Growth estimates of derivatives of a polynomial

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Abstract

This paper deals with the problem of finding an upper bound for the maximal modulus of the derivative and higher order derivatives of a complex polynomial on a disk under the assumption that the polynomial has no zeros in another disk. The estimates involve the maximal modulus of the polynomial in some disk, the degree, the coefficients and the radii of the disks.

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1 Introduction and statement of results

Let $P(z)$ be a polynomial of degree n and $P^{(s)}(z)$ is its s^{th} derivative. If $M(P,r) := \max_{|z|=r}|P(z)|$, then

$$M(P',1) \leq nM(P,1), \tag{1.1}$$

and

$$M(P,\rho) \leq \rho^n M(P,1), \quad \rho > 1. \tag{1.2}$$

Inequality (1.1) is a well known result of S. Bernstein (for reference see [12]), whereas inequality (1.2) is a simple deduction from the maximum modulus principle, (see [11]). If we restrict ourselves to the class of polynomials having no zeros in $|z| < 1$, then

$$M(P',1) \leq \frac{n}{2}M(P,1), \tag{1.3}$$

and

$$M(P,\rho) \leq \frac{\rho^n + 1}{2}M(P,1), \quad \rho > 1. \tag{1.4}$$

Inequality (1.3) was conjecture by Erdös and later proved by Lax [9], whereas inequality (1.4) is due to Ankeny and Rivlin [1]. As an extension of (1.3), Malik [10] proved that if $P(z)$ is a polynomial of degree n such that $P(z) \neq 0$, in $|z| < k$, $k \geq 1$, then

$$M(P',1) \leq \frac{n}{1+k}M(P,1). \tag{1.5}$$

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As a generalization of (1.5), Dewan and Bidkham [3] proved that if $P(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial of degree n having no zeros in $|z| < k$, $k \geq 1$, then for $1 \leq \rho \leq k$,

$$M(P', \rho) \leq \frac{n(\rho + k)^{n-1}}{(1+k)^n} M(P, 1). \tag{1.6}$$

As a generalization and refinement of (1.6), Dewan and Mir [4] proved under the same hypothesis for $0 \leq r \leq \rho \leq k$, that

$$M(P', \rho) \leq \frac{n(\rho + k)^{n-1}}{(k + r)^n} \left\{ 1 - \frac{k(k-\rho)(n|a_0| - k|a_1|)n}{(k^2 + \rho^2)n|a_0| + 2k^2\rho|a_1|} \left(\frac{\rho - r}{k + \rho} \right) \left(\frac{k + r}{k + \rho} \right)^{n-1} \right\} M(P, r). \tag{1.7}$$

The above inequalities (1.4) and (1.6) were further generalized to the s^{th} derivative by Jain [7], by proving the following result.

Theorem 1. If $P(z)$ is a polynomial of degree n having no zeros in $|z| < k$, $k \geq 1$, then for $0 \leq s < n$,

$$M(P^{(s)}, \rho) \leq \frac{1}{2} \left\{ \frac{d^s}{d\rho^s} (\rho^n + k^n) \right\} \left(\frac{2}{1+k} \right)^n M(P, 1), \quad \rho \geq k, \tag{1.8}$$

and

$$M(P^{(s)}, \rho) \leq \left(\frac{1}{\rho^s + k^s} \right) \left\{ \left(\frac{d^s}{dx^s} (1 + x^n) \right)_{x=1} \right\} \left(\frac{\rho + k}{1 + k} \right)^n M(P, 1), \; 1 \leq \rho \leq k. \tag{1.9}$$

Equality holds in (1.8) (with $k = 1$ and $s = 0$) for $P(z) = z^n + 1$ and in (1.9) (with $s = 1$) for $P(z) = (z + k)^n$.

In this paper, we have generalized and refined the inequalities (1.6) and (1.7) by considering polynomials not vanishing in $|z| < k$, $k > 0$ and simultaneously have taken into consideration the s^{th} derivative ($0 \leq s < n$) of the polynomial $P(z)$. Besides, our results also sharpens (1.8) and (1.9) due to Jain [7]. More precisely, we have proved the following results.

