$.

Proof. By applying the operator $\delta_{b_1 b_2}^k$ to the series $f(b_1 z) = \sum\limits_{n=0}^{\infty} S_n(a_1+a_2)\, b_1^n z^n$, we have

$$\begin{aligned}
\delta_{b_1 b_2}^k \left(\sum_{n=0}^{\infty} S_n(a_1+a_2)\, b_1^n z^n \right) &= \frac{\sum\limits_{n=0}^{\infty} S_n(a_1+a_2)\, b_1^n b_1^k z^n - \sum\limits_{n=0}^{\infty} S_n(a_1+a_2)\, b_2^n b_2^k z^n}{b_1-b_2} \\
&= \sum_{n=0}^{\infty} S_n(a_1+a_2) \left(\frac{b_1^{n+k} - b_2^{n+k}}{b_1-b_2} \right) z^n \\
&= \sum_{n=0}^{\infty} S_n(a_1+a_2)\, S_{n+k-1}(b_1+b_2)\, z^n.
\end{aligned}$$

On the other hand, since

$$f(b_1 z) = \frac{1}{\prod\limits_{a\in A}(1-ab_1 z)},$$

we have

$$\begin{aligned}
\delta_{b_1 b_2}^k \left(\frac{1}{\prod\limits_{a\in A}(1-ab_1 z)} \right) &= \frac{\frac{b_1^k}{\prod\limits_{a\in A}(1-ab_1 z)} - \frac{b_2^k}{\prod\limits_{a\in A}(1-ab_2 z)}}{b_1-b_2} \\
&= \frac{b_1^k \prod\limits_{a\in A}(1-ab_2 z) - b_2^k \prod\limits_{a\in A}(1-ab_1 z)}{(b_1-b_2)\prod\limits_{a\in A}(1-ab_1 z)\prod\limits_{a\in A}(1-ab_2 z)} \\
&= \frac{b_1^k - b_2^k - (a_1+a_2)\left(b_1^k b_2 - b_2^k b_1\right) z - a_1 a_2 \left(b_2^k b_1^2 - b_1^k b_2^2\right) z^2}{(b_1-b_2)\prod\limits_{a\in A}(1-ab_1 z)\prod\limits_{a\in A}(1-ab_2 z)}.
\end{aligned}$$

This completes the proof. ∎

Theorem 2.2. Let A, P and C be three alphabets, respectively, $\{a_1, a_2\}$, $\{p_1, p_2\}$ and $\{c_1, c_2\}$, then we have

$$\sum_{n=0}^{\infty} S_n(A)\, S_{n+k-1}(P)\, S_{n+k-1}(C)\, z^n =$$

$$\frac{p_1^k p_2^k \left(\begin{array}{c} \left(\sum_{n=0}^{\infty} S_n(-A)\, p_2^n c_1^n z^n\right)\left(\sum_{n=0}^{\infty} S_n(-A)\, p_1^n c_1^n z^n\right)\left(\sum_{n=0}^{\infty} S_n(-A)\, S_{n-k-1}(P)\, c_2^{n+k} z^n\right) \\ -\left(\sum_{n=0}^{\infty} S_n(-A)\, p_2^n c_2^n z^n\right)\left(\sum_{n=0}^{\infty} S_n(-A)\, p_1^n c_2^n z^n\right)\left(\sum_{n=0}^{\infty} S_n(-A)\, S_{n-k-1}(P)\, c_1^{n+k} z^n\right) \end{array} \right)}{\left(\begin{array}{c} (c_1-c_2)\prod_{a\in A}(1-ap_1c_1z)\prod_{a\in A}(1-ap_2c_1z)\times \\ \prod_{a\in A}(1-ap_1c_2z)\prod_{a\in A}(1-ap_2c_2z) \end{array} \right)},$$

for all $k \in \mathbb{N}_0$ and $k < n$.

Proof. By applying the operator $\delta_{c_1c_2}^k \delta_{p_1p_2}^k$ to the series $f(p_1c_1z) = \sum_{n=0}^{\infty} S_n(A)\, p_1^n c_1^n z^n$, we have

