Families of exact solutions of Biswas-Milovic equation by an exponential rational function method

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Abstract

In this paper, we introduce generalized exponential rational function method (GERFM) to obtain an exact solutions for the Biswas-Milovic (BM) equation with quadratic-cubic and parabolic nonlinearities. A wide range of closed solutions are acquired. The most important feature of the new method is that it is very effective and simple. The main merits of the proposed is that it gives more general solutions with some free parameters and can be applied to other types of nonlinear partial differential equations.Some interesting Figures for the physical features of some of the obtained solutions are also presented.

2010 Mathematics Subject Classification. **35R03**. 35C08
Keywords. Exact solutions, Optical solutions, Exponential rational function method, Nonlinear optical fibers, Nonlinear Biswas-Milovic equation.

1 Introduction

During the last decades,a wide variety of analytic methods have been proposed to solve partial differential equation arising in various physical aspects of applied sciences [1]-[22].
In this research, we consider the dimensionless form of the nonlinear Biswas-Milovic equation (BME) in nonlinear optical fibers, giving by [1]-[8]

$$i\left(\psi^m\right)_t + a\left(\psi^m\right)_{xx} + b\mathcal{F}\left(|\psi|^2\right)\psi^m = 0, i = \sqrt{-1}. \tag{1}$$

where x denotes the non-dimensional distance along the fiber and t is the temporal variable of the problem. $\psi(x,t)$ is the dependent variable of the equation, and the terms a and b are the coefficients of nonlinearity and group velocity dispersion (GVD), respectively. The function $\mathcal{F}(|\psi|^2)\psi$ is a real-valued algebraic function and is k times continuously differentiable, so that

$$\mathcal{F}(|\psi|^2)\psi \in \mathbb{C}^k\left((-n,n)\times(-m,m);\mathbb{R}^2\right) \tag{2}$$

This paper is organized as follows. Section 2 describes the main steps of a new method and then its application to solve BME with two nonlinearity laws is given in section 3. Finally, some concluding remarks are drawn in Section 5.

Tbilisi Mathematical Journal 13(2) (2020), pp. 39–65.
Tbilisi Centre for Mathematical Sciences.
Received by the editors: 12 February 2019.
Accepted for publication: 30 December 2019.

2 Description of the method

The key steps of the new proposed method can be summarized as follows:

1. We suppose that given nonlinear partial differential equation for $\psi(x,t)$ to be in the form

$$\mathcal{N}(\psi, \psi_x, \psi_t, \psi_{xx}, \ldots) = 0. \tag{3}$$

Using the transformations $u = u(\xi)$ and $\xi = \lambda(x + \nu t)$, we reduce the nonlinear partial differential equation to the following ordinary differential equation

$$\mathcal{N}(u, u', u'', \ldots) = 0, \tag{4}$$

where λ and ν are constants to be determined later.

2. The key step of the new method is to suppose that Eq. (4) has the formal solution

$$\Phi(\xi) = \frac{p_1 e^{q_1 \xi} + p_2 e^{q_2 \xi}}{p_3 e^{q_3 \xi} + p_4 e^{q_4 \xi}}. \tag{5}$$

where p_1, p_2, p_3, p_4 and q_1, q_2, q_3, q_4 are real (or complex) numbers such that the travelling wave solution of Eq.3 can be expressed as follows:

$$u(\xi) = A_0 + \sum_{k=1}^{N} A_k \Phi(\xi)^k + \sum_{k=1}^{N} B_k \Phi(\xi)^{-k}. \tag{6}$$

Unknown coefficients $A_0, A_k, B_k (1 \leq k \leq N)$ and $p_n, q_n (1 \leq n \leq 4)$ are constants to be determined, such that solution 6 satisfies the nonlinear ordinary differential equation 4. Note that, the positive integer m can be determined by the balancing principle.

3. Substituting 6 into Eq. 4 and collecting all terms, the left-hand side of (**??**) is converted into polynomial equation $P(Y_1, Y_2, Y_3, Y_4) = 0$ in terms of $Y_i = e^{q_i \xi}$ for $i = 1, \ldots, 4$. Setting each coefficient of P to zero, we derive a set of algebraic equations for $p_n, q_n (1 \leq n \leq 4)$, and $k, \omega, \lambda, A_0, A_1, B_1$ with the aid of symbolic computation in Maple.

4. Solving the algebraic equations in Step 3., and then substituting non-trivial solutions in (6), one can obtain the soliton solutions of Eq.(1).

Remark 2.1. It is worth mentioning that for $p_1 = 1, p_2 = 0, p_3 = p_4 = 1$, $q_1 = q_2 = q_3 = 0, q_4 = 1$ and $B_i, i = 1, \ldots, N$ in (5) and (6), we have

$$\Phi(\xi) = \frac{1}{1 + e^{\xi}}. \tag{7}$$

$$u(\xi) = A_0 + \sum_{k=1}^{N} \frac{A_k}{(1 + e^{\xi})^k}. \tag{8}$$

which is the well-known exponential rational function method [9]-[12]. So, we call our method as generalized exponential rational function method (GERFM).

3 Application of GERFM to BME

To solve Eq. (1), first we need to apply the travelling wave transformation

$$\psi(x,t) = u(\xi)e^{i\varphi(x,t)}, \quad \xi = K(x+\nu t) \tag{9}$$

where

$$\varphi = -kx + \omega t + \theta. \tag{10}$$

where $\varphi(x,t)$ represents the phase component, ω is the frequency, k represents the wave number, q represents the phase constant, ν is the velocity and l is the width of the soliton. Assuming Eq. (9) into Eq. (1) converts the nonlinear Schrdinger equation into two equations of real and imaginary components.
From the imaginary component of the equation, we have

$$\nu = 2akm,$$

and also the real part reads as

$$ak^2muu'' - m\left(ak^2mu^2 + \omega\right)u^2 + b\mathcal{F}(u^2)u^2 - a(1-m)mK^2\left(u'\right)^2 = 0. \tag{11}$$

In what follows, we will employ GERFM to solve RNLSE with two nonlinearity laws namely quadratic-cubic law and parabolic law.

3.1 Quadratic-cubic law

This form of nonlinearity can be written as $\mathcal{F}(s) = b_1\sqrt{s} + b_2 s$, where b_1 and b_2 are constants. For this nonlinearity, Eq. (1) reduce as follows

$$i\left(\psi^m\right)_t + a\left(\psi^m\right)_{xx} + \left(|b_1|\psi| + b_2|\psi|^2\right)\psi^m = 0. \tag{12}$$

while Eq. (11) reduces to

$$-\left(ak^2m + \omega\right)mu^2 + b_1u^3 + b_2u^4 - (1-m)aK^2m(u')^2 + aK^2muu'' = 0. \tag{13}$$

Balancing the terms of u^4 and uu'' in Eq. (13) gives $N = 1$. Hence, from Eq. (6), we obtain:

$$u(\xi) = A_0 + A_1\Phi(\xi) + \frac{B_1}{\Phi(\xi)}. \tag{14}$$

where $\Phi(\xi)$ is giving by (5). According to method described in Section 2, by substituting (14) into 13 and collecting all terms, the left-hand side of (13) is converted into polynomial equation $P(Y_1, Y_2, Y_3, Y_4) = 0$ in terms of $Y_i = e^{q_i\xi}, i = 1, \ldots, 4$. Setting each coefficient of P to zero, we derive a set of algebraic equations for $p_n, q_n (1 \le n \le 4)$, and $k, \omega, \lambda, A_0, A_1, B_1$. Solving such algebraic polynomial system by the aid of Maple, we have the following cases and non-trivial solutions of (12) as given below:

Family 1.1: $p_1 = 1, p_2 = -1$, $p_3 = p_4 = 1$, and $q_1 = 1, q_2 = -1$, $q_3 = 1, q_4 = -1$, which gives

$$\Phi(\xi) = \frac{\sinh(\xi)}{\cosh(\xi)}. \tag{15}$$

Case 1:

$$K=\sqrt{-\left(18\,am^2b_2-18\,amb_2\right)^{-1}}b_1, k=k, \omega=-\frac{9\,k^2am^2b_2+2\,{b_1}^2}{9mb_2},$$

$$A_0=-\frac{b_1}{3b_2}, A_1=\frac{b_1}{3b_2}, B_1=0.$$

Substituting above values into Eqs.(14) and (15), we have

$$u\left(\xi\right)=\frac{b_1\left(-\cosh\left(\xi\right)+\sinh\left(\xi\right)\right)}{3b_2\cosh\left(\xi\right)}.$$

Therefore an exact solution of the equation (12) is obtained as

$$\psi_{1,1}\left(x,t\right)=\left(\frac{b_1\left(-\cosh\left(\xi\right)+\sinh\left(\xi\right)\right)}{3b_2\cosh\left(\xi\right)}\right)\times e^{i\left(-kx-\frac{9\,k^2am^2b_2+2\,{b_1}^2}{9mb_2}t+\theta\right)}. \tag{16}$$

where

$$\xi=\sqrt{-\left(18\,am^2b_2-18\,amb_2\right)^{-1}}b_1\left(x+2akm\,t\right).$$

Case 2:

$$K=\sqrt{-\left(18\,am^2b_2-18\,amb_2\right)^{-1}}b_1, k=k, \omega=-\frac{9\,k^2am^2b_2+2\,{b_1}^2}{9mb_2},$$

$$A_0=-\frac{b_1}{3b_2}, A_1=0, B_1=\frac{b_1}{3b_2}.$$

Substituting above values into Eqs. (20) and (14), we have

$$u\left(\xi\right)=-\frac{b_1\left(-\cosh\left(\xi\right)+\sinh\left(\xi\right)\right)}{3b_2\sinh\left(\xi\right)}.$$

Therefore an exact solution of the equation (12) is obtained as

$$\psi_{1,2}\left(x,t\right)=\left(-\frac{b_1\left(-\cosh\left(\xi\right)+\sinh\left(\xi\right)\right)}{3b_2\sinh\left(\xi\right)}\right)\times e^{i\left(-kx-\frac{9\,k^2am^2b_2+2\,{b_1}^2}{9mb_2}t+\theta\right)}. \tag{17}$$

where

$$\xi=\sqrt{-\left(18\,am^2b_2-18\,amb_2\right)^{-1}}b_1\left(x+2akm\,t\right).$$

Case 3:

$$K=\sqrt{-\left(72\,am^2b_2-72\,amb_2\right)^{-1}}b_1, k=k, \omega=-\frac{9\,k^2am^2b_2+2\,{b_1}^2}{9mb_2},$$

$$A_0=-\frac{b_1}{3b_2}, A_1=-\frac{b_1}{6b_2}, B_1=-\frac{b_1}{6b_2}.$$

Substituting above values into Eqs.(14) and (15), we have

$$u\left(\xi\right)=-\frac{b_1\left(2\cosh^2\left(\xi\right)+2\cosh\left(\xi\right)\sinh\left(\xi\right)-1\right)}{6b_2\sinh\left(\xi\right)\cosh\left(\xi\right)}.$$