Theorem 2. Let $P(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial of degree n having no zeros in $|z| < k$, $k \geq 1$ and let $\lambda = \frac{ka_1}{na_0}$, then for $0 \leq s < n$ and $\rho \geq k$,

$$M\big(P^{(s)}, \rho\big) \leq \frac{1}{2} \left\{ \frac{d^s}{d\rho^s} (\rho^n + k^n) \right\} \left(\frac{2 + 2|\lambda|}{1 + 2|\lambda|k + k^2} \right)^{\frac{n}{2}} M(P, 1). \tag{1.10}$$

Equality holds in (1.10) (with $s = 0$ and $k = 1$) for $P(z) = z^n + 1$.

Remark 1. Since if $P(z) = \sum_{\nu=0}^{n} a_\nu z^\nu \neq 0$ in $|z| < k$, $k \geq 1$, then

by Lemma 7 (for $s = 1$),we have $|\lambda| = \left| \frac{ka_1}{na_0} \right| \leq 1$. We show that in general

$$\left(\frac{2 + 2|\lambda|}{1 + 2|\lambda|k + k^2} \right)^{\frac{n}{2}} \leq \Big(\frac{2}{1+k} \Big)^n,$$

which is equivalent to showing

$$\left(\frac{2+2|\lambda|}{1+2|\lambda|k+k^2}\right) \leq \left(\frac{2}{1+k}\right)^2,$$

that is,

$$(1+|\lambda|)(1+k)^2 \leq 2(1+2|\lambda|k+k^2),$$

which is equivalent to

$$(|\lambda|-1)(k-1)^2 \leq 0,$$

which clearly holds as $|\lambda| \leq 1$. This shows that (1.10) is a refinement of (1.8).

Theorem 3. If $P(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial of degree n having no zeros in $|z| < k$, $k > 0$, then for $0 \leq r \leq \rho \leq k$ and $0 \leq s < n$,

$$M(P^{(s)}, \rho) \leq \left(\frac{c(n,s)|a_0|\rho + |a_s|k^{s+1}}{c(n,s)|a_0|(k^{s+1}+\rho^{s+1}) + |a_s|(\rho^s + \rho k^{s-1})k^{s+1}}\right)\left\{\left(\frac{d^s}{dx^s}(1+x^n)\right)_{x=1}\right\}$$
$$\times \left(\frac{\rho+k}{r+k}\right)^n \left\{1 - \frac{k(k-\rho)(n|a_0|-k|a_1|)n}{(k^2+\rho^2)n|a_0|+2k^2\rho|a_1|}\left(\frac{\rho-r}{k+\rho}\right)\left(\frac{k+r}{k+\rho}\right)^{n-1}\right\} M(P,r). \tag{1.11}$$

Equality holds in (1.11) (with $s = 1$) for $P(z) = (z+k)^n$.

Remark 2. The above Theorem 3 is in general an improvement and extension of inequality (1.7). To see this, we first note that if $P(z) \neq 0$ in $|z| < k$, $k > 0$, then by Lemma 7, we have for $0 \leq s < n$, $\frac{1}{c(n,s)}\left|\frac{a_s}{a_0}\right| k^s \leq 1$, which can also be taken equivalent to

$$\frac{c(n,s)t^{s+1} + \left|\frac{a_s}{a_0}\right| k^{s+1}t^s}{c(n,s)(k^{s+1}+t^{s+1}) + \left|\frac{a_s}{a_0}\right|(k^{s+1}t^s + t\,k^{2s})} \leq \frac{t^s}{t^s+k^s}, \quad \text{where } 0 < t \leq k. \tag{1.12}$$

Since $\rho \leq k$, if we put $t = \rho$ in (1.12), we get

$$\frac{c(n,s)\rho|a_0| + |a_s|k^{s+1}}{c(n,s)|a_0|(k^{s+1}+\rho^{s+1}) + |a_s|(\rho^s + \rho\, k^{s-1})k^{s+1}} \leq \frac{1}{\rho^s + k^s}, \tag{1.13}$$

hence (for $s = 1$) (1.11) is an improvement of (1.6) and (1.7).