$$\begin{aligned}
\delta_{c_1c_2}^k \delta_{p_1p_2}^k \left(\sum_{n=0}^{\infty} S_n(A)\, p_1^n c_1^n z^n\right) &= \delta_{c_1c_2}^k \left(\frac{p_1^k \sum_{n=0}^{\infty} S_n(A)\, p_1^n c_1^n z^n - p_2^k \sum_{n=0}^{\infty} S_n(A)\, p_2^n c_1^n z^n}{p_1 - p_2} \right) \\
&= \delta_{c_1c_2}^k \left(\sum_{n=0}^{\infty} S_n(A)\, \frac{p_1^{n+k} - p_2^{n+k}}{p_1 - p_2} c_1^n z^n \right) \\
&= \delta_{c_1c_2}^k \left(\sum_{n=0}^{\infty} S_n(A)\, S_{n+k-1}(P)\, c_1^n z^n \right) \\
&= \frac{c_1^k \sum_{n=0}^{\infty} S_n(A)\, S_{n+k-1}(P)\, c_1^n z^n - c_2^k \sum_{n=0}^{\infty} S_n(A)\, S_{n+k-1}(P)\, c_2^n z^n}{c_1 - c_2} \\
&= \sum_{n=0}^{\infty} S_n(A)\, S_{n+k-1}(P) \left(\frac{c_1^{n+k} - c_2^{n+k}}{c_1 - c_2} \right) z^n \\
&= \sum_{n=0}^{\infty} S_n(A)\, S_{n+k-1}(P)\, S_{n+k-1}(C)\, z^n.
\end{aligned}$$

On the other hand, since

$$f(p_1c_1z) = \frac{1}{\prod_{a\in A}(1-ap_1c_1z)},$$

we have

$$\begin{aligned}
\delta_{c_1c_2}^k \delta_{p_1p_2}^k \left(\frac{1}{\prod_{a\in A}(1-ap_1c_1z)} \right) &= \delta_{c_1c_2}^k \left(\frac{\frac{p_1^k}{\prod_{a\in A}(1-ap_1c_1z)} - \frac{p_2^k}{\prod_{a\in A}(1-ap_2c_1z)}}{p_1 - p_2} \right) \\
&= \delta_{c_1c_2}^k \left(\frac{p_1^k \prod_{a\in A}(1-ap_2c_1z) - p_2^k \prod_{a\in A}(1-ap_1c_1z)}{(p_1-p_2)\prod_{a\in A}(1-ap_1c_1z)\prod_{a\in A}(1-ap_2c_1z)} \right)
\end{aligned}$$

Using the fact that: $\sum_{n=0}^{\infty} S_n(-A)\, p_1^n c_1^n z^n = \prod_{a\in A}(1-ap_1c_1z)$, then

$$\begin{aligned}
\delta_{c_1c_2}^k \delta_{p_1p_2}^k \left(\frac{1}{\prod_{a\in A}(1-ap_1c_1z)}\right) &= \delta_{c_1c_2}^k \left(\frac{\sum_{n=0}^{\infty} S_n(-A)\, p_1^k p_2^n c_1^n z^n - \sum_{n=0}^{\infty} S_n(-A)\, p_2^k p_1^n c_1^n z^n}{(p_1-p_2)\prod_{a\in A}(1-ap_1c_1z)\prod_{a\in A}(1-ap_2c_1z)}\right) \\
&= \delta_{c_1c_2}^k \left(\frac{-p_1^k p_2^k \sum_{n=0}^{\infty} S_n(-A)\left(\frac{p_1^{n-k}-p_2^{n-k}}{p_1-p_2}\right) c_1^n z^n}{\prod_{a\in A}(1-ap_1c_1z)\prod_{a\in A}(1-ap_2c_1z)}\right) \\
&= \delta_{c_1c_2}^k \left(\frac{-p_1^k p_2^k \sum_{n=0}^{\infty} S_n(-A)\, S_{n-k-1}(P)\, c_1^n z^n}{\prod_{a\in A}(1-ap_1c_1z)\prod_{a\in A}(1-ap_2c_1z)}\right)
\end{aligned}$$

$$= \frac{1}{c_1-c_2}\left(\frac{-c_1^k p_1^k p_2^k \sum_{n=0}^{\infty} S_n(-A)\, S_{n-k-1}(P)\, c_1^n z^n}{\prod_{a\in A}(1-ap_1c_1z)\prod_{a\in A}(1-ap_2c_1z)} + \frac{c_2^k p_1^k p_2^k \sum_{n=0}^{\infty} S_n(-A)\, S_{n-k-1}(P)\, c_2^n z^n}{\prod_{a\in A}(1-ap_1c_2z)\prod_{a\in A}(1-ap_2c_2z)}\right)$$

$$= \frac{p_1^k p_2^k \left(\begin{array}{c} \left(\sum_{n=0}^{\infty} S_n(-A)\, p_1^n c_1^n z^n\right)\left(\sum_{n=0}^{\infty} S_n(-A)\, p_2^n c_1^n z^n\right)\left(\sum_{n=0}^{\infty} S_n(-A)\, S_{n-k-1}(P)\, c_2^{n+k} z^n\right) \\ -\left(\sum_{n=0}^{\infty} S_n(-A)\, p_1^n c_2^n z^n\right)\left(\sum_{n=0}^{\infty} S_n(-A)\, p_2^n c_2^n z^n\right)\left(\sum_{n=0}^{\infty} S_n(-A)\, S_{n-k-1}(P)\, c_1^{n+k} z^n\right) \end{array}\right)}{\left(\begin{array}{c} (c_1-c_2)\prod_{a\in A}(1-ap_1c_1z)\prod_{a\in A}(1-ap_2c_1z)\times \\ \prod_{a\in A}(1-ap_1c_2z)\prod_{a\in A}(1-ap_2c_2z) \end{array}\right)}.$$

This completes the proof. ∎

3 On the generating functions of some numbers

In this part, we consider theorem 1 in order to derive a generating functions of some known numbers if $k = 0, 1$.