Therefore an exact solution of the equation (12) is obtained as

$$\psi_{1,3}(x,t)=\left(-\frac{b_1\left(2\cosh^2(\xi)+2\cosh(\xi)\sinh(\xi)-1\right)}{6b_2\sinh(\xi)\cosh(\xi)}\right)\times e^{i\left(-kx-\frac{9\,k^2am^2b_2+2\,{b_1}^2}{9mb_2}t+\theta\right)}. \tag{18}$$

where

$$\xi=\sqrt{-\left(72\,am^2b_2-72\,amb_2\right)^{-1}}b_1\left(x+2akm\,t\right).$$

Case 4:

$$K=\sqrt{-\left(72\,am^2b_2-72\,amb_2\right)^{-1}}b_1, k=k, \omega=-\frac{9\,k^2am^2b_2+2\,{b_1}^2}{9mb_2},$$

$$A_0=-\frac{b_1}{3b_2}, A_1=\frac{b_1}{6b_2}, B_1=\frac{b_1}{6b_2}.$$

Substituting above values into Eqs.(14) and (**??**), we have

$$u(\xi)=-\frac{b_1\left(2\,\cosh(\xi)\sinh(\xi)-2\,\cosh^2(\xi)+1\right)}{6b_2\sinh(\xi)\cosh(\xi)}.$$

Therefore an exact solution of the equation (12) is obtained as

$$\psi_{1,4}(x,t)=\left(-\frac{b_1\left(2\,\cosh(\xi)\sinh(\xi)-2\,\cosh^2(\xi)+1\right)}{6b_2\sinh(\xi)\cosh(\xi)}\right)\times e^{i\left(-kx-\frac{9\,k^2am^2b_2+2\,{b_1}^2}{9mb_2}t+\theta\right)}. \tag{19}$$

where

$$\xi=\sqrt{-\left(72\,am^2b_2-72\,amb_2\right)^{-1}}b_1\left(x+2akm\,t\right).$$

Family 1.2: $p_1=i, p_2=-i$, $p_3=p_4=1$, and $q_1=i, q_2=-i$, $q_3=i, q_4=-i$, which gives

$$\Phi(\xi)=-\frac{\sin(\xi)}{\cos(\xi)}. \tag{20}$$

Case 1:

$$K=\sqrt{\left(18\,am^2b_2-18\,amb_2\right)^{-1}}b_1, k=k, \omega=-\frac{9\,k^2am^2b_2+2\,{b_1}^2}{9mb_2},$$

$$A_0=-\frac{b_1}{3b_2}, A_1=0, B_1=\frac{ib_1}{3b_2}.$$

Substituting above values into Eqs. (20) and (14), we have

$$u(\xi)=\frac{b_1\left(i\cos(\xi)-\sin(\xi)\right)}{3b_2\sin(\xi)}.$$

Therefore an exact solution of the equation (12) is obtained as

$$\psi_{1,5}(x,t)=\left(\frac{b_1\left(i\cos(\xi)-\sin(\xi)\right)}{3b_2\sin(\xi)}\right)\times e^{i\left(-kx-\frac{9\,k^2am^2b_2+2\,{b_1}^2}{9mb_2}t+\theta\right)}. \tag{21}$$

where

$$\xi=\sqrt{\left(18\,am^2b_2-18\,amb_2\right)^{-1}}b_1\left(x+2akm\,t\right).$$

Case 2:

$$K = \sqrt{\left(18\,am^2b_2 - 18\,amb_2\right)^{-1}}b_1, k = k, \omega = -\frac{9\,k^2am^2b_2 + 2\,{b_1}^2}{9mb_2},$$

$$A_0 = -\frac{b_1}{3b_2}, A_1 = \frac{ib_1}{3b_2}, B_1 = 0.$$

Substituting above values into Eqs. (20) and (14), we have

$$u\left(\xi\right) = \frac{b_1\left(i\sin\left(\xi\right) - \cos\left(\xi\right)\right)}{3b_2\cos\left(\xi\right)}.$$

Therefore an exact solution of the equation (12) is obtained as

$$\psi_{1,6}\left(x,t\right) = \left(\frac{b_1\left(i\sin\left(\xi\right) - \cos\left(\xi\right)\right)}{3b_2\cos\left(\xi\right)}\right) \times e^{i\left(-kx - \frac{9\,k^2am^2b_2 + 2\,{b_1}^2}{9mb_2}t + \theta\right)}. \tag{22}$$

where

$$\xi = \sqrt{\left(18\,am^2b_2 - 18\,amb_2\right)^{-1}}b_1\left(x + 2akm\,t\right).$$

Case 3:

$$K = \sqrt{-\left(36\,am^2b_2 - 36\,amb_2\right)^{-1}}b_1, k = k, \omega = -\frac{9\,k^2am^2b_2 + 2\,{b_1}^2}{9mb_2},$$

$$A_0 = -\frac{b_1}{3b_2}, A_1 = \frac{\sqrt{2}b_1}{6b_2}, B_1 = \frac{\sqrt{2}b_1}{6b_2}.$$

Substituting above values into Eqs.(14) and (20), we have

$$u\left(\xi\right) = \frac{b_1\left(-2\,\sin\left(\xi\right)\cos\left(\xi\right) + \sqrt{2}\right)}{6b_2\sin\left(\xi\right)\cos\left(\xi\right)}.$$

Therefore an exact solution of the equation (12) is obtained as

$$\psi_{1,7}\left(x,t\right) = \left(\frac{b_1\left(-2\,\sin\left(\xi\right)\cos\left(\xi\right) + \sqrt{2}\right)}{6b_2\sin\left(\xi\right)\cos\left(\xi\right)}\right) \times e^{i\left(-kx - \frac{9\,k^2am^2b_2 + 2\,{b_1}^2}{9mb_2}t + \theta\right)}. \tag{23}$$

where

$$\xi = \sqrt{-\left(36\,am^2b_2 - 36\,amb_2\right)^{-1}}b_1\left(x + 2akm\,t\right).$$

Case 4:

$$K = \sqrt{\left(72\,am^2b_2 - 72\,amb_2\right)^{-1}}b_1, k = k, \omega = -\frac{9\,k^2am^2b_2 + 2\,{b_1}^2}{9mb_2},$$

$$A_0 = -\frac{b_1}{3b_2}, A_1 = -\frac{ib_1}{6b_2}, B_1 = \frac{ib_1}{6b_2}.$$

Substituting above values into Eqs.(14) and (15), we have

$$u\left(\xi\right) = \frac{b_1\left(2\,i\cos^2\left(\xi\right) - 2\,\sin\left(\xi\right)\cos\left(\xi\right) - i\right)}{6b_2\sin\left(\xi\right)\cos\left(\xi\right)}.$$

Therefore an exact solution of the equation (12) is obtained as

$$\psi_{1,8}(x,t)=\left(\frac{b_1\left(2\,i\cos^2(\xi)-2\,\sin(\xi)\cos(\xi)-i\right)}{6b_2\sin(\xi)\cos(\xi)}\right)\times e^{i\left(-kx-\frac{9\,k^2am^2b_2+2\,{b_1}^2}{9mb_2}t+\theta\right)}. \quad (24)$$

where

$$\xi=\sqrt{(72\,am^2b_2-72\,amb_2)^{-1}}b_1\,(x+2akm\,t).$$

Family 1.3: $p_1=1, p_2=2$, $p_3=p_4=1$, and $q_1=0, q_2=1$, $q_3=0, q_4=1$, which gives

$$\Phi(\xi)=\frac{1+2\,\mathrm{e}^{\xi}}{1+\mathrm{e}^{\xi}}. \quad (25)$$

Case 1:

$$K=\sqrt{-\frac{-3\sqrt{73}+41}{2304\,am^2b_2-2304\,amb_2}}b_1, k=\frac{1}{m}\sqrt{-\frac{75\,{b_1}^2\sqrt{73}+4608\,mb_2\omega+511\,{b_1}^2}{4608\,ab_2}}, \omega=\omega,$$

$$A_0=\frac{\left(-41+3\sqrt{73}\right)b_1}{96b_2}, A_1=\frac{b_1\left(-1272897+147723\sqrt{73}\right)}{72\,b_2\left(-72553+8643\sqrt{73}\right)}, B_1=-\frac{b_1\left(-3+\sqrt{73}\right)}{48b_2}.$$

Substituting above values into Eqs.(14) and (25), we have

$$u(\xi)=-\frac{b_1\left(-18870\,\mathrm{e}^{2\,\xi}\sqrt{73}-224098\,\mathrm{e}^{2\,\xi}+70177\,\mathrm{e}^{\xi}\sqrt{73}+512451\,\mathrm{e}^{\xi}-9435\sqrt{73}-112049\right)}{48b_2\left(72553+8643\sqrt{73}\right)(1+\mathrm{e}^{\xi})(2\,\mathrm{e}^{\xi}+1)}.$$

Therefore an exact solution of the equation (12) is obtained as

$$\psi_{1,9}(x,t)=u(\xi)\times e^{i\left(-\frac{1}{m}\sqrt{-\frac{75\,{b_1}^2\sqrt{73}+4608\,mb_2\omega+511\,{b_1}^2}{4608\,ab_2}}x+\omega t+\theta\right)}. \quad (26)$$

where

$$\xi=\sqrt{-\frac{-3\sqrt{73}+41}{2304\,am^2b_2-2304\,amb_2}}b_1\left(x+2a\sqrt{-\frac{75\,{b_1}^2\sqrt{73}+4608\,mb_2\omega+511\,{b_1}^2}{4608\,ab_2}}\,t\right).$$

Case 2:

$$K=\sqrt{-2\,(9\,am^2b_2-9\,amb_2)^{-1}}b_1, k=\frac{1}{m}\sqrt{-\frac{9\,mb_2\omega+2\,{b_1}^2}{9ab_2}}, \omega=\omega,$$

$$A_0=\frac{2b_1}{3b_2}, A_1=-\frac{2b_1}{3b_2}, B_1=0.$$

Substituting above values into Eqs. (25) and (14), we have

$$u(\xi)=-\frac{2b_1\mathrm{e}^{\xi}}{3b_2(1+\mathrm{e}^{\xi})}.$$

Therefore an exact solution of the equation (12) is obtained as

$$\psi_{1,10}(x,t)=\left(-\frac{2b_1\mathrm{e}^{\xi}}{3b_2(1+\mathrm{e}^{\xi})}\right)\times e^{i\left(-\frac{1}{m}\sqrt{-\frac{9\,mb_2\omega+2\,{b_1}^2}{9ab_2}}x+\omega t+\theta\right)}. \quad (27)$$

where

$$\xi = \sqrt{-2\,\left(9\,am^2b_2 - 9\,amb_2\right)^{-1}}b_1\left(x + 2a\sqrt{-\frac{9\,mb_2\omega + 2\,{b_1}^2}{9ab_2}}\,t\right).$$