Taking $r = 1$, and using (1.13) in (1.11), we immediately obtain the following interesting result which provides a refinement of (1.9) due to Jain [7].

Corollary 1. If $P(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial of degree n having no zeros in $|z| < k$, $k \geq 1$, then for $1 \leq \rho \leq k$ and $0 \leq s < n$,

$$M(P^{(s)}, \rho) \leq \frac{1}{\rho^s + k^s} \left\{ \left(\frac{d^s}{dx^s}(1+x^n) \right)_{x=1} \right\} \left(\frac{\rho + k}{1+k} \right)^n \times \left\{ 1 - \frac{k(k-\rho)(n|a_0| - k|a_1|)n}{(k^2+\rho^2)n|a_0| + 2k^2\rho|a_1|} \left(\frac{\rho - 1}{k+\rho} \right) \left(\frac{k+1}{k+\rho} \right)^{n-1} \right\} M(P, 1) \tag{1.14}$$

Equality in (1.14) holds (with $s = 1$) for $P(z) = (z + k)^n$.
Since if $P(z) = \sum_{v=0}^{n} a_v z^v \neq 0$ in $|z| < k$, $k \geq 1$, then $\left| \frac{a_1}{a_0} \right| \leq n/k$ and hence

$$\left\{ 1 - \frac{k(k-\rho)(n|a_0| - k|a_1|)n}{(k^2+\rho^2)n|a_0| + 2k^2\rho|a_1|} \left(\frac{\rho - 1}{k+\rho} \right) \left(\frac{k+1}{k+\rho} \right)^{n-1} \right\}$$

is always non-negative, the above Corollary gives a bound that is much better than obtainable from inequality (1.9) of Theorem 1.
Although, Theorem 3 in general gives the bound sharper than obtainable from (1.9) of Theorem 1, in some cases the improvement can be considerably significant and this we show by means of the following examples.
Example 1 Consider $P(z) = z^4 + z^3 + z^2 + z + 1000$. Clearly, $n = 4$. We take $s = \rho = 2$ and $k = 5.4$, since we find numerically that $P(z) \neq 0$ in $|z| < 5.79837$. For this polynomial, we have by (1.9) of Theorem 1,

$$M(P^{(s)}, \rho) \leq 0.6468055\, M(P, 1),$$

while as Theorem 3 gives,

$$M(P^{(s)}, \rho) \leq 0.23821945\, M(P, 1).$$

Example 2. Consider $P(z) = z^6 - z^5 + z^4 + z^3 + z + 100$, which is a polynomial of degree 6. Here we take $s = 3, \rho = 1.5$ and $k = 1.9$, since we find numerically that $P(z) \neq 0$ in $|z| < 1.9878$. For this polynomial we have by (1.9) of Theorem 1,

$$M(P^{(s)}, \rho) \leq 30.4523852\, M(P, 1),$$

while as Theorem 3 gives,

$$M(P^{(s)}, \rho) \leq 24.537504\, M(P, 1).$$

2 Lemmas

For the proof of these theorems, we need the following lemmas. The following lemma is due to Govil, Qazi and Rahman [6].

Lemma 1. Let $P(z) = \sum_{\nu=0}^{n} a_\nu z^\nu \neq 0$ for $|z| < k$ where $k \geq 1$ and let $\lambda = \frac{ka_1}{na_0}$. Then for $1 \leq R \leq k^2$,

$$M(P,R) \leq \left(\frac{R^2 + 2|\lambda|Rk + k^2}{1 + 2|\lambda|k + k^2}\right)^{\frac{n}{2}} M(P,1).$$

Lemma 2. If $P(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial of degree n having no zeros in $|z| < k$, $k \geq 1$, then

$$M(P',1) \leq n\Big\{\frac{n|a_0| + k^2|a_1|}{(1+k^2)n|a_0| + 2k^2|a_1|}\Big\}M(P,1).$$

The above lemma is due to Govil, Rahman and Schmeisser [5].