Lemma 3.1. ([2]) Given an alphabet $A = \{a_1, a_2\}$, we have

$$\sum_{n=0}^{\infty} S_n(a_1+a_2)\, z^n = \frac{1}{(1-a_1z)(1-a_2z)}. \tag{3.1}$$

From(3.1) we can deduce

$$\sum_{n=0}^{\infty} S_{n-1}(a_1+a_2)\, z^n = \frac{z}{(1-a_1z)(1-a_2z)}. \tag{3.2}$$

By replacing a_2 by $(-a_2)$ in (3.1) and (3.2), we obtain

$$\sum_{n=0}^{\infty} S_n\left(a_1+[-a_2]\right) z^n = \frac{1}{\left(1-a_1 z\right)\left(1+a_2 z\right)}, \tag{3.3}$$

$$\sum_{n=0}^{\infty} S_{n-1}\left(a_1+[-a_2]\right) z^n = \frac{z}{\left(1-a_1 z\right)\left(1+a_2 z\right)}, \tag{3.4}$$

respectively.

Choosing a_1 and a_2 such that $\begin{cases} a_1 a_2 = 2 \\ a_1 - a_2 = 1 \end{cases}$ and substituting in (3.3) and (3.4), we get

$$\sum_{n=0}^{\infty} S_n\left(a_1+[-a_2]\right) z^n = \frac{1}{1-z-2z^2}, \tag{3.5}$$

$$\sum_{n=0}^{\infty} S_{n-1}\left(a_1+[-a_2]\right) z^n = \frac{z}{1-z-2z^2} = \sum_{n=0}^{\infty} J_n z^n, \tag{3.6}$$

which represents a generating function for Jacobsthal numbers.

- Multiplying the equation (3.5) by 2 and added to (3.6) by (-1), we obtain

$$\begin{aligned} \sum_{n=0}^{\infty}\left(2S_n\left(a_1+[-a_2]\right)-S_{n-1}\left(a_1+[-a_2]\right)\right) z^n &= \frac{2-z}{1-z-2z^2} \\ &= \sum_{n=0}^{\infty} j_n z^n, \end{aligned}$$

which represents a generating function for Jacobsthal-Lucas numbers.

In this part, we now derive the generating functions of the products of some known numbers. Indeed, we consider theorem 1 in order to derive of binary products of some special numbers if $k = 0, 1$.

Theorem 3.2. (Theorem1 ([12])) Given two alphabets $A = \{a_1, a_2\}$ and $B = \{b_1, b_2\}$ we have

$$\sum_{n=0}^{\infty} S_n\left(a_1+a_2\right) S_{n-1}\left(b_1+b_2\right) z^n = \frac{\left(a_1+a_2\right) z - a_1 a_2\left(b_1+b_2\right) z^2}{\prod\limits_{a\in A}\left(1-ab_1 z\right)\prod\limits_{a\in A}\left(1-ab_2 z\right)}. \tag{3.7}$$

Theorem 3.3. (Theorem4 ([5])) Given two alphabets $A = \{a_1, a_2\}$ and $B = \{b_1, b_2\}$ we have

$$\sum_{n=0}^{\infty} S_n\left(a_1+a_2\right) S_n\left(b_1+b_2\right) z^n = \frac{1-a_1 a_2 b_1 b_2 z^2}{\prod\limits_{a\in A}\left(1-ab_1 z\right)\prod\limits_{a\in A}\left(1-ab_2 z\right)}. \tag{3.8}$$

From (3.8) we deduce the following proposition.

