Arithmetic properties of 2-color overpartition pairs

M. S. Mahadeva Naika[1] and S. Shivaprasada Nayaka[2]

[1]Department of Mathematics, Bengaluru Central University, Bengaluru-560 001, Karnataka, India
[2]Department of Mathematics, BMS Institute of Technology and Management, Yelahanka, Bengaluru-560 064, Karnataka, India.

E-mail: msmnaika@rediffmail.com[1], shivprasadnayaks@gmail.com[2]

Abstract

Let $\overline{p}_{k,-2}(n)$ counts the number of overpartition pairs of n with 2-color in which one of the colors appears only in parts that are multiples of k. We establish several infinite families of congruences modulo powers of 2 and 3 for $\overline{p}_{k,-2}(n)$ where k= 2 and 3. For example, for all $n \geq 0$ and $\alpha \geq 0$, we have

$$\overline{p}_{2,-2}(16 \cdot 3^{4\alpha+2}n + 34 \cdot 3^{4\alpha+1}) \equiv 0 \pmod{128},$$

$$\overline{p}_{3,-2}(6 \cdot 25^{\alpha+2}n + 20 \cdot 25^{\alpha+2}) \equiv 0 \pmod{27}.$$

2010 Mathematics Subject Classification. **11P81**. 11P83
Keywords. congruences, 2-color overpartition pairs, dissections.

1 Introduction

Ramanujan's general theta function $E(a, b)$ is defined as

$$E(a, b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, |ab| < 1.$$

The product representation of $E(a, b)$ arises from Jacobi's triple product identity [4, p. 35, Entry 19], [17] as

$$E(a, b) = (-a; ab)_\infty (-b; ab)_\infty (ab; ab)_\infty,$$

where

$$(a; q)_\infty := \prod_{n=0}^{\infty} (1 - aq^n), |q| < 1.$$

Throughout this paper, we will use

$$E_k := (q^k; q^k)_\infty.$$

Some special cases of $E(a, b)$, known as Ramanujan's theta functions, are

$$\varphi(q) := E(q, q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q; q^2)_\infty^2 (q^2; q^2)_\infty = \frac{E_2^5}{E_1^2 E_4^2},$$

Tbilisi Mathematical Journal 13(2) (2020), pp. 67–85.
Tbilisi Centre for Mathematical Sciences.

Received by the editors: 09 July 2019.
Accepted for publication: 02 January 2020.

$$\psi(q) := E(q, q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} = \frac{E_2^2}{E_1}$$

and

$$E(-q) := E(-q, -q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty = E_1.$$

A partition of a positive integer n is a finite non-increasing sequence of positive integers whose sum is n. For example, $\kappa = (5, 2, 1)$ is a partition of 8 since $8 = 5 + 2 + 1$. Let $p(n)$ denote the number of partitions of n. The generating function is given by

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty}. \tag{1.1}$$

An overpartition of a nonnegative integer n is a non-increasing sequence of natural numbers whose sum is n, and where the first occurrence of a number may be overlined. Let $\overline{p}(n)$ counts the number of overpartitions of n. Due to Corteel and Lovejoy [10], the generating function of $\overline{p}(n)$ is given by

$$\sum_{n=0}^{\infty} \overline{p}(n)q^n = \frac{(-q;q)_\infty}{(q;q)_\infty}. \tag{1.2}$$

Let $p_0(n) := p(n)$ and for a positive integer k, Ahmed et al.[3], define by

$$\sum_{n=0}^{\infty} p_k(n)q^n = \frac{1}{(q;q)_\infty (q^k;q^k)_\infty}, \tag{1.3}$$

where $p_k(n)$ be the number of 2-color partitions of n where one of the colors appears only in parts that are multiplies of k.

Baruah and Sarmah [6] studied some arithmetic properties of $p_k(n)$ for $k = 1$. In [7] Chan obtained some congruences modulo 3 for $p_k(n)$ where $k = 2$. Chen and Lin [8] proved some congruences modulo 5 and 7 with aid of modular forms for $p_k(n)$ where $k = 2$. Recently Ahmed et al. [3] found some of congruences modulo 5 for $p_k(n)$ for $k \in \{2, 3, 4\}$ by Ramanujan's theta function identities and some identities for the Rogers-Ramanujan continued fraction. More recently, Chern [20] proved a conjecture made by Ahmed et al. [3] on congruences modulo 5 for $p_k(n)$ also obtained some congruences modulo 7 for $p_k(n)$ where $k = 4$. See [1, 2, 18, 19].

We define,

$$\sum_{n=0}^{\infty} \overline{p}_k(n)q^n = \frac{(-q;q)_\infty (-q^k;q^k)_\infty}{(q;q)_\infty (q^k;q^k)_\infty}, \tag{1.4}$$

where $\overline{p}_k(n)$ be the number of overpartitions of n with 2-color in which one of the colors appears only in parts that are multiples of k.

Naika et al. [15] established some infinite families of congruences modulo small powers of 2 and 3 for $\overline{p}_k(n)$ where $k = 3$ with aid of q-series technique.

In this paper, we define $\overline{p}_{k,-2}(n)$ is given by

$$\sum_{n=0}^{\infty}\overline{p}_{k,-2}(n)q^n=\frac{(-q;q)_\infty^2(-q^k;q^k)_\infty^2}{(q;q)_\infty^2(q^k;q^k)_\infty^2}. \tag{1.5}$$

Note that $\overline{p}_{k,-2}(n)$ be the number of overpartition pairs of n with 2-color in which one of the colors appears only in parts that are multiples of k. We obtain several infinite families of congruences modulo 27, 32, 64 and 128 for $\overline{p}_{k,-2}(n)$ where k=2 and 3. The main results of this paper can be stated as follows.