Lemma 3. If $P(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial of degree n having no zeros in $|z| < k$, $k > 0$, then for $0 \leq r \leq \rho \leq k$,

$$M(P,r) \geq \left(\frac{r+k}{\rho+k}\right)^n M(P,\rho). \tag{2.1}$$

The above lemma is due to Jain [8].
There is equality in (2.1) for $P(z) = (z+k)^n$.

Lemma 4 . If $P(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial of degree n having no zeros in $|z| < k$, $k > 0$, then for $0 < r \leq \rho \leq k$,

$$M(P,r) \geq \left(\frac{r+k}{\rho+k}\right)^n \left\{1 - \frac{k(k-\rho)(n|a_0| - k|a_1|)n}{(k^2+\rho^2)n|a_0| + 2k^2\rho|a_1|}\left(\frac{\rho - r}{k+\rho}\right)\left(\frac{k+r}{k+\rho}\right)^{n-1}\right\}^{-1} M(P,\rho).$$

Proof of Lemma 4. Recall that $P(z)$ has no zeros in $|z| < k$, $k > 0$, therefore, the polynomial $T(z) = P(tz)$ where $0 < t \leq k$ has no zeros in $|z| < k/t$, where $k/t \geq 1$. Using Lemma 2 with the polynomial $T(z)$, we get

$$\max_{|z|=1}|T'(z)| \leq n\left\{\frac{n|a_0| + k^2/t^2|ta_1|}{(1+k^2/t^2)n|a_0| + 2(k^2/t^2)|ta_1|}\right\}\max_{|z|=1}|T(z)|,$$

which implies

$$\max_{|z|=t}|P'(z)| \leq n\left\{\frac{n|a_0|t + k^2|a_1|}{(t^2+k^2)n|a_0| + 2k^2t|a_1|}\right\}\max_{|z|=t}|P(z)|. \tag{2.2}$$

Now for $0 \leq r \leq \rho \leq k$ and $0 \leq \theta < 2\pi$, we have by (2.1) and (2.2),

$$
\begin{aligned}
|P(\rho e^{i\theta}) - P(re^{i\theta})| &\leq \int_r^\rho |P'(te^{i\theta})|dt \\
&\leq \int_r^\rho n\left\{\frac{n|a_0|t + k^2|a_1|}{(t^2+k^2)n|a_0| + 2k^2t|a_1|}\right\}\max_{|z|=t}|P(z)|dt \\
&\leq \int_r^\rho n\left\{\frac{n|a_0|t + k^2|a_1|}{(t^2+k^2)n|a_0| + 2k^2t|a_1|}\right\}\left(\frac{k+t}{k+r}\right)^n M(P,r)dt \\
&= \frac{nM(P,r)}{(k+r)^n}\int_r^\rho \left\{\frac{n|a_0|t + k^2|a_1|}{(t^2+k^2)n|a_0| + 2k^2t|a_1|}\right\}(k+t)^n dt.
\end{aligned}
$$