Family 1.4: $p_1 = -2, p_2 = -3$, $p_3 = p_4 = 1$, and $q_1 = 0, q_2 = 1$, $q_3 = 0, q_4 = 1$, which gives

$$\Phi\left(\xi\right) = \frac{-2 - 3\,\mathrm{e}^{\xi}}{1 + \mathrm{e}^{\xi}}. \tag{28}$$

Case 1:

$$K = \sqrt{-2\,\left(9\,am^2b_2 - 9\,amb_2\right)^{-1}}b_1, k = \frac{1}{m}\sqrt{-\frac{9\,mb_2\omega + 2\,{b_1}^2}{9ab_2}}, \omega = \omega,$$

$$A_0 = \frac{4b_1}{3b_2}, A_1 = 0, B_1 = \frac{4b_1}{b_2}.$$

Substituting above values into Eqs.(14) and (28), we have

$$u\left(\xi\right) = -\frac{4b_1}{3b_2\left(2 + 3\,\mathrm{e}^{\xi}\right)}.$$

Therefore an exact solution of the equation (12) is obtained as

$$\psi_{1,11}\left(x,t\right) = \left(-\frac{4b_1}{3b_2\left(2 + 3\,\mathrm{e}^{\xi}\right)}\right) \times e^{i\left(-\frac{1}{m}\sqrt{-\frac{9\,mb_2\omega + 2\,{b_1}^2}{9ab_2}}x + \omega t + \theta\right)}. \tag{29}$$

where

$$\xi = \sqrt{-2\,\left(9\,am^2b_2 - 9\,amb_2\right)^{-1}}b_1\left(x + 2a\sqrt{-\frac{9\,mb_2\omega + 2\,{b_1}^2}{9ab_2}}\,t\right).$$

Case 2:

$$K = \sqrt{-2\,\left(9\,am^2b_2 - 9\,amb_2\right)^{-1}}b_1, k = \frac{1}{m}\sqrt{-\frac{9\,mb_2\omega + 2\,{b_1}^2}{9ab_2}}, \omega = \omega,$$

$$A_0 = -\frac{2b_1}{b_2}, A_1 = 0, B_1 = -\frac{4b_1}{b_2}.$$

Substituting above values into Eqs. (28) and (14), we have

$$u\left(\xi\right) = -2\,\frac{b_1\mathrm{e}^{\xi}}{b_2\left(2 + 3\,\mathrm{e}^{\xi}\right)}.$$

Therefore an exact solution of the equation (12) is obtained as

$$\psi_{1,12}\left(x,t\right) = \left(-2\,\frac{b_1\mathrm{e}^{\xi}}{b_2\left(2 + 3\,\mathrm{e}^{\xi}\right)}\right) \times e^{i\left(-\frac{1}{m}\sqrt{-\frac{9\,mb_2\omega + 2\,{b_1}^2}{9ab_2}}x + \omega t + \theta\right)}.$$

where

$$\xi = \sqrt{-2\,\left(9\,am^2b_2 - 9\,amb_2\right)^{-1}}b_1\left(x + 2a\sqrt{-\frac{9\,mb_2\omega + 2\,{b_1}^2}{9ab_2}}\,t\right).$$

Family 1.5: $p_1 = 3, p_2 = 2$, $p_3 = p_4 = 1$, and $q_1 = 0, q_2 = 1$, $q_3 = 0, q_4 = 1$, which gives

$$\Phi(\xi) = \frac{3 + 2\,\mathrm{e}^{\xi}}{1 + \mathrm{e}^{\xi}}. \tag{30}$$

Case 1:

$$K = \sqrt{-2\,\left(9\,am^2b_2 - 9\,amb_2\right)^{-1}}b_1, k = \frac{1}{m}\sqrt{-\frac{9\,mb_2\omega + 2\,{b_1}^2}{9ab_2}},$$

$$\omega = \omega, A_0 = \frac{4b_1}{3b_2}, A_1 = 0, B_1 = -\frac{4b_1}{b_2}.$$

Substituting above values into Eqs.(14) and (30), we have

$$u(\xi) = -\frac{4b_1\mathrm{e}^{\xi}}{3b_2\left(3 + 2\,\mathrm{e}^{\xi}\right)}.$$

Therefore an exact solution of the equation (12) is obtained as

$$\psi_{1,13}(x,t) = \left(-\frac{4b_1\mathrm{e}^{\xi}}{3b_2\left(3 + 2\,\mathrm{e}^{\xi}\right)}\right) \times e^{i\left(-\frac{1}{m}\sqrt{-\frac{9\,mb_2\omega + 2\,{b_1}^2}{9ab_2}}x + \omega t + \theta\right)}. \tag{31}$$

where

$$\xi = \sqrt{-2\,\left(9\,am^2b_2 - 9\,amb_2\right)^{-1}}b_1\left(x + 2a\sqrt{-\frac{9\,mb_2\omega + 2\,{b_1}^2}{9ab_2}}\,t\right).$$

Case 2:

$$K = \sqrt{-\frac{5\sqrt{217}+121}{20736\,am^2b_2 - 20736\,amb_2}}b_1, k = \frac{1}{m}\sqrt{-\frac{-365\,{b_1}^2\sqrt{217} + 41472\,mb_2\omega + 4991\,{b_1}^2}{41472\,ab_2}},$$

$$\omega = \omega, A_0 = -\frac{b_1\left(70445 + 4777\sqrt{217}\right)}{144\,b_2\left(725 + 49\sqrt{217}\right)}, A_1 = \frac{\left(5 + \sqrt{217}\right)b_1}{144b_2}, B_1 = \frac{\left(5 + \sqrt{217}\right)b_1}{24b_2}.$$

Substituting above values into Eqs. (30) and (14), we have

$$u(\xi) = -\frac{b_1\left(-146\sqrt{217}\mathrm{e}^{2\,\xi} + 1690\,\mathrm{e}^{2\,\xi} + 605\sqrt{217}\mathrm{e}^{\xi} - 10033\,\mathrm{e}^{\xi} - 219\sqrt{217} + 2535\right)}{144\,b_2\left(-725 + 49\sqrt{217}\right)\left(1 + \mathrm{e}^{\xi}\right)\left(3 + 2\,\mathrm{e}^{\xi}\right)}.$$

Therefore an exact solution of the equation (12) is obtained as

$$\psi_{1,14}(x,t) = u(\xi) \times e^{i\left(-\frac{1}{m}\sqrt{-\frac{-365\,{b_1}^2\sqrt{217} + 41472\,mb_2\omega + 4991\,{b_1}^2}{41472\,ab_2}}x + \omega t + \theta\right)}. \tag{32}$$

where

$$\xi = \sqrt{-\frac{5\sqrt{217} + 121}{20736\,am^2b_2 - 20736\,amb_2}}b_1\left(x + 2a\sqrt{-\frac{-365\,{b_1}^2\sqrt{217} + 41472\,mb_2\omega + 4991\,{b_1}^2}{41472\,ab_2}}\,t\right).$$

Family 1.6: $p_1 = 1-i, p_2 = -1-i$, $p_3 = -1, p_4 = 1$, and $q_1 = i, q_2 = -i$, $q_3 = i, q_4 = -i$, which gives

$$\Phi(\xi) = \frac{\cos(\xi) - \sin(\xi)}{\sin(\xi)}. \tag{33}$$

Case 1:

$$K = \sqrt{(18\,am^2b_2 - 18\,amb_2)^{-1}}b_1, k = k, \omega = -\frac{9\,k^2am^2b_2 + 2\,{b_1}^2}{9mb_2},$$

$$A_0 = \frac{(-1+i)\,b_1}{3b_2}, A_1 = 0, B_1 = \frac{2i\,b_1}{3b_2}.$$

Substituting above values into Eqs.(14) and (33), we have

$$u(\xi) = -\frac{b_1\left(2\,i\sin(\xi)\cos(\xi) + 2\,\cos^2(\xi) - 1 + i\right)}{6\,b_2\cos^2(\xi) - 3\,b_2}.$$

Therefore an exact solution of the equation (12) is obtained as

$$\psi_{1,15}(x,t) = \left(-\frac{b_1\left(2\,i\sin(\xi)\cos(\xi) + 2\,\cos^2(\xi) - 1 + i\right)}{6\,b_2\cos^2(\xi) - 3\,b_2}\right) \times e^{i\left(-kx - \frac{9\,k^2am^2b_2 + 2\,{b_1}^2}{9mb_2}t + \theta\right)}. \tag{34}$$

where

$$\xi = \sqrt{(18\,am^2b_2 - 18\,amb_2)^{-1}}b_1\,(x + 2akm\,t).$$

Case 2:

$$K = \sqrt{-(18\,am^2b_2 - 18\,amb_2)^{-1}}b_1, k = k, \omega = -\frac{9\,k^2am^2b_2 - 2\,{b_1}^2}{9mb_2},$$

$$A_0 = -\frac{2b_1}{3b_2}, A_1 = -\frac{b_1}{3b_2}, B_1 = -\frac{2b_1}{3}b_2.$$

Substituting above values into Eqs. (33) and (14), we have

$$u(\xi) = \frac{b_1}{3b_2\sin(\xi)\,(-\cos(\xi) + \sin(\xi))}.$$

Therefore an exact solution of the equation (12) is obtained as

$$\psi_{1,16}(x,t) = \left(\frac{b_1}{3b_2\sin(\xi)\,(-\cos(\xi) + \sin(\xi))}\right) \times e^{i\left(-kx - \frac{9\,k^2am^2b_2 - 2\,{b_1}^2}{9mb_2}t + \theta\right)}. \tag{35}$$

where

$$\xi = \sqrt{-(18\,am^2b_2 - 18\,amb_2)^{-1}}b_1\,(x + 2akm\,t).$$

Family 1.7: $p_1 = 2-i, p_2 = -2-i$, $p_3 = -1, p_4 = 1$, and $q_1 = i, q_2 = -i$, $q_3 = i, q_4 = -i$, which gives

$$\Phi(\xi) = \frac{\cos(\xi) - 2\,\sin(\xi)}{\sin(\xi)}. \tag{36}$$

Case 1:

$$K=\sqrt{\left(18\,am^2b_2-18\,amb_2\right)^{-1}}b_1, k=k, \omega=-\frac{9\,k^2am^2b_2+2\,{b_1}^2}{9mb_2},$$

$$A_0=\frac{\left(-1+2\,i\right)b_1}{3b_2}, A_1=0, B_1=\frac{5\,ib_1}{3b_2}.$$

Substituting above values into Eqs.(14) and (33), we have

$$u\left(\xi\right)=-\frac{b_1\left(5\,i\sin\left(\xi\right)\cos\left(\xi\right)+5\,\cos^2\left(\xi\right)-4+2\,i\right)}{15\,b_2\cos^2\left(\xi\right)-12\,b_2}.$$