Proposition 3.4. Given two alphabets $A = \{a_1, a_2\}$ and $B = \{b_1, b_2\}$ we have

$$\sum_{n=0}^{\infty} S_{n-1}(a_1 + a_2) S_{n-1}(b_1 + b_2) z^n = \frac{z - a_1 a_2 b_1 b_2 z^3}{\prod\limits_{a \in A}(1 - a b_1 z) \prod\limits_{a \in A}(1 - a b_2 z)}. \tag{3.9}$$

Proof. We multiply both sides of the equation (3.8) by z , we get

$$\sum_{n=0}^{\infty} S_n(a_1 + a_2) S_n(b_1 + b_2) z^{n+1} = \frac{z - a_1 a_2 b_1 b_2 z^3}{\prod\limits_{a \in A}(1 - a b_1 z) \prod\limits_{a \in A}(1 - a b_2 z)},$$

therefore

$$\sum_{n=1}^{\infty} S_{n-1}(a_1 + a_2) S_{n-1}(b_1 + b_2) z^n = \frac{z - a_1 a_2 b_1 b_2 z^3}{\prod\limits_{a \in A}(1 - a b_1 z) \prod\limits_{a \in A}(1 - a b_2 z)},$$

then

$$\sum_{n=0}^{\infty} S_{n-1}(a_1 + a_2) S_{n-1}(b_1 + b_2) z^n = \frac{z - a_1 a_2 b_1 b_2 z^3}{\prod\limits_{a \in A}(1 - a b_1 z) \prod\limits_{a \in A}(1 - a b_2 z)}.$$

This completes the proof. ∎

Replacing a_2 by $(-a_2)$ and b_2 by $(-b_2)$ in (3.7), (3.8) and (3.9), we obtain

$$\sum_{n=0}^{\infty} S_n(a_1 + [-a_2]) S_{n-1}(b_1 + [-b_2]) z^n = \frac{(a_1 - a_2) z + a_1 a_2 (b_1 - b_2) z^2}{(1 - a_1 b_1 z)(1 + a_1 b_2 z)(1 + a_2 b_1 z)(1 - a_2 b_2 z)}, \tag{3.10}$$

$$\sum_{n=0}^{\infty} S_n(a_1 + [-a_2]) S_n(b_1 + [-b_2]) z^n = \frac{1 - a_1 a_2 b_1 b_2 z^2}{(1 - a_1 b_1 z)(1 + a_1 b_2 z)(1 + a_2 b_1 z)(1 - a_2 b_2 z)}, \tag{3.11}$$

$$\sum_{n=0}^{\infty} S_{n-1}(a_1 + [-a_2]) S_{n-1}(b_1 + [-b_2]) z^n = \frac{z - a_1 a_2 b_1 b_2 z^3}{(1 - a_1 b_1 z)(1 + a_1 b_2 z)(1 + a_2 b_1 z)(1 - a_2 b_2 z)}, \tag{3.12}$$

respectively.

This case consists of five related parts.

Case 3.5. The substitutions of $\begin{cases} a_1 - a_2 = 1 \\ a_1 a_2 = 2 \end{cases}$ and $\begin{cases} b_1 - b_2 = 1 \\ b_1 b_2 = 2 \end{cases}$ in (3.10), (3.11) and (3.12), we get

$$\sum_{n=0}^{\infty} S_n(a_1 + [-a_2]) S_{n-1}(b_1 + [-b_2]) z^n = \frac{z + 2z^2}{1 - z - 12z^2 - 4z^3 + 16z^4}, \tag{3.13}$$

$$\sum_{n=0}^{\infty} S_n\left(a_1+[-a_2]\right) S_n\left(b_1+[-b_2]\right) z^n = \frac{1-4z^2}{1-z-12z^2-4z^3+16z^4}, \tag{3.14}$$

$$\sum_{n=0}^{\infty} S_{n-1}\left(a_1+[-a_2]\right) S_{n-1}\left(b_1+[-b_2]\right) z^n = \frac{z-4z^3}{1-z-12z^2-4z^3+16z^4}, \tag{3.15}$$

which represents a new generating functions respectively.

- Multiplying the equation (3.13) by 2 and added to (3.15) by (-1), we obtain the following proposition.

Proposition 3.6. ([12]) For $n \in \mathbb{N}$, the generating function of the product of Jacobsthal numbers and Jacobsthal-Lucas numbers is given by

$$\sum_{n=0}^{\infty} J_n j_n z^n = \frac{z}{1-5z+4z^2}, \tag{3.16}$$

We can state the following corollary.

Corollary 3.7. The following identity holds true:

$$J_n j_n = S_{n-1}\left(b_1+[-b_2]\right)\left(2S_n\left(a_1+[-a_2]\right)-S_{n-1}\left(a_1+[-a_2]\right)\right).$$

- Multiplying the equation (3.13) by 4 and added to (3.14), we obtain

$$\sum_{n=0}^{\infty} S_n\left(a_1+[-a_2]\right)\left(4S_{n-1}\left(b_1+[-b_2]\right)+S_n\left(b_1+[-b_2]\right)\right) z^n = \frac{1}{1-5z+4z^2}, \tag{3.17}$$

which represent a new generating function.

- Multiplying the equation (3.16) by 5 and subtract it from (3.17) by 2, we have the following proposition ([9]).