which gives, for $0 \leq r \leq \rho \leq k$,

$$
\begin{aligned}
M(P,\rho) &\leq \left[1 + \frac{n}{(k+r)^n}\int_r^\rho \left\{\frac{n|a_0|t + k^2|a_1|}{(t^2+k^2)n|a_0| + 2k^2t|a_1|}\right\}(k+t)^n dt\right] M(P,r) \\
&\leq \left[1 + \frac{n(k+\rho)}{(k+r)^n}\left\{\frac{n|a_0|\rho + k^2|a_1|}{(\rho^2+k^2)n|a_0| + 2k^2\rho|a_1|}\right\}\int_r^\rho (k+t)^{n-1}dt\right] M(P,r) \\
&= \left[1 - \left\{\frac{(k+\rho)(n|a_0|\rho + k^2|a_1|)}{(\rho^2+k^2)n|a_0| + 2k^2\rho|a_1|}\right\} + \left\{\frac{(k+\rho)(n|a_0|\rho + k^2|a_1|)}{(\rho^2+k^2)n|a_0| + 2k^2\rho|a_1|}\right\}\right. \\
&\qquad\qquad \left.\times\left(\frac{k+\rho}{k+r}\right)^n\right] M(P,r) \\
&= \left[\frac{k(k-\rho)(n|a_0| - k|a_1|)}{(\rho^2+k^2)n|a_0| + 2k^2\rho|a_1|} + \left\{1 - \frac{k(k-\rho)(n|a_0| - k|a_1|)}{(\rho^2+k^2)n|a_0| + 2k^2\rho|a_1|}\right\}\right. \\
&\qquad\qquad \left.\times\left(\frac{k+\rho}{k+r}\right)^n\right] M(P,r)
\end{aligned}
$$

$$= \left(\frac{k+\rho}{k+r}\right)^n \left[1 - \frac{k(k-\rho)(n|a_0| - k|a_1|)}{(\rho^2+k^2)n|a_0| + 2k^2\rho|a_1|}\left\{1 - \left(\frac{k+r}{k+\rho}\right)^n\right\}\right] M(P,r)$$

$$= \left(\frac{k+\rho}{k+r}\right)^n \left[1 - \frac{k(k-\rho)(n|a_0| - k|a_1|)}{(\rho^2+k^2)n|a_0| + 2k^2\rho|a_1|} \times \frac{\rho - r}{(k+\rho)\{1 - (\frac{k+r}{k+\rho})\}}\right.$$

$$\left. \times \left\{1 - \left(\frac{k+r}{k+\rho}\right)^n\right\}\right] M(P,r)$$

$$\leq \left(\frac{k+\rho}{k+r}\right)^n \left[1 - \frac{k(k-\rho)(n|a_0| - k|a_1|)n}{(\rho^2+k^2)n|a_0| + 2k^2\rho|a_1|}\left(\frac{\rho - r}{k+\rho}\right)\left(\frac{k+r}{k+\rho}\right)^{n-1}\right] M(P,r)$$

from which Lemma 4 follows.

Lemma 5. If $P(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial of degree n having no zeros in $|z| < k$, $k \geq 1$, then for $1 \leq s < n$,

$$M(P^{(s)}, 1) \leq n(n-1)\dots(n-s+1)\left\{\frac{c(n,s) + \left|\frac{a_s}{a_0}\right| k^{s+1}}{c(n,s)(1+k^{s+1}) + \left|\frac{a_s}{a_0}\right|(k^{s+1}+k^{2s})}\right\} M(P,1),$$

where $c(n,j)$ are the binomial coefficients defined by

$$c(n,j) = \frac{n!}{j!(n-j)!},\ 0! = 1.$$

The above lemma is due to Aziz and Rather [2].

From Lemma 5, we easily get the following:

Lemma 6. If $P(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial of degree n having no zeros in $|z| < k$, $k \geq 1$, then for $0 \leq s < n$,

$$M(P^{(s)}, 1) \leq \left\{\frac{c(n,s) + \left|\frac{a_s}{a_0}\right| k^{s+1}}{c(n,s)(1+k^{s+1}) + \left|\frac{a_s}{a_0}\right|(k^{s+1}+k^{2s})}\right\}\left\{\left(\frac{d^s}{dx^s}(1+x^n)\right)_{x=1}\right\} M(P,1).$$