Therefore an exact solution of the equation (12) is obtained as

$$\psi_{1,17}\left(x,t\right)=\left(-\frac{b_1\left(5\,i\sin\left(\xi\right)\cos\left(\xi\right)+5\,\cos^2\left(\xi\right)-4+2\,i\right)}{15\,b_2\cos^2\left(\xi\right)-12\,b_2}\right)\times e^{i\left(-kx-\frac{9\,k^2am^2b_2+2\,{b_1}^2}{9mb_2}t+\theta\right)}. \tag{37}$$

where

$$\xi=\sqrt{\left(18\,am^2b_2-18\,amb_2\right)^{-1}}b_1\left(x+2akm\,t\right).$$

Case 2:

$$K=\sqrt{-\frac{-\sqrt{11}+6}{900\,am^2b_2-900\,amb_2}}b_1, k=k,$$

$$\omega=-\frac{92025\,\left(-1+\sqrt{11}\right)ak^2m^2b_2-139500\,k^2am^2b_2+6294\,\left(-1+\sqrt{11}\right){b_1}^2+3830\,{b_1}^2}{225\,b_2m\left(-1029+409\sqrt{11}\right)},$$

$$A_0=-\frac{b_1\left(-17+7\sqrt{11}\right)}{15b_2\left(-1+\sqrt{11}\right)}, A_1=\frac{\left(-1+\sqrt{11}\right)b_1}{30b_2}, B_1=-\frac{b_1\left(-375+62\sqrt{11}\right)}{3b_2\left(-67+57\sqrt{1}\right)}.$$

Substituting above values into Eqs. (33) and (14), we have

$$u\left(\xi\right)=\frac{b_1\left(198\,\sqrt{11}\sin\left(\xi\right)\cos\left(\xi\right)+396\,\sqrt{11}\cos^2\left(\xi\right)-1412\,\sin\left(\xi\right)\cos\left(\xi\right)-A\right)}{30b_2\sin\left(\xi\right)\left(\cos\left(\xi\right)-2\,\sin\left(\xi\right)\right)\left(347+62\,\sqrt{11}\right)},$$

where $A=2824\,\cos^2\left(\xi\right)-805\,\sqrt{11}+1795$ Therefore an exact solution of the equation (12) is obtained as

$$\psi_{1,18}\left(x,t\right)=u\left(\xi\right)\times e^{i\left(-kx-\frac{92025\left(-1+\sqrt{11}\right)ak^2m^2b_2-139500\,k^2am^2b_2+6294\left(-1+\sqrt{11}\right){b_1}^2+3830\,{b_1}^2}{225\,b_2m\left(-1029+409\sqrt{11}\right)}t+\theta\right)}. \tag{38}$$

where

$$\xi=\sqrt{-\frac{-\sqrt{11}+6}{900\,am^2b_2-900\,amb_2}}b_1\left(x+2akm\,t\right).$$

Family 1.8: $p_1=-1, p_2=-2$, $p_3=p_4=1$, and $q_1=0, q_2=1$, $q_3=0, q_4=1$, which gives

$$\Phi\left(\xi\right)=\frac{-1-2\,\mathrm{e}^{\xi}}{1+\mathrm{e}^{\xi}}. \tag{39}$$

Case 1:

$$K=\sqrt{-2\left(81\,am^2b_2-81\,amb_2\right)^{-1}}b_1, k=\frac{1}{m}\sqrt{-\frac{81\,mb_2\omega+2\,{b_1}^2}{81\,ab_2}}, \omega=\omega,$$

$$A_0=-\frac{2b_1}{3b_2}, A_1=\frac{2b_1}{9b_2}, B_1=\frac{4b_1}{9b_2}.$$

Substituting above values into Eqs.(14) and (39), we have

$$u\left(\xi\right)=-\frac{2b_1\mathrm{e}^{\xi}}{9b_2\left(1+\mathrm{e}^{\xi}\right)\left(2\,\mathrm{e}^{\xi}+1\right)}.$$

Therefore an exact solution of the equation (12) is obtained as

$$\psi_{1,19}\left(x,t\right)=\left(-\frac{2b_1\mathrm{e}^{\xi}}{9b_2\left(1+\mathrm{e}^{\xi}\right)\left(2\,\mathrm{e}^{\xi}+1\right)}\right)\times e^{i\left(-\frac{1}{m}\sqrt{-\frac{81\,mb_2\omega+2\,{b_1}^2}{81\,ab_2}}x+\omega t+\theta\right)}. \tag{40}$$

where

$$\xi=\sqrt{-2\left(81\,am^2b_2-81\,amb_2\right)^{-1}}b_1\left(x+2a\sqrt{-\frac{81\,mb_2\omega+2\,{b_1}^2}{81\,ab_2}}\,t\right).$$

Family 1.9: $p_1=2, p_2=p_3=p_4=1$, and $q_1=1, q_2=0$, $q_3=1, q_4=0$, which gives

$$\Phi\left(\xi\right)=\frac{2+\mathrm{e}^{\xi}}{1+\mathrm{e}^{\xi}}. \tag{41}$$

Case 1:

$$K=\sqrt{-2\left(9\,am^2b_2-9\,amb_2\right)^{-1}}b_1, k=\frac{1}{m}\sqrt{-\frac{9\,mb_2\omega+2\,{b_1}^2}{9ab_2}}, \omega=\omega,$$

$$A_0=-\frac{4b_1}{3b_2}, A_1=\frac{2b_1}{3b_2}, B_1=0.$$

Substituting above values into Eqs.(14) and (41), we have

$$u\left(\xi\right)=-\frac{2b_1}{3b_2\left(1+\mathrm{e}^{\xi}\right)}.$$

Therefore an exact solution of the equation (12) is obtained as

$$\psi_{1,20}\left(x,t\right)=\left(-\frac{2b_1}{3b_2\left(1+\mathrm{e}^{\xi}\right)}\right)\times e^{i\left(-\frac{1}{m}\sqrt{-\frac{9\,mb_2\omega+2\,{b_1}^2}{9ab_2}}x+\omega t+\theta\right)}. \tag{42}$$

where

$$\xi=\sqrt{-2\left(9\,am^2b_2-9\,amb_2\right)^{-1}}b_1\left(x+2a\sqrt{-\frac{9\,mb_2\omega+2\,{b_1}^2}{9ab_2}}\,t\right).$$

Family 1.10: $p_1 = -2 - i, p_2 = 2 - i$, $p_3 = -1, p_4 = 1$, and $q_1 = i, q_2 = -i$, $q_3 = i, q_4 = -i$, which gives

$$\Phi(\xi) = \frac{\cos(\xi) + 2\sin(\xi)}{\sin(\xi)}. \tag{43}$$

Case 1:

$$K = \sqrt{-(72\,am^2b_2 - 72\,amb_2)^{-1}}b_1, k = k, \omega = -\frac{18\,k^2am^2b_2 - {b_1}^2}{18mb_2},$$

$$A_0 = -\frac{2b_1}{3b_2}, A_1 = \frac{b_1}{6b_2}, B_1 = \frac{5b_1}{6b_2}.$$

Substituting above values into Eqs.(14) and (43), we have

$$u(\xi) = \frac{b_1}{6b_2\sin(\xi)(\cos(\xi) + 2\sin(\xi))}.$$

Therefore an exact solution of the equation (12) is obtained as

$$\psi_{1,21}(x,t) = \left(\frac{b_1}{6b_2\sin(\xi)(\cos(\xi) + 2\sin(\xi))}\right) \times e^{i\left(-kx - \frac{18\,k^2am^2b_2 - {b_1}^2}{18mb_2}t + \theta\right)}. \tag{44}$$

where

$$\xi = \sqrt{-(72\,am^2b_2 - 72\,amb_2)^{-1}}b_1(x + 2akm\,t).$$

Family 1.11: $p_1 = p_2 = i$, $p_3 = 1, p_4 = -1$, and $q_1 = i, q_2 = -i$, $q_3 = i, q_4 = -i$, which gives

$$\Phi(\xi) = \frac{\cos(\xi)}{\sin(\xi)}. \tag{45}$$

Case 1:

$$K = \sqrt{(18\,am^2b_2 - 18\,amb_2)^{-1}}b_1, k = k, \omega = -\frac{9\,k^2am^2b_2 + 2\,{b_1}^2}{9mb_2},$$

$$A_0 = -\frac{b_1}{3b_2}, A_1 = 0, B_1 = \frac{ib_1}{3b_2}.$$

Substituting above values into Eqs.(14) and (47), we have

$$u(\xi) = -\frac{b_1(i\sin(\xi) + \cos(\xi))}{3\cos(\xi)b_2}.$$

Therefore an exact solution of the equation (12) is obtained as

$$\psi_{1,22}(x,t) = \left(-\frac{b_1(i\sin(\xi) + \cos(\xi))}{3\cos(\xi)b_2}\right) \times e^{i\left(-kx - \frac{9\,k^2am^2b_2 + 2\,{b_1}^2}{9mb_2}t + \theta\right)}. \tag{46}$$

where

$$\xi = \sqrt{(18\,am^2b_2 - 18\,amb_2)^{-1}}b_1(x + 2akm\,t).$$

Family 1.12: $p_1 = -1 - i, p_2 = 1 - i$, $p_3 = -1, p_4 = 1$, and $q_1 = i, q_2 = -i$, $q_3 = i, q_4 = -i$, which gives

$$\Phi(\xi) = \frac{\sin(\xi) + \cos(\xi)}{\sin(\xi)}. \tag{47}$$

Case 1:

$$K = \sqrt{-\frac{\sqrt{17}+9}{576\,am^2b_2 - 576\,amb_2}}\,b_1, k = k,$$

$$\omega = -\frac{15696\,\left(1/8+1/8\sqrt{17}\right)ak^2m^2b_2 + 3600\,k^2am^2b_2 + 2007\,\left(1/8+1/8\sqrt{17}\right)b_1{}^2 + 155\,b_1{}^2}{144\,mb_2\left(\frac{309}{8}+\frac{109\sqrt{17}}{8}\right)},$$

$$A_0 = -\frac{b_1\left(13+5\sqrt{17}\right)}{12b_2\left(1+\sqrt{17}\right)}, A_1 = \frac{\left(1+\sqrt{17}\right)b_1}{24b_2}, B_1 = \frac{b_1\left(193+25\sqrt{17}\right)}{6b_2\left(29+21\sqrt{17}\right)}.$$