Proposition 3.8. For $n \in \mathbb{N}$, The generating function of even indices of Jacobsthal-Lucas numbers is given by

$$\sum_{n=0}^{\infty} j_{2n} z^n = \frac{2-5z}{1-5z+4z^2}. \tag{3.18}$$

We have the following corollary.

Corollary 3.9. The following identity holds true:

$$\begin{aligned} j_{2n} &= 2S_n\left(a_1+[-a_2]\right) S_n\left(b_1+[-b_2]\right) - 2S_n\left(a_1+[-a_2]\right) S_{n-1}\left(b_1+[-b_2]\right) \\ &\quad +5S_{n-1}\left(a_1+[-a_2]\right) S_{n-1}(b_1+[-b_2]). \end{aligned}$$

- Multiplying the equation (3.16) by 2 and subtract it from (3.17), we have the following proposition.

Proposition 3.10. ([9]) For $n \in \mathbb{N}$, the generating function of odd indices of Jacobsthal numbers is given by

$$\sum_{n=0}^{\infty} J_{2n+1} z^n = \frac{1-2z}{1-5z+4z^2}. \tag{3.19}$$

By comparing the coefficients z^n on both sides of (3.19), we have the following corollary.

Corollary 3.11. The following identity holds true:

$$J_{2n+1} = S_n\left(a_1+[-a_2]\right) S_n\left(b_1+[-b_2]\right) + 2S_{n-1}\left(a_1+[-a_2]\right) S_{n-1}\left(b_1+[-b_2]\right).$$

- Multiplying the equation (3.16) by 2 and added it from (3.17), we have the following proposition ([9]).

Proposition 3.12. For $n \in \mathbb{N}$, the generating function of odd indices of Jacobsthal-Lucas numbers is given by

$$\sum_{n=0}^{\infty} j_{2n+1} z^n = \frac{1+2z}{1-5z+4z^2}. \tag{3.20}$$

Thus we get the following both corollary.

Corollary 3.13. The following identity holds true:

$$\begin{aligned} j_{2n+1} &= S_n\left(a_1+[-a_2]\right) S_n\left(b_1+[-b_2]\right) + 8S_n\left(a_1+[-a_2]\right) S_{n-1}\left(b_1+[-b_2]\right) \\ &\quad -2S_{n-1}\left(a_1+[-a_2]\right) S_{n-1}\left(b_1+[-b_2]\right). \end{aligned}$$

Case 3.14. The substitutions of $\begin{cases} a_1-a_2=1 \\ a_1a_2=2 \end{cases}$ and $\begin{cases} b_1-b_2=1 \\ b_1b_2=1 \end{cases}$ in (3.10) and (3.12), we obtain

$$\sum_{n=0}^{\infty} S_n\left(a_1+[-a_2]\right) S_{n-1}\left(b_1+[-b_2]\right) z^n = \frac{z+2z^2}{1-z-7z^2-2z^3+4z^4}, \tag{3.21}$$

$$\sum_{n=0}^{\infty} S_{n-1}\left(a_1+[-a_2]\right) S_{n-1}\left(b_1+[-b_2]\right) z^n = \frac{z-2z^3}{1-z-7z^2-2z^3+4z^4}, \tag{3.22}$$

which represents a new generating functions respectively.

- Multiplying the equation (3.21) by (2) and added to (3.22) by (-1), we have the following proposition.

Proposition 3.15. ([9]) For $n \in \mathbb{N}$, The generating function for the products of Jacobsthal-Lucas numbers and Fibonacci numbers is given by

$$\sum_{n=0}^{\infty} F_n j_n z^n = \frac{z + 4z^2 + 2z^3}{1 - z - 7z^2 - 2z^3 + 4z^4},$$

such that $F_n j_n = S_{n-1}(b_1 + [-b_2])(2S_n(a_1 + [-a_2]) - S_{n-1}(a_1 + [-a_2]))$.

Case 3.16. The substitutions of $\begin{cases} a_1 - a_2 = 2, \\ a_1 a_2 = 1, \\ b_1 - b_2 = 1, \\ b_1 b_2 = 2, \end{cases}$ and $\begin{cases} a_1 - a_2 = 1, \\ a_1 a_2 = 1, \\ b_1 - b_2 = 1, \\ b_1 b_2 = 2, \end{cases}$ in (3.11) and (3.12), we deduce the following propositions and corollaries.