Lemma 7. If $P(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial of degree n having no zeros in $|z| < k$, $k > 0$, then for $0 \leq s < n$,

$$\frac{1}{c(n,s)}\left|\frac{a_s}{a_0}\right| k^s \leq 1. \tag{2.3}$$

Proof of Lemma 7. Since $P(z) = \sum_{v=0}^{n} a_v z^v \neq 0$ in $|z| < k$, $k > 0$, let $z_1, z_2, \ldots, z_n$ be the zeros of $P(z)$, then $|z_v| \geq k$; $v = 1, 2, \ldots, n$, and we have

$$\begin{aligned}
(-1)\frac{a_{n-1}}{a_n} &= \Gamma(n,1) = \Sigma z_1, \\
(-1)^2\frac{a_{n-2}}{a_n} &= \Gamma(n,2) = \Sigma z_1 z_2, \\
&\vdots \\
(-1)^{n-s}\frac{a_s}{a_n} &= \Gamma(n,n-s) = \Sigma z_1 z_2 \ldots z_{n-s}, \qquad (2.4) \\
&\vdots \\
(-1)^n\frac{a_0}{a_n} &= \Gamma(n,n) = z_1 z_2 \ldots z_n, \qquad (2.5)
\end{aligned}$$

where $\Gamma(n,s)$ is the sum of all possible products of $z_1, z_2, \ldots, z_n$ taken s at a time. From (2.4) and (2.5), we get

$$\begin{aligned}
\left|\frac{a_s}{a_0}\right| &= \left|\frac{a_s}{a_n}\right|\left|\frac{a_n}{a_0}\right| = \left|\frac{\Gamma(n,n-s)}{\Gamma(n,n)}\right| = \left|\frac{\Sigma z_1 z_2 \ldots z_{n-s}}{z_1 z_2 \ldots z_n}\right| \\
&= \left|\Sigma\frac{1}{z_1 z_2 \ldots z_s}\right| \leq \Sigma\left|\frac{1}{z_1}\right|\left|\frac{1}{z_2}\right|\ldots\left|\frac{1}{z_s}\right| \leq c(n,s)\frac{1}{k^s},
\end{aligned}$$

which is (2.3) and this completes the proof of Lemma 7.
The following two lemmas are due to Jain [7].

Lemma 8. Let $P(z)$ be a polynomial of degree n having all its zeros in $|z| \leq 1$. If $T(z)$ is a polynomial of degree at most n such that

$$|T(z)| \leq |P(z)|, \text{ for } |z| = 1,$$

then for $0 \leq s < n$,

$$\left|T^{(s)}(z)\right| \leq \left|P^{(s)}(z)\right|, \text{ for } |z| \geq 1.$$

Lemma 9. If $P(z)$ is a polynomial of degree at most n, then for $0 \leq s < n$,

$$\left|P^{(s)}(z)\right| + \left|Q^{(s)}(z)\right| \leq \left\{\left|\frac{d^s}{dz^s}(1)\right| + \left|\frac{d^s}{dz^s}(z^n)\right|\right\} M(P,1), \quad \text{for } |z| \geq 1,$$

where $Q(z) = z^n\overline{P(\frac{1}{\bar{z}})}$.

3 Proofs of Theorems

Proof of Theorem 2. Since $P(z) \neq 0$ in $|z| < k$, $k \geq 1$, the polynomial $T(z) = P(kz)$ has no zeros in $|z| < 1$. Let $H(z) = z^n\overline{T(\frac{1}{\bar{z}})}$, then

$$|T(z)| \leq |H(z)|, \quad \text{for } |z| = 1,$$

and $H(z)$ has all its zeros in $|z| \leq 1$. Therefore, on applying Lemma 8 to the polynomials $T(z)$ and $H(z)$, we get for $0 \leq s < n$ and $r \geq 1$,

$$\left|T^{(s)}(re^{i\theta})\right| \leq \left|H^{(s)}(re^{i\theta})\right|, \ 0 \leq \theta < 2\pi. \tag{3.1}$$

Also, by Lemma 9, we have for $r \geq 1$ and $0 \leq s < n$,

$$\left|T^{(s)}(re^{i\theta})\right| + \left|H^{(s)}(re^{i\theta})\right| \leq \left\{\frac{d^s}{dr^s}(r^n+1)\right\}M(T,1), \quad 0 \leq \theta < 2\pi,$$

which on combining with (3.1) gives

$$\left|T^{(s)}(re^{i\theta})\right| \leq \frac{1}{2}\left\{\frac{d^s}{dr^s}(r^n+1)\right\}M(T,1),$$

which further implies

$$\left|P^{(s)}(kre^{i\theta})\right| \leq \frac{1}{2k^s}\left\{\frac{d^s}{dr^s}(r^n+1)\right\}M(P,k). \tag{3.2}$$