Substituting above values into Eqs.(14) and (20), we have

$$u(\xi) = -\frac{b_1\left(-9\sqrt{17}\sin(\xi)\cos(\xi) + 9\sqrt{17}\left(\cos(\xi)\right)^2 - 463\,\sin(\xi)\cos(\xi) + B\right)}{(300\,\sin(\xi) + 300\,\cos(\xi))\sin(\xi)\,b_2\left(-\frac{193}{25}+\sqrt{17}\right)},$$

where $B = 463\,(\cos(\xi))^2 - 118\sqrt{17} - 154$. Therefore an exact solution of the equation (12) is obtained as

$$\psi_{1,23}(x,t) = u(\xi) \times e^{i\left(-kx - \frac{15696\left(1/8+1/8\sqrt{17}\right)ak^2m^2b_2 + 3600\,k^2am^2b_2 + 2007\left(1/8+1/8\sqrt{17}\right)b_1{}^2 + 155\,b_1{}^2}{144\,mb_2\left(\frac{309}{8}+\frac{109\sqrt{17}}{8}\right)}t + \theta\right)}. \tag{48}$$

where

$$\xi = \sqrt{-\frac{\sqrt{17}+9}{576\,am^2b_2 - 576\,amb_2}}\,b_1\,(x + 2akm\,t).$$

3.2 Parabolic law

This form of nonlinearity can be written as $\mathcal{F}(s) = b_3 s + b_4 s^2$, where b_3, b_4 are constants. For this nonlinearity, Eq. (1) reduce as follows

$$i\,(\psi^m)_t + a\,(\psi^m)_{xx} + \left(b_3|\psi|^2 + b_4|\psi|^4\right)\psi^m = 0. \tag{49}$$

while Eq. (11) reduces to

$$-\left(ak^2m + \omega\right)mu^2 + b_3u^4 + b_4u^6 - aK^2m(u')^2 + aK^2m^2(u')^2 + aK^2muu'' = 0. \tag{50}$$

Setting

$$u = U^{\frac{1}{2}}$$

Eq. (50) is transformed to

$$-4m\left(ak^2m + \omega\right)U^2 + 4b_3U^3 + 4b_4U^4 + aK^2(-2+m)m(U')^2 + 2aK^2mUU'' = 0. \tag{51}$$

Balancing the terms of U^4 and UU'' in Eq. (51) gives $N = 1$. Hence, from Eq. (6), we obtain:

$$u(\xi) = A_0 + A_1\Phi(\xi) + \frac{B_1}{\Phi(\xi)}. \tag{52}$$

Applying the method described above, we conclude that Eq. (**??**), has the following non-trivial solutions as given below:

Family 2.1: $p_1 = -2, p_2 = -3$, $p_3 = p_4 = 1$, and $q_1 = 0, q_2 = 1$, $q_3 = 0, q_4 = 1$, which gives

$$\Phi(\xi) = \frac{-2 - 3\,\mathrm{e}^{\xi}}{1 + \mathrm{e}^{\xi}}. \tag{53}$$

Case 1:

$$K = \frac{b_3}{m+1}\sqrt{-\frac{m+2}{amb_4}}, k = \frac{1}{m+1}\sqrt{-\frac{4\,\omega\,m^2b_4 + 8\,\omega\,mb_4 + mb_3{}^2 + 4\,\omega\,b_4 + 2\,b_3{}^2}{4amb_4}}, \omega = \omega,$$

$$A_0 = \frac{b_3\,(m+2)}{b_4\,(m+1)}, A_1 = 0, B_1 = 3\,\frac{b_3\,(m+2)}{b_4\,(m+1)}.$$

Substituting above values into Eqs.(52) and (53), we have

$$U(\xi) = -\frac{b_3\,(m+2)}{b_4\,(m+1)\,(2+3\,\mathrm{e}^{\xi})}.$$

Therefore an exact solution of the equation (49) is obtained as

$$\psi_{2,1}(x,t) = \left(-\frac{b_3\,(m+2)}{b_4\,(m+1)\,(2+3\,\mathrm{e}^{\xi})}\right)^{1/2} \times e^{i\left(-\frac{1}{m+1}\sqrt{-\frac{4\,\omega\,m^2b_4+8\,\omega\,mb_4+mb_3{}^2+4\,\omega\,b_4+2\,b_3{}^2}{4amb_4}}x+\omega t+\theta\right)}. \tag{54}$$

where

$$\xi = \frac{b_3}{m+1}\sqrt{-\frac{m+2}{amb_4}}\left(x + \frac{2am}{m+1}\sqrt{-\frac{4\,\omega\,m^2b_4 + 8\,\omega\,mb_4 + mb_3{}^2 + 4\,\omega\,b_4 + 2\,b_3{}^2}{4amb_4}}\,t\right).$$

Case 2:

$$K = \frac{b_3}{m+1}\sqrt{-\frac{m+2}{amb_4}}, k = \frac{1}{m+1}\sqrt{-\frac{4\,\omega\,m^2b_4 + 8\,\omega\,mb_4 + mb_3{}^2 + 4\,\omega\,b_4 + 2\,b_3{}^2}{4amb_4}}, \omega = \omega,$$

$$A_0 = -\frac{3b_3\,(m+2)}{2b_4\,(m+1)}, A_1 = 0, B_1 = -\frac{3b_3\,(m+2)}{b_4\,(m+1)}.$$

Substituting above values into Eqs. (53) and (52), we have

$$U(\xi) = -\frac{3b_3\,(m+2)\,\mathrm{e}^{\xi}}{2b_4\,(m+1)\,(2+3\,\mathrm{e}^{\xi})}.$$

Therefore an exact solution of the equation (49) is obtained as

$$\psi_{2,2}(x,t)=\left(-\frac{3b_3\,(m+2)\,\mathrm{e}^{\xi}}{2b_4\,(m+1)\,(2+3\,\mathrm{e}^{\xi})}\right)^{1/2}\times e^{i\left(-\frac{1}{m+1}\sqrt{-\frac{4\,\omega\,m^2b_4+8\,\omega\,mb_4+mb_3{}^2+4\,\omega\,b_4+2\,b_3{}^2}{4amb_4}}x+\omega t+\theta\right)}. \tag{55}$$

where

$$\xi=\frac{b_3}{m+1}\sqrt{-\frac{m+2}{amb_4}}\left(x+\frac{2am}{m+1}\sqrt{-\frac{4\,\omega\,m^2b_4+8\,\omega\,mb_4+mb_3{}^2+4\,\omega\,b_4+2\,b_3{}^2}{4amb_4}}\,t\right).$$

Case 3:

$$K=\frac{b_3}{m+1}\sqrt{-\frac{m+2}{25amb_4}},k=\frac{1}{m+1}\sqrt{-\frac{100\,\omega\,m^2b_4+200\,\omega\,mb_4+mb_3{}^2+100\,\omega\,b_4+2\,b_3{}^2}{100\,amb_4}},$$

$$\omega=\omega,A_0=-\frac{b_3\,(m+2)}{2b_4\,(m+1)},A_1=-\frac{b_3\,(m+2)}{10b_4\,(m+1)},B_1=-3\frac{b_3\,(m+2)}{5b_4\,(m+1)}.$$

Substituting above values into Eqs.(52) and (53), we have

$$U(\xi)=-\frac{b_3\,(m+2)\,\mathrm{e}^{\xi}}{30\,(m+1)\,(1+\mathrm{e}^{\xi})\,(2/3+\mathrm{e}^{\xi})\,b_4}.$$

Therefore an exact solution of the equation (49) is obtained as

$$\psi_{2,3}(x,t)=\left(-\frac{b_3\,(m+2)\,\mathrm{e}^{\xi}}{30\,(m+1)\,(1+\mathrm{e}^{\xi})\,(2/3+\mathrm{e}^{\xi})\,b_4}\right)^{1/2}\times e^{i(-E_0)}, \tag{56}$$

where

$$E_0=\frac{1}{m+1}\sqrt{-\frac{100\,\omega\,m^2b_4+200\,\omega\,mb_4+mb_3{}^2+100\,\omega\,b_4+2\,b_3{}^2}{100\,amb_4}}x+\omega t+\theta$$

, and

$$\xi=\frac{b_3}{m+1}\sqrt{-\frac{m+2}{25amb_4}}\left(x+\frac{2am}{m+1}\sqrt{-\frac{100\,\omega\,m^2b_4+200\,\omega\,mb_4+mb_3{}^2+100\,\omega\,b_4+2\,b_3{}^2}{100\,amb_4}}\,t\right).$$

Family 2.2: $p_1=1-i,p_2=-1-i,\,p_3=-1,p_4=1$, and $q_1=i,q_2=-i,\,q_3=i,q_4=-i$, which gives

$$\Phi(\xi)=\frac{\cos(\xi)-\sin(\xi)}{\sin(\xi)}. \tag{57}$$

Case 1:

$$K=\frac{b_3}{m+1}\sqrt{\frac{m+2}{4amb_4}},k=\frac{1}{m+1}\sqrt{-\frac{4\,\omega\,m^2b_4+8\,\omega\,mb_4+mb_3{}^2+4\,\omega\,b_4+2\,b_3{}^2}{4amb_4}},\omega=\omega,$$

$$A_0 = \frac{(-1+i)\,b_3\,(m+2)}{4b_4\,(m+1)}, A_1 = 0, B_1 = \frac{(1+i)\,((-1+i)\,m+2+2\,i+2\,m)\,b_3}{4b_4\,(m+1)}.$$

Substituting above values into Eqs.(52) and (57), we have

$$U(\xi) = -\frac{b_3\,(m+2)\left(2\,i\sin(\xi)\cos(\xi)+2\,\cos^2(\xi)-1+i\right)}{4b_4\,(m+1)\,(2\,\cos^2(\xi)-1)}.$$

Therefore an exact solution of the equation (49) is obtained as

$$\psi_{2,4}(x,t) = \left(-\frac{b_3\,(m+2)\left(2\,i\sin(\xi)\cos(\xi)+2\,\cos^2(\xi)-1+i\right)}{4b_4\,(m+1)\,(2\,\cos^2(\xi)-1)}\right)^{1/2} \times e^{i\left(-\frac{1}{m+1}\sqrt{-\frac{4\,\omega\,m^2b_4+8\,\omega\,mb_4+mb_3{}^2+4\,\omega\,b_4+2\,b_3{}^2}{4amb_4}}x+\omega t+\theta\right)}. \tag{58}$$

where

$$\xi = \frac{b_3}{m+1}\sqrt{\frac{m+2}{4amb_4}}\left(x+\frac{2am}{m+1}\sqrt{-\frac{4\,\omega\,m^2b_4+8\,\omega\,mb_4+mb_3{}^2+4\,\omega\,b_4+2\,b_3{}^2}{4amb_4}}\,t\right).$$