Proposition 3.17. ([9]) For $n \in \mathbb{N}$, the generating function for the products of Jacobsthal numbers and Pell numbers is given by

$$\sum_{n=0}^{\infty} P_n J_n = \frac{z\left(1 - 2z^2\right)}{(1 + 2z - z^2)(1 - 4z - 4z^2)}.$$

Corollary 3.18. The following identity holds true:

$$P_n J_n = S_{n-1}(a_1 + [-a_2]) S_{n-1}(b_1 + [-b_2]).$$

Proposition 3.19. ([12]) For $n \in \mathbb{N}$, the generating function for the products of Jacobsthal numbers and Fibonacci numbers is given by

$$\sum_{n=0}^{\infty} F_n J_n z^n = \frac{1 - 2z^2}{(1 + z - z^2)(1 - 2z - 4z^2)}. \tag{3.25}$$

Corollary 3.20. The following identity holds true:

$$F_n J_n = S_n(a_1 + [-a_2]) S_n(b_1 + [-b_2]).$$

Case 3.21. Taking $\begin{cases} a_1 - a_2 = 1, \\ a_1 a_2 = 1, \\ b_1 - b_2 = 1, \\ b_1 b_2 = 2, \end{cases}$ in (3.10) and (3.12), we get

$$\sum_{n=0}^{\infty} S_n(a_1 + [-a_2]) S_{n-1}(b_1 + [-b_2]) z^n = \frac{z(1 + z)}{(1 + z - z^2)(1 - 2z - 4z^2)}, \tag{3.26}$$

$$\sum_{n=0}^{\infty} S_{n-1}(a_1 + [-a_2]) S_{n-1}(b_1 + [-b_2]) z^n = \frac{z\left(1 - 2z^2\right)}{(1 + z - z^2)(1 - 2z - 4z^2)}, \tag{3.27}$$

which represents a new generating functions respectively.

- Multiplying the equation (3.26) by (2) and added to (3.27) by (-1), we have the following proposition and corollary.

Proposition 3.22. ([9]) For $n \in \mathbb{N}$, the generating function for the products of Jacobsthal numbers with Lucas numbers is given by

$$\sum_{n=0}^{\infty} J_n L_n z^n = \frac{z\left(1+2z+2z^2\right)}{\left(1+z-z^2\right)\left(1-2z-4z^2\right)}. \tag{3.28}$$

Corollary 3.23. The following identity holds true:

$$J_n L_n = S_{n-1}\left(b_1+[-b_2]\right)\left(2S_n\left(a_1+[-a_2]\right)-S_{n-1}\left(a_1+[-a_2]\right)\right).$$

Case 3.24. Assuming that $\begin{cases} a_1-a_2=1, \\ b_1-b_2=2, \\ a_1a_2=2, \\ b_1b_2=1, \end{cases}$ in (3.10) and (3.12), we get

$$\sum_{n=0}^{\infty} S_n\left(a_1+[-a_2]\right) S_{n-1}\left(b_1+[-b_2]\right) z^n = \frac{z\left(1+4z\right)}{\left(1+2z-z^2\right)\left(1-4z-4z^2\right)}, \tag{3.29}$$

$$\sum_{n=0}^{\infty} S_{n-1}\left(a_1+[-a_2]\right) S_{n-1}\left(b_1+[-b_2]\right) z^n = \frac{z\left(1-2z^2\right)}{\left(1+2z-z^2\right)\left(1-4z-4z^2\right)}, \tag{3.30}$$

respectively.

- Multiplying the equation(2.30) by (-1) and added to (3.29) by 2, we obtain the following proposition and corollary.

Proposition 3.25. ([9]) For $n \in \mathbb{N}$, the generating function for the products of Jacobsthal-Lucas numbers and Pell numbers is given by

$$\sum_{n=0}^{\infty} j_n P_n z^n = \frac{z\left(1+8z+2z^2\right)}{\left(1+2z-z^2\right)\left(1-4z-4z^2\right)}. \tag{3.31}$$

Corollary 3.26. The following identity holds true:

$$j_n P_n = S_{n-1}\left(b_1+[-b_2]\right)\left(2S_n\left(a_1+[-a_2]\right)-S_{n-1}\left(a_1+[-a_2]\right)\right).$$

4 On the generating functions of powers of some numbers

In this part , we consider theorem 2 in order to derive a new generating function for the products of Fibonacci numbers, Jacobsthal numbers and Pell numbers if $k=1$.