The above inequality (3.2) in conjunction with Lemma 1 gives

$$\left|P^{(s)}(kre^{i\theta})\right| \leq \frac{1}{2k^s}\left\{\frac{d^s}{dr^s}(r^n+1)\right\}\left(\frac{k^2+2|\lambda|k^2+k^2}{1+2|\lambda|k+k^2}\right)^{\frac{n}{2}}M(P,1). \tag{3.3}$$

Now taking $r = \frac{\rho}{k} \geq 1$ in (3.3), we get

$$\left|P^{(s)}(\rho e^{i\theta})\right| \leq \frac{1}{2}\left\{\frac{d^s}{d\rho^s}(\rho^n+k^n)\right\}\left(\frac{2+2|\lambda|}{1+2|\lambda|k+k^2}\right)^{\frac{n}{2}}M(P,1).$$

This completes the proof of Theorem 2.

Proof of Theorem 3. Since $P(z) \neq 0$ in $|z| < k$, $k > 0$, the polynomial $P(\rho z)$ has no zeros in $|z| < k/\rho$, $k/\rho \geq 1$. Hence using Lemma 6, we have for $0 \leq s < n$,

$$\rho^s M(P^{(s)},\rho) \leq \left\{\frac{c(n,s)+\left|\frac{a_s}{a_0}\right|\rho^s\left(k/\rho\right)^{s+1}}{c(n,s)\left(1+\left(k/\rho\right)^{s+1}\right)+\left|\frac{a_s}{a_0}\right|\rho^s\left(\left(k/\rho\right)^{s+1}+\left(k/\rho\right)^{2s}\right)}\right\}$$
$$\times\left\{\left(\frac{d^s}{dx^s}(1+x^n)\right)_{x=1}\right\}M(P,\rho).$$

This gives

$$M(P^{(s)},\rho) \leq \left\{ \frac{c(n,s)\rho + \left|\frac{a_s}{a_0}\right| k^{s+1}}{c(n,s)(k^{s+1}+\rho^{s+1}) + \left|\frac{a_s}{a_0}\right| (\rho^s k^{s+1} + \rho k^{2s})} \right\} \times \left\{ \left(\frac{d^s}{dx^s}(1+x^n) \right)_{x=1} \right\} M(P,\rho). \tag{3.4}$$

Now if $0 < r \leq \rho \leq k$, then by Lemma 4, we obtain

$$M(P,\rho) \leq \left(\frac{\rho+k}{r+k}\right)^n \left\{ 1 - \frac{k(k-\rho)(n|a_0| - k|a_1|)n}{(k^2+\rho^2)n|a_0| + 2k^2\rho|a_1|} \left(\frac{\rho-r}{k+\rho}\right) \times \left(\frac{k+r}{k+\rho}\right)^{n-1} \right\} M(P,r). \tag{3.5}$$

Combining (3.4) and (3.5), we get

$$M(P^{(s)},\rho) \leq \left\{ \frac{c(n,s)\rho + \left|\frac{a_s}{a_0}\right| k^{s+1}}{c(n,s)(k^{s+1}+\rho^{s+1}) + \left|\frac{a_s}{a_0}\right| (\rho^s k^{s+1} + \rho k^{2s})} \right\} \left\{ \left(\frac{d^s}{dx^s}(1+x^n) \right)_{x=1} \right\} \times \left(\frac{\rho+k}{r+k}\right)^n \left\{ 1 - \frac{k(k-\rho)(n|a_0| - k|a_1|)n}{(k^2+\rho^2)n|a_0| + 2k^2\rho|a_1|} \left(\frac{\rho-r}{k+\rho}\right) \left(\frac{k+r}{k+\rho}\right)^{n-1} \right\} M(P,r),$$

which completes the proof of Theorem 3.

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