Case 2:

$$K = \frac{b_3}{m+1}\sqrt{-\frac{m+2}{4amb_4}}, k = \frac{1}{m+1}\sqrt{-\frac{4\,\omega\,m^2b_4+8\,\omega\,mb_4-mb_3{}^2+4\,\omega\,b_4-2\,b_3{}^2}{4amb_4}}, \omega = \omega,$$

$$A_0 = -\frac{b_3\,(m+2)}{2b_4\,(m+1)}, A_1 = -\frac{b_3\,(m+2)}{4b_4\,(m+1)}, B_1 = -\frac{b_3\,(m+2)}{2b_4\,(m+1)}.$$

Substituting above values into Eqs. (57) and (52), we have

$$U(\xi) = \frac{b_3\,(m+2)}{4b_4\,(m+1)\sin(\xi)\,(-\cos(\xi)+\sin(\xi))}.$$

Therefore an exact solution of the equation (49) is obtained as

$$\psi_{2,5}(x,t) = \left(\frac{b_3\,(m+2)}{4b_4\,(m+1)\sin(\xi)\,(-\cos(\xi)+\sin(\xi))}\right)^{1/2} \times e^{i\left(-\frac{1}{m+1}\sqrt{-\frac{4\,\omega\,m^2b_4+8\,\omega\,mb_4-mb_3{}^2+4\,\omega\,b_4-2\,b_3{}^2}{4amb_4}}x+\omega t+\theta\right)}. \tag{59}$$

where

$$\xi = \frac{b_3}{m+1}\sqrt{-\frac{m+2}{4amb_4}}\left(x+\frac{2am}{m+1}\sqrt{-\frac{4\,\omega\,m^2b_4+8\,\omega\,mb_4-mb_3{}^2+4\,\omega\,b_4-2\,b_3{}^2}{4amb_4}}\,t\right).$$

Family 2.3: $p_1 = 1, p_2 = -1$, $p_3 = p_4 = 1$, and $q_1 = 1, q_2 = -1$, $q_3 = 1, q_4 = -1$, which gives

$$\Phi(\xi) = \frac{\sinh(\xi)}{\cosh(\xi)}. \tag{60}$$

Case 1:

$$K = \frac{b_3}{m+1}\sqrt{-\frac{m+2}{4amb_4}}, k = \frac{1}{m+1}\sqrt{-\frac{4\,\omega\,m^2b_4 + 8\,\omega\,mb_4 + mb_3{}^2 + 4\,\omega\,b_4 + 2\,b_3{}^2}{4amb_4}}, \omega = \omega,$$

$$A_0 = -\frac{b_3(m+2)}{4(m+1)b_4}, A_1 = \frac{b_3(m+2)}{4(m+1)b_4}, B_1 = 0.$$

Substituting above values into Eqs.(52) and (60), we have

$$U(\xi) = \frac{b_3(m+2)(\sinh(\xi) - \cosh(\xi))}{4b_4(m+1)\cosh(\xi)}.$$

Therefore an exact solution of the equation (49) is obtained as

$$\psi_{2,6}(x,t) = \left(\frac{b_3(m+2)(\sinh(\xi) - \cosh(\xi))}{4b_4(m+1)\cosh(\xi)}\right)^{1/2} \times$$

$$e^{i\left(-\frac{1}{m+1}\sqrt{-\frac{4\,\omega\,m^2b_4+8\,\omega\,mb_4+mb_3{}^2+4\,\omega\,b_4+2\,b_3{}^2}{4amb_4}}x+\omega t+\theta\right)}. \tag{61}$$

where

$$\xi = \frac{b_3}{m+1}\sqrt{-\frac{m+2}{4amb_4}}\left(x + \frac{2am}{m+1}\sqrt{-\frac{4\,\omega\,m^2b_4 + 8\,\omega\,mb_4 + mb_3{}^2 + 4\,\omega\,b_4 + 2\,b_3{}^2}{4amb_4}}\,t\right).$$

Case 2:

$$K = \frac{b_3}{m+1}\sqrt{-\frac{m+2}{16amb_4}}, k = \frac{1}{m+1}\sqrt{-\frac{4\,\omega\,m^2b_4 + 8\,\omega\,mb_4 + mb_3{}^2 + 4\,\omega\,b_4 + 2\,b_3{}^2}{4amb_4}}, \omega = \omega,$$

$$A_0 = -\frac{b_3(m+2)}{4(m+1)b_4}, A_1 = \frac{b_3(m+2)}{8(m+1)b_4}, B_1 = \frac{b_3(m+2)}{8(m+1)b_4}.$$

Substituting above values into Eqs. (60) and (52), we have

$$U(\xi) = -\frac{b_3(m+2)\left(2\cosh(\xi)\sinh(\xi) - 2\cosh^2(\xi) + 1\right)}{8b_4(m+1)\cosh(\xi)\sinh(\xi)}.$$

Therefore an exact solution of the equation (49) is obtained as

$$\psi_{2,7}(x,t) = \left(-\frac{b_3(m+2)\left(2\cosh(\xi)\sinh(\xi) - 2\cosh^2(\xi) + 1\right)}{8b_4(m+1)\cosh(\xi)\sinh(\xi)}\right)^{1/2} \times$$

$$e^{i\left(-\frac{1}{m+1}\sqrt{-\frac{4\,\omega\,m^2b_4+8\,\omega\,mb_4+mb_3{}^2+4\,\omega\,b_4+2\,b_3{}^2}{4amb_4}}x+\omega t+\theta\right)}. \tag{62}$$

where

$$\xi=\frac{b_3}{m+1}\sqrt{-\frac{m+2}{16amb_4}}\left(x+\frac{2am}{m+1}\sqrt{-\frac{4\,\omega\,m^2b_4+8\,\omega\,mb_4+mb_3{}^2+4\,\omega\,b_4+2\,b_3{}^2}{4amb_4}}\,t\right).$$

Family 2.4: $p_1=1, p_2=2$, $p_3=p_4=1$, and $q_1=0, q_2=1$, $q_3=0, q_4=1$, which gives

$$\Phi(\xi)=\frac{1+2\,\mathrm{e}^{\xi}}{1+\mathrm{e}^{\xi}}. \tag{63}$$

Case 1:

$$K=\frac{b_3}{m+1}\sqrt{-\frac{m+2}{amb_4}}, k=\frac{1}{m+1}\sqrt{-\frac{4\,\omega\,m^2b_4+8\,\omega\,mb_4+mb_3{}^2+4\,\omega\,b_4+2\,b_3{}^2}{4amb_4}}, \omega=\omega,$$

$$A_0=-\frac{b_3\,(m+2)}{(m+1)\,b_4}, A_1=\frac{b_3\,(m+2)}{2\,(m+1)\,b_4}, B_1=0.$$

Substituting above values into Eqs.(52) and (63), we have

$$U(\xi)=-\frac{b_3\,(m+2)}{2b_4\,(m+1)\,(1+\mathrm{e}^{\xi})}.$$

Therefore an exact solution of the equation (49) is obtained as

$$\psi_{2,8}(x,t)=\left(-\frac{b_3\,(m+2)}{2b_4\,(m+1)\,(1+\mathrm{e}^{\xi})}\right)^{1/2}\times e^{i\left(-\frac{1}{m+1}\sqrt{-\frac{4\,\omega\,m^2b_4+8\,\omega\,mb_4+mb_3{}^2+4\,\omega\,b_4+2\,b_3{}^2}{4amb_4}}x+\omega t+\theta\right)}.$$

where

$$\xi=\frac{b_3}{m+1}\sqrt{-\frac{m+2}{amb_4}}\left(x+\frac{2am}{m+1}\sqrt{-\frac{4\,\omega\,m^2b_4+8\,\omega\,mb_4+mb_3{}^2+4\,\omega\,b_4+2\,b_3{}^2}{4amb_4}}\,t\right).$$

Family 2.5: $p_1=2, p_2=1$, $p_3=p_4=1$, and $q_1=0, q_2=1$, $q_3=0, q_4=1$, which gives

$$\Phi(\xi)=\frac{2+\mathrm{e}^{\xi}}{1+\mathrm{e}^{\xi}}. \tag{64}$$

Case 1:

$$K=\frac{b_3}{m+1}\sqrt{-\frac{m+2}{9amb_4}}, k=\frac{1}{m+1}\sqrt{-\frac{36\,\omega\,m^2b_4+72\,\omega\,mb_4+mb_3{}^2+36\,\omega\,b_4+2\,b_3{}^2}{36amb_4}},$$

$$\omega=\omega, A_0=-\frac{b_3\,(m+2)}{2\,(m+1)\,b_4}, A_1=\frac{b_3\,(m+2)}{6\,(m+1)\,b_4}, B_1=\frac{b_3\,(m+2)}{3\,(m+1)\,b_4}.$$

Substituting above values into Eqs.(52) and (64), we have

$$U(\xi) = -\frac{b_3\,(m+2)\,\mathrm{e}^{\xi}}{6b_4\,(m+1)\,(1+\mathrm{e}^{\xi})\,(2+\mathrm{e}^{\xi})}.$$

Therefore an exact solution of the equation (49) is obtained as

$$\psi_{2,9}(x,t) = \left(-\frac{b_3\,(m+2)\,\mathrm{e}^{\xi}}{6b_4\,(m+1)\,(1+\mathrm{e}^{\xi})\,(2+\mathrm{e}^{\xi})}\right)^{1/2} \times e^{i\left(-\frac{1}{m+1}\sqrt{-\frac{36\,\omega\,m^2b_4+72\,\omega\,mb_4+{mb_3}^2+36\,\omega\,b_4+2\,{b_3}^2}{36amb_4}}x+\omega t+\theta\right)}. \tag{65}$$

where

$$\xi = \frac{b_3}{m+1}\sqrt{-\frac{m+2}{9amb_4}}\left(x+\frac{2am}{m+1}\sqrt{-\frac{36\,\omega\,m^2b_4+72\,\omega\,mb_4+{mb_3}^2+36\,\omega\,b_4+2\,{b_3}^2}{36amb_4}}\,t\right).$$

Family 2.6: $p_1 = -2-i, p_2 = 2-i$, $p_3 = -1, p_4 = 1$, and $q_1 = i, q_2 = -i$, $q_3 = i, q_4 = -i$, which gives

$$\Phi(\xi) = \frac{\cos(\xi) + 2\,\sin(\xi)}{\sin(\xi)}. \tag{66}$$

Case 1:

$$K = \frac{b_3}{m+1}\sqrt{-\frac{m+2}{16amb_4}}, k = \frac{1}{m+1}\sqrt{-\frac{16\,\omega\,m^2b_4+32\,\omega\,mb_4-{mb_3}^2+16\,\omega\,b_4-2\,{b_3}^2}{16amb_4}},$$

$$\omega = \omega, A_0 = -\frac{b_3\,(m+2)}{2\,(m+1)\,b_4}, A_1 = \frac{b_3\,(m+2)}{8\,(m+1)\,b_4}, B_1 = \frac{5b_3\,(m+2)}{8\,(m+1)\,b_4}.$$