Theorem 4.1. Given three alphabets $A=\{a_1,a_2\}, P=\{p_1,p_2\}$ and $C=\{c_1,c_2\}$ we have

$$\sum_{n=0}^{\infty} S_n\left(a_1+a_2\right) S_n\left(p_1+p_2\right) S_n\left(c_1+c_2\right) z^n = \frac{N_2}{D},\ n \in \mathbb{N}. \tag{4.1}$$

with

$$\begin{aligned} N_2 &= 1-(a_1a_2c_1c_2\,(p_1+p_2)^2+a_1a_2p_1p_2\,(c_1+c_2)^2+p_1p_2c_1c_2\,(a_1+a_2)^2-3p_1p_2a_1a_2c_1c_2)z^2\\ &\quad +2p_1p_2a_1a_2c_1c_2\,(a_1+a_2)\,(c_1+c_2)\,(p_1+p_2)\,z^3-(p_1p_2a_1^2a_2^2c_1^2c_2^2\,(p_1+p_2)^2+\\ &\quad c_1c_2a_1^2a_2^2p_1^2p_2^2\,(c_1+c_2)^2+a_1a_2c_1^2c_2^2p_1^2p_2^2\,(a_1+a_2)^2-3a_1^2a_2^2p_1^2p_2^2c_1^2c_2^2)z^4\\ &\quad +a_1^3a_2^3c_1^3c_2^3p_1^3p_2^3z^6. \end{aligned}$$

$$\begin{aligned} D &= 1-(a_1+a_2)\,(c_1+c_2)\,(p_1+p_2)\;z+(\;p_1p_2\,(a_1+a_2)^2\,(c_1+c_2)^2+((p_1+p_2)^2-2p_1p_2)\\ &\quad ((a_1+a_2)^2\,c_1c_2-2a_1a_2c_1c_2+a_1a_2\,(c_1+c_2)^2))\;z^2-(a_1+a_2)\,(c_1+c_2)\,(p_1+p_2)\\ &\quad (\;p_1p_2c_1c_2\,(a_1+a_2)^2+p_1p_2a_1a_2\,(c_1+c_2)^2+a_1a_2c_1c_2\,(p_1+p_2)^2-5p_1p_2a_1a_2c_1c_2)z^3\\ &\quad +(a_1^2a_2^2c_1^2c_2^2\,(p_1+p_2)^4+c_1^2c_2^2p_1^2p_2^2\,(a_1+a_2)^4+a_1^2a_2^2p_1^2p_2^2\,(c_1+c_2)^4-a_1a_2c_1c_2p_1p_2\\ &\quad (4c_1c_2p_1p_2\,(a_1+a_2)^2+4a_1a_2c_1c_2\,(p_1+p_2)^2+4a_1a_2p_1p_2\,(c_1+c_2)^2-(a_1+a_2)^2\,(p_1+p_2)^2\\ &\quad (c_1+c_2)^2)+6a_1^2a_2^2c_1^2c_2^2p_1^2p_2^2)\;z^4-a_1a_2c_1c_2p_1p_2\,(a_1+a_2)\,(c_1+c_2)\,(p_1+p_2)\,(a_1a_2c_1c_2\,(p_1+p_2)^2\\ &\quad +c_1c_2p_1p_2\,(a_1+a_2)^2+p_1p_2a_1a_2\,(c_1+c_2)^2-5a_1a_2c_1c_2p_1p_2)z^5+(a_1^2a_2^2c_1^2c_2^2p_1^3p_2^3\,(a_1+a_2)^2\\ &\quad (c_1+c_2)^2+a_1^2a_2^2c_1^2c_2^2p_1^2p_2^2((p_1+p_2)^2-2p_1p_2)((a_1+a_2)^2\,c_1c_2-2a_1a_2c_1c_2+a_1a_2\,(c_1+c_2)^2))\;z^6\\ &\quad -p_1^3p_2^3a_1^3a_2^3c_1^3c_2^3\,(a_1+a_2)\,(c_1+c_2)\,(p_1+p_2)\;z^7+p_1^4p_2^4a_1^4a_2^4c_1^4c_2^4z^8. \end{aligned}$$

From Theorem 5 we can deduce

$$\sum_{n=0}^{\infty}S_{n-1}\,(a_1+a_2)\,S_{n-1}\,(p_1+p_2)\,S_{n-1}\,(c_1+c_2)\,z^n=\frac{N_3}{D}. \tag{4.2}$$

with

$$\begin{aligned} N_3 &= z-(a_1a_2c_1c_2\,(p_1+p_2)^2+a_1a_2p_1p_2\,(c_1+c_2)^2+p_1p_2c_1c_2\,(a_1+a_2)^2-3p_1p_2a_1a_2c_1c_2)z^3\\ &\quad +2p_1p_2a_1a_2c_1c_2\,(a_1+a_2)\,(c_1+c_2)\,(p_1+p_2)\,z^4-(p_1p_2a_1^2a_2^2c_1^2c_2^2\,(p_1+p_2)^2+\\ &\quad c_1c_2a_1^2a_2^2p_1^2p_2^2\,(c_1+c_2)^2+a_1a_2c_1^2c_2^2p_1^2p_2^2\,(a_1+a_2)^2-3a_1^2a_2^2p_1^2p_2^2c_1^2c_2^2)z^5\\ &\quad +a_1^3a_2^3c_1^3c_2^3p_1^3p_2^3z^7. \end{aligned}$$