Substituting above values into Eqs.(52) and (66), we have

$$U(\xi) = \frac{b_3\,(m+2)}{8\,(m+1)\,b_4\sin(\xi)\,(\cos(\xi)+2\,\sin(\xi))}.$$

Therefore an exact solution of the equation (49) is obtained as

$$\psi_{2,10}(x,t) = \left(\frac{b_3\,(m+2)}{8\,(m+1)\,b_4\sin(\xi)\,(\cos(\xi)+2\,\sin(\xi))}\right)^{1/2} \times e^{i\left(-\frac{1}{m+1}\sqrt{-\frac{16\,\omega\,m^2b_4+32\,\omega\,mb_4-{mb_3}^2+16\,\omega\,b_4-2\,{b_3}^2}{16amb_4}}x+\omega t+\theta\right)}. \tag{67}$$

where

$$\xi = \frac{b_3}{m+1}\sqrt{-\frac{m+2}{16amb_4}}\left(x+\frac{2am}{m+1}\sqrt{-\frac{16\,\omega\,m^2b_4+32\,\omega\,mb_4-{mb_3}^2+16\,\omega\,b_4-2\,{b_3}^2}{16amb_4}}\,t\right).$$

Family 2.7: $p_1 = -1 - i, p_2 = 1 - i$, $p_3 = -1, p_4 = 1$, and $q_1 = i, q_2 = -i$, $q_3 = i, q_4 = -i$, which gives

$$\Phi(\xi) = \frac{\sin(\xi) + \cos(\xi)}{\sin(\xi)}. \tag{68}$$

Case 1:

$$K = \frac{b_3}{m+1}\sqrt{\frac{m+2}{4amb_4}}, k = \frac{1}{m+1}\sqrt{-\frac{4\,\omega\,m^2b_4 + 8\,\omega\,mb_4 + mb_3{}^2 + 4\,\omega\,b_4 + 2\,b_3{}^2}{4amb_4}},$$

$$\omega = \omega, A_0 = \frac{(-1+i)\,b_3\,(m+2)}{4\,(m+1)\,b_4}, A_1 = 0, B_1 = \frac{(-1-i)\,((-1+i)\,m + 2 + 2\,i + 2\,m)\,b_3}{4\,(m+1)\,b_4}.$$

Substituting above values into Eqs.(52) and (68), we have

$$U(\xi) = \frac{b_3\,(m+2)\left(2\,i\sin(\xi)\cos(\xi) - 2\,\cos^2(\xi) + 1 - i\right)}{4\,(2\cos^2(\xi) - 1)\,(m+1)\,b_4}.$$

Therefore an exact solution of the equation (49) is obtained as

$$\psi_{2,11}(x,t) = \left(\frac{b_3\,(m+2)\left(2\,i\sin(\xi)\cos(\xi) - 2\,\cos^2(\xi) + 1 - i\right)}{4\,(2\cos^2(\xi) - 1)\,(m+1)\,b_4}\right)^{1/2} \times$$

$$e^{i\left(-\frac{1}{m+1}\sqrt{-\frac{4\,\omega\,m^2b_4 + 8\,\omega\,mb_4 + mb_3{}^2 + 4\,\omega\,b_4 + 2\,b_3{}^2}{4amb_4}}\,x + \omega t + \theta\right)}. \tag{69}$$

where

$$\xi = \frac{b_3}{m+1}\sqrt{\frac{m+2}{4amb_4}}\left(x + \frac{2am}{m+1}\sqrt{-\frac{4\,\omega\,m^2b_4 + 8\,\omega\,mb_4 + mb_3{}^2 + 4\,\omega\,b_4 + 2\,b_3{}^2}{4amb_4}}\,t\right).$$

Family 2.8: $p_1 = 2 - i, p_2 = -2 - i$, $p_3 = -1, p_4 = 1$, and $q_1 = i, q_2 = -i$, $q_3 = i, q_4 = -i$, which gives

$$\Phi(\xi) = \frac{\cos(\xi) - 2\,\sin(\xi)}{\sin(\xi)}. \tag{70}$$

– [Case 1:]

$$K = \frac{b_3}{m+1}\sqrt{\frac{m+2}{4amb_4}}, k = \frac{1}{m+1}\sqrt{-\frac{4\,\omega\,m^2b_4 + 8\,\omega\,mb_4 + mb_3{}^2 + 4\,\omega\,b_4 + 2\,b_3{}^2}{4amb_4}},$$

$$\omega = \omega, A_0 = \frac{(-1+2i)\,(m+2)\,b_3}{4\,(m+1)\,b_4}, A_1 = 0,$$

$$B_1 = \frac{5}{2}\frac{b_3\left((-763750+1527500\,i)m^7+(6210625+12421250\,i)m^6+253125\,m^7+(8421225+16842450\,i)m^5+7731250\,m^6+\right.}{(-510625+1021250\,i)m^7+(1343125+2686250\,i)m^6+3565625\,m^7+(7872225+15744450\,i)m^5+35218750\,m^6+}$$

$$\frac{(49120220+98240440\,i)m^4+83221625\,m^5+(183843700+367687400\,i)m^3+291142800\,m^4+}{(176220980+352441960\,i)m^4+93674625\,m^5+(140062420+280124840\,i)m^3+16479800\,m^4+16479800\,m^4+}$$

$$\frac{434325500\,m^3+(107948736-215897472\,i)m-224820960\,m^2+69226496-}{(35300384-70600768\,i)m^2-406391700\,m^3+(99556672-199113344\,i)-}$$

$$\left.\frac{18102272\,i+56551360\,m(230478304-460956608\,i)m^2}{599680160\,m^2-18102272-69226496\,i-324505280\,m)b_4}\right).$$

Substituting above values into Eqs.(52) and (70), we have

$$U(\xi) = \frac{\left(-\frac{2487381}{6640625}-\frac{3326007\,i}{26562500}\right)b_3\,(m+2)\left(\left(\frac{4078}{4071}+i\right)\cos(\xi)-\left(1-\frac{4078\,i}{4071}\right)\sin(\xi)\right)}{(m+1)\,(\cos(\xi)-2\,\sin(\xi))\,b_4}.$$

Therefore an exact solution of the equation (49) is obtained as

$$\psi_{2,12}(x,t) = (U(\xi))^{1/2}\times e^{i\left(-\frac{1}{m+1}\sqrt{-\frac{4\,\omega\,m^2b_4+8\,\omega\,mb_4+mb_3{}^2+4\,\omega\,b_4+2\,b_3{}^2}{4amb_4}}\,x+\omega t+\theta\right)}. \tag{71}$$

where

$$\xi = \frac{b_3}{m+1}\sqrt{\frac{m+2}{4amb_4}}\left(x+\frac{2am}{m+1}\sqrt{-\frac{4\,\omega\,m^2b_4+8\,\omega\,mb_4+mb_3{}^2+4\,\omega\,b_4+2\,b_3{}^2}{4amb_4}}\,t\right).$$

Family 2.9: $p_1 = 0, p_2 = p_3 = p_4 = 1$, and $q_1 = 0, q_2 = 1$, $q_3 = 0, q_4 = 1$, which gives

$$\Phi(\xi) = \frac{e^{\xi}}{1+e^{\xi}}. \tag{72}$$

Case 1:

$$K = \frac{b_3}{m+1}\sqrt{-\frac{m+2}{amb_4}}, k = k, \omega = -\frac{4\,ak^2m^3b_4+8\,ak^2m^2b_4+4\,ak^2mb_4+mb_3{}^2+2\,b_3{}^2}{4b_4\,(m^2+2\,m+1)},$$

$$A_0 = 0, A_1 = -\frac{(m+2)\,b_3}{2\,(m+1)\,b_4}, B_1 = 0.$$

Substituting above values into Eqs.(52) and (72), we have

$$U(\xi) = -\frac{b_3\,(m+2)\,e^{\xi}}{2b_4\,(m+1)\,(1+e^{\xi})}.$$

Therefore an exact solution of the equation (49) is obtained as

$$\psi_{2,13}(x,t) = \left(-\frac{b_3\,(m+2)\,e^{\xi}}{2b_4\,(m+1)\,(1+e^{\xi})}\right)^{1/2}\times e^{i\left(-kx-\frac{4\,ak^2m^3b_4+8\,ak^2m^2b_4+4\,ak^2mb_4+mb_3{}^2+2\,b_3{}^2}{4b_4\left(m^2+2\,m+1\right)}t+\theta\right)}. \tag{73}$$

where

$$\xi = \frac{b_3}{m+1}\sqrt{-\frac{m+2}{amb_4}}\,(x+2akm\,t).$$

Family 2.10: $p_1 = i, p_2 = -i$, $p_3 = p_4 = 1$, and $q_1 = i, q_2 = -i$, $q_3 = i, q_4 = -i$, which gives

$$\Phi(\xi) = -\frac{\sin(\xi)}{\cos(\xi)}. \tag{74}$$

Case 1:

$$K = \frac{b_3}{m+1}\sqrt{\frac{m+2}{4amb_4}}, k = \frac{1}{m+1}\sqrt{-\frac{4\,\omega\,m^2b_4 + 8\,\omega\,mb_4 + mb_3{}^2 + 4\,\omega\,b_4 + 2\,b_3{}^2}{4amb_4}},$$

$$\omega = \omega, A_0 = -\frac{(m+2)\,b_3}{4\,(m+1)\,b_4}, A_1 = 0, B_1 = \frac{ib_3\,(m+2)}{4\,(m+1)\,b_4}.$$

Substituting above values into Eqs.(52) and (74), we have

$$U(\xi) = \frac{b_3\,(m+2)\,(i\cos(\xi) - \sin(\xi))}{4\,(m+1)\,b_4\sin(\xi)}.$$

Therefore an exact solution of the equation (49) is obtained as

$$\psi_{2,14}(x,t) = \left(\frac{b_3\,(m+2)\,(i\cos(\xi) - \sin(\xi))}{4\,(m+1)\,b_4\sin(\xi)}\right)^{1/2} \times$$

$$e^{i\left(-\frac{1}{m+1}\sqrt{-\frac{4\,\omega\,m^2b_4+8\,\omega\,mb_4+mb_3{}^2+4\,\omega\,b_4+2\,b_3{}^2}{4amb_4}}x+\omega t+\theta\right)}. \tag{75}$$

where

$$\xi = \frac{b_3}{m+1}\sqrt{\frac{m+2}{4amb_4}}\left(x + \frac{2am}{m+1}\sqrt{-\frac{4\,\omega\,m^2b_4 + 8\,\omega\,mb_4 + mb_3{}^2 + 4\,\omega\,b_4 + 2\,b_3{}^2}{4amb_4}}\,t\right).$$

Family 2.11: $p_1 = i, p_2 = i$, $p_3 = 1, p_4 = -1$, and $q_1 = i, q_2 = -i$, $q_3 = i, q_4 = -i$, which gives

$$\Phi(\xi) = \frac{\cos(\xi)}{\sin(\xi)}. \tag{76}$$

Case 1:

$$K = \frac{b_3}{m+1}\sqrt{\frac{m+2}{4amb_4}}, k = \frac{1}{m+1}\sqrt{-\frac{4\,\omega\,m^2b_4 + 8\,\omega\,mb_4 + mb_3{}^2 + 4\,\omega\,b_4 + 2\,b_3{}^2}{4amb_4}},$$

$$\omega = \omega, A_0 = -\frac{b_3\,(m+2)}{4\,(m+1)\,b_4}, A_1 = \frac{ib_3\,(m+2)}{4b_4\,(m+1)}, B_1 = 0.$$

Substituting above values into Eqs.(52) and (76), we have

$$U(\xi) = -\frac{b_3\,(m+2)\,(i\cos(\xi) + \sin(\xi))}{4\,(m+1)\,b_4\sin(\xi)}.$$

Therefore an exact solution of the equation (49) is obtained as

$$\psi_{2,15}(x,t) = \left(-\frac{b_3(m+2)(i\cos(\xi)+\sin(\xi))}{4(m+1)b_4\sin(\xi)}\right)^{1/2} \times e^{i(-E_1)}, \tag{77}$$

where

$$E_1 = \frac{1}{m+1}\sqrt{-\frac{4\omega m^2 b_4 + 8\omega m b_4 + m{b_3}^2 + 4\omega b_4 + 2{b_3}^2}{4amb_4}}x + \omega t + \theta$$

, and

$$\xi = \frac{b_3}{m+1}\sqrt{\frac{m+2}{4amb_4}}\left(x + \frac{2am}{m+1}\sqrt{-\frac{4\omega m^2 b_4 + 8\omega m b_4 + m{b_3}^2 + 4\omega b_4 + 2{b_3}^2}{4amb_4}}\,t\right).$$

4 Physical interpretation

In this section, we present some three dimensional Figures of the modulus of some of the obtained solutions presented in the immediate section. The construction of the Figures is carried out by taking suitable values of the parameters in order to see the mechanism of the original Eqs. (1) and (2). The physical features of the solutions (17), (18), (48) and (69) are shown in Figures 1-4, respectively.

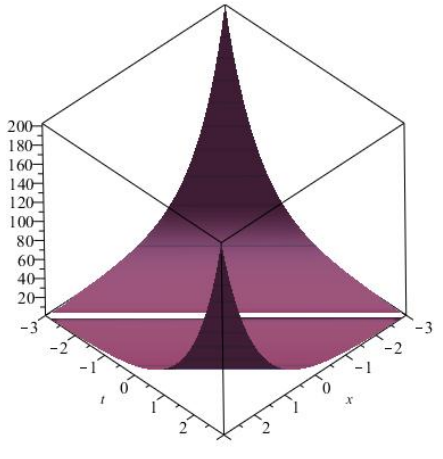


FIGURE 1. 3D plot of the modulus of (17) with $b_1 = B_2 = a = 1$, $m = 0.5$, $k = 1.7$.

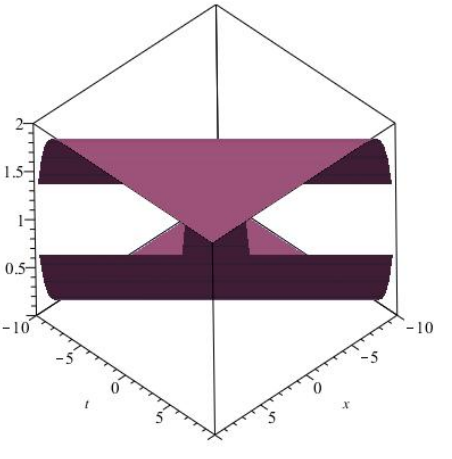


FIGURE 2. 3D plot of the modulus of (18) with $b_1 = B_2 = a = 1$, $m = 0.5$, $k = 1.7$.

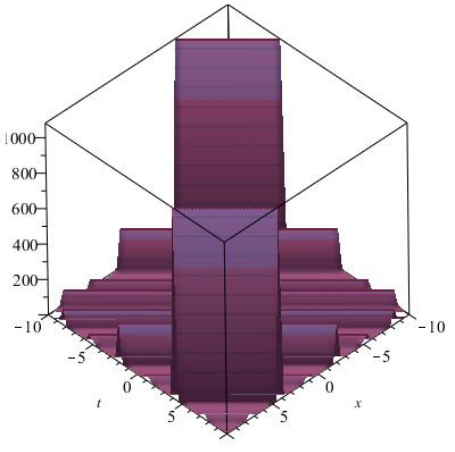


FIGURE 3. 3D plot of the modulus of (48) with $b_1 = B_2 = a = 1$, $m = 0.5$, $k = 1.7$.

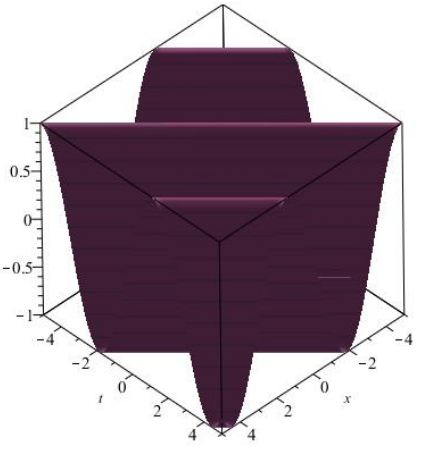


FIGURE 4. 3D plot of the modulus of (69) with $b_3 = B_4 = \omega = 1$, $m = 0.5$, $k = 1.7$.

5 Conclusion

This paper applied the newly proposed GERFM to obtain an exact solutions for the BM equation with quadratic-cubic and parabolic nonlinearities. A wide range of closed solutions are acquired. The most important feature of the method is that it is very efficient and easy to use. The main merits of the proposed is that it gives more general solutions with some free parameters and can be applied to other types of nonlinear partial differential equations. Some interesting Figures for the physical features of some of the obtained solutions are also presented. Finally, we have checked the entire set of solutions for all cases with Maple, and found that all the obtained solutions satisfied the original equation.

References

[1] A. Biswas, D. Milovic, *Bright and dark solitons of the generalized nonlinear Schrodinger's equation*, Commun. Nonlinear Sci. Numer. Simul. 15 (2010) 1473-1484.

[2] C. M. Khalique, *Stationary solutions for the Biswas-Milovic equation*, Appl. Math. Comput. 217 (2011) 7400-7404.

[3] Q. Zhou, M. Ekici, A. Sonmezoglu, M. Mirzazadeh, *Optical solitons with Biswas-Milovic equation by extended G/G-expansion method*. Optik127 (2016) 6277-6290.

[4] H. Jafari, A. Soorakia, C.M. Khalique, *Dark solitons of the BiswasMilovic equation by the first integral method*, Optik 124 (2013) 3929-3932.

[5] J. Manafian, M. Lakestani, *Optical solitons with BiswasMilovic equation for Kerr law nonlinearity*, Eur. Phys. J. Plus 130 (2015) 61.

[6] Q. Zhou, *Optical solitons for BiswasMilovic model with Kerr law and parabolic law nonlinearities*, Nonlinear Dyn. 84 (2016) 677-681.

[7] Q.Zhou, M.Ekici, A.Sonmezoglu, M.Mirzazadeh, M.Eslami, *Optical solitons with Biswas-Milovic equation by extended trial equation method*. Nonlinear Dyn. 84,(2016) 1883-1900.

[8] M. Inc, A.I. Aliyu, A. Yusuf, D. Baleanu, *Optical solitons for Biswas-Milovic Model in nonlinear optics by Sine-Gordon equation method*, Optik 157 (2018) 267-274.

[9] A. Bekir, M. Kaplan, *Exponential rational function method for solving nonlinear equations arising in various physical models*, Chin. J. Phys. 54 (2016) 365-370.

[10] M. Kaplan, K. Hosseini, *Investigation of exact solutions for the Tzitzica type equations in nonlinear optics*, Optik 154 (2018) 393-397.

[11] M. Kaplan, P. Mayeli, K. Hosseini, *Exact traveling wave solutions of the WuZhang system describing (1 + 1)-dimensional dispersive long wave*, Opt Quant Electron 49 (2017) 404.

[12] S. T. Mohyud-Din, S. Bibi, *Exact solutions for nonlinear fractional differential equations using exponential rational function method*, Opt Quant Electron 49 (2017) 64.

[13] M. M. Al Qurashi, A. Yusuf, A. I. Aliyu, M. Inc. *Soliton solutions and Conservation laws for Lossy Nonlinear Transmission line equation*. Superlattices and Microstructures, 107, 320-336 (2017).

[14] F. Tchier, AI. Aliyu, A. Yusuf, M. Inc. *Dynamics of solitons to the ill-posed Boussinesq equation*. European. Physical. Journal. Plus. 132, 136 (2017).

[15] F. Tchier, A. Yusuf, A.I Aliyu, M. Inc. *Optical and other solitons for the fourth-order dispersive nonlinear Shr**ö**dinger's equation with dual-power law nonlinearity*, Superlattices and Microstructures, 105 183-197 (2017).

[16] M. Inc, A.I Aliyu, A. Yusuf. *Traveling wave solutions and conservation laws of some fifth-order nonlinear equations*. European. Physical. Journal. Plus, 132 224 (2017).

[17] M. Inc, A. Yusuf, A.I. Aliyu. *Dark optical and other Soliton solutions for the three different nonlinear Shr**ö**dinger's equations*. Optical and quantum electronics, Opt Quant Electron (2017) 49:354.

[18] D. Baleanu, M. Inc, A. Yusuf, AI. Aliyu. *Traveling wave solutions and conservation laws for nonlinear evolution equation*. Journal of Mathematical Physics, **59** (2018) 023506.

[19] M. Inc, A. Yusuf, A.I. Aliyu, D. Baleanu. *Time-fractional CahnAllen and time-fractional Klein Gordon equations: Lie symmetry analysis, explicit solutions and convergence analysis*. Physica A 493 (2018) 94106.

[20] D. Baleanu, M. Inc, A. Yusuf, A.I.Aliyu. *Lie symmetry analysis, exact solutions and conservation laws for the time fractional modified ZakharovKuznetsov equation*. Nonlinear Analysis: Modelling and Control, Vol. 22, No. 6, 861876.

[21] D. Baleanu, M. Inc, A. Yusuf, A.I. Aliyu. *Optical solitons, nonlinear self-adjointness and conservation laws for Kundu Eckhaus equation.* Chinese Journal of Physics 55 (2017) 23412355.

[22] M. Inc, AI. Aliyu, A. Yusufa, D. Baleanu. *Combined optical solitary waves and conservation laws for nonlinear Chen-Lee-Liu equation in optical fibers.* Optik 158 (2018) 297-304.