- By replacing a_2 by $(-a_2)$, c_2 by $(-c_2)$ and p_2 by $(-p_2)$ in (4.2) we obtain the following table:

	p_1–p_2	c_1–c_2	a_1a_2	p_1p_2	c_1c_2	Coefficient of z^n	Symmetric Functions
1	1	1	1	1	1	F_n^3	$S_{n-1}\,(a_1+[-a_2])\,S_{n-1}\,(c_1+[-c_2])\,S_{n-1}\,(p_1+[-p_2])$
2	2	2	1	1	1	P_n^3	$S_{n-1}\,(a_1+[-a_2])\,S_{n-1}\,(c_1+[-c_2])\,S_{n-1}\,(p_1+[-p_2])$
1	1	1	2	2	2	J_n^3	$S_{n-1}\,(a_1+[-a_2])\,S_{n-1}\,(c_1+[-c_2])\,S_{n-1}\,(p_1+[-p_2])$
1	1	2	1	1	1	$F_n^2P_n$	$S_{n-1}\,(a_1+[-a_2])\,S_{n-1}\,(c_1+[-c_2])\,S_{n-1}\,(p_1+[-p_2])$
1	1	1	1	1	2	$F_n^2J_n$	$S_{n-1}\,(a_1+[-a_2])\,S_{n-1}\,(c_1+[-c_2])\,S_{n-1}\,(p_1+[-p_2])$
1	2	2	1	1	1	$P_n^2F_n$	$S_{n-1}\,(a_1+[-a_2])\,S_{n-1}\,(c_1+[-c_2])\,S_{n-1}\,(p_1+[-p_2])$
1	1	1	1	2	2	$J_n^2F_n$	$S_{n-1}\,(a_1+[-a_2])\,S_{n-1}\,(c_1+[-c_2])\,S_{n-1}\,(p_1+[-p_2])$
2	2	1	1	1	2	$P_n^2J_n$	$S_{n-1}\,(a_1+[-a_2])\,S_{n-1}\,(c_1+[-c_2])\,S_{n-1}\,(p_1+[-p_2])$
1	2	2	2	1	1	$J_n^2P_n$	$S_{n-1}\,(a_1+[-a_2])\,S_{n-1}\,(c_1+[-c_2])\,S_{n-1}\,(p_1+[-p_2])$

Table 4.1. The symmetric functions for some sequences of number.

5 Conclusion

In this paper, a new theorems has been proposed in order to determine the generating functions. The proposed theorems is based on the symmetric functions.The obtained results agree with the results obtained in some previous works.

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References

[1] A. Boussayoud, A. Abderrezzak, P.B. Zhang, *Symmetric functions for families of generating functions*, Neural, Parallel and Scientific Computations., 26 (2018), 53-64.

[2] A. Boussayoud, M. Kerada, N. Harrouche, *On the k-Lucas numbers and Lucas Polynomials*, Turkish Journal of Analysis and Number., 5(3) (2017), 121-125.

[3] A. Boussayoud, *On some identities and generating functions for Pell-Lucas numbers*, Online.J. Anal. Comb. 12 (2017), 1-10.

[4] A. Boussayoud, A. Abderrezzak, M. Kerada, *Some applications of symmetric functions*, Integers., 15, A#48 (2015) 1–7.

[5] A. Boussayoud, M. Kerada, *Symmetric and Generating Functions*, Int. Electron. J. Pure Appl. Math., **7** (2014) 195–203.

[6] A. Boussayoud, M. Kerada, R. Sahali, W. Rouibah, *Some Applications on Generating Functions*, J. Concr. Appl. Math., 12 (2014) 321–330.

[7] C. Cook , M. Bacon, *Some identities for Jacobsthal and Jacobsthal-Lucas numbers satisfying higher order recurrence relations*, Ann. Math. Inform., 41 (2013), 27-39.

[8] S. Falcón, Á. Plaza, *On the Fibonacci k*-numbers, Chaos Solitons Fractals., 32(5) (2007), 1615-1624.

[9] I. Mezo, *Several Generating Functions for Second-Order Recurrence Sequences*, J. Integer Seq., 12 (2009), 1-16.

[10] D. Marques, *The Order of Appearance of the Product of Consecutive Lucas Numbers*, Fibonacci Q., 51 (1) (2013), 38-43

[11] M. Shattuk, *Combinatorial Proofs of Determinant Formulas for the Fibonacci and Lucas Polynomials*, *Fibonacci Q.*, 51 (1) (2013), 63-71.

[12] K. H. Boubellouta, A. Boussayoud, M. Kerada, *Symmetric Functions for k-Fibonacci Numbers and Orthogonal Polynomials*, Turkish Journal of Analysis and Number Theory., 6 (3) (2018), 98-102.