Theorem 1.1. For each $n\geq 0$ and $\alpha\geq 0$, we have

$$\overline{p}_{2,-2}(16n+14)\equiv 0 \pmod{128}, \tag{1.6}$$
$$\overline{p}_{2,-2}(72n+42)\equiv 0 \pmod{128}, \tag{1.7}$$
$$\overline{p}_{2,-2}(72n+66)\equiv 0 \pmod{128}, \tag{1.8}$$
$$\overline{p}_{2,-2}(8\cdot 9^{\alpha+2}n+42\cdot 9^{\alpha+1})\equiv 0 \pmod{128}. \tag{1.9}$$

Theorem 1.2. For α, β and $\gamma\geq 0$, we have

$$\sum_{n=0}^{\infty}\overline{p}_{2,-2}\left(16\cdot 3^{4\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma}n+6\cdot 3^{4\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma}\right)q^n\equiv 64E_1^9 \pmod{128}, \tag{1.10}$$

$$\sum_{n=0}^{\infty}\overline{p}_{2,-2}\left(16\cdot 3^{4\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}n+14\cdot 3^{4\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}\right)q^n\equiv 64qE_5^9 \pmod{128}, \tag{1.11}$$

$$\sum_{n=0}^{\infty}\overline{p}_{2,-2}\left(16\cdot 3^{4\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma+1}n+10\cdot 3^{4\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma+1}\right)q^n\equiv 16q^2E_7^9 \pmod{128}, \tag{1.12}$$

$$\overline{p}_{2,-2}(16\cdot 3^{4\alpha+2}n+34\cdot 3^{4\alpha+1})\equiv 0 \pmod{128}, \tag{1.13}$$
$$\overline{p}_{2,-2}(16\cdot 3^{4\alpha+4}n+2\cdot 3^{4\alpha+3})\equiv 0 \pmod{128}, \tag{1.14}$$
$$\overline{p}_{2,-2}(16\cdot 3^{4\alpha+4}n+34\cdot 3^{4\alpha+3})\equiv 0 \pmod{128}, \tag{1.15}$$

$$\overline{p}_{2,-2}\left(16\cdot 3^{4\alpha+2}n+2\cdot 3^{4\alpha+1}\right) \equiv \begin{cases} 64 \pmod{128} & \text{if } n=k(3k+1)/2 \text{ for some } k\in Z, \\ 0 \pmod{128} & \text{otherwise.} \end{cases} \tag{1.16}$$

Corollary 1.3. For α, β and $\gamma \geq 0$, we have

$$\overline{p}_{2,-2}\left(16\cdot 3^{4\alpha}\cdot 5^{2\beta+2}\cdot 7^{2\gamma}n+14\cdot 3^{4\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}\right)\equiv 0 \pmod{128}, \tag{1.17}$$

$$\overline{p}_{2,-2}\left(16\cdot 3^{4\alpha}\cdot 5^{2\beta+2}\cdot 7^{2\gamma}n+46\cdot 3^{4\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}\right)\equiv 0 \pmod{128}, \tag{1.18}$$

$$\overline{p}_{2,-2}\left(16\cdot 3^{4\alpha}\cdot 5^{2\beta+2}\cdot 7^{2\gamma}n+46\cdot 3^{4\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}\right)\equiv 0 \pmod{128}, \tag{1.19}$$

$$\overline{p}_{2,-2}\left(16\cdot 3^{4\alpha}\cdot 5^{2\beta+2}\cdot 7^{2\gamma}n+78\cdot 3^{4\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}\right)\equiv 0 \pmod{128}, \tag{1.20}$$

$$\overline{p}_{2,-2}\left(16\cdot 3^{4\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma+2}n+26\cdot 3^{4\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma+1}\right)\equiv 0 \pmod{128}, \tag{1.21}$$

$$\overline{p}_{2,-2}\left(16\cdot 3^{4\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma+2}n+58\cdot 3^{4\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma+1}\right)\equiv 0 \pmod{128}, \tag{1.22}$$

$$\overline{p}_{2,-2}\left(16\cdot 3^{4\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma+2}n+74\cdot 3^{4\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma+1}\right)\equiv 0 \pmod{128}, \tag{1.23}$$

$$\overline{p}_{2,-2}\left(16\cdot 3^{4\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma+2}n+80\cdot 3^{4\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma+1}\right)\equiv 0 \pmod{128}, \tag{1.24}$$

$$\overline{p}_{2,-2}\left(16\cdot 3^{4\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma+2}n+106\cdot 3^{4\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma+1}\right)\equiv 0 \pmod{128}. \tag{1.25}$$

Theorem 1.4. For each $n \geq 0$ and $\alpha \geq 0$, we have

$$\overline{p}_{2,-2}(144n+66)\equiv 0 \pmod{64}, \tag{1.26}$$

$$\overline{p}_{2,-2}(144n+114)\equiv 0 \pmod{64}, \tag{1.27}$$

$$\overline{p}_{2,-2}(600n+170)\equiv 0 \pmod{64}, \tag{1.28}$$

$$\overline{p}_{2,-2}(600n+290)\equiv 0 \pmod{64}, \tag{1.29}$$

$$\overline{p}_{2,-2}(600n+410)\equiv 0 \pmod{64}, \tag{1.30}$$

$$\overline{p}_{2,-2}(600n+530)\equiv 0 \pmod{64}, \tag{1.31}$$

$$\overline{p}_{2,-2}(16\cdot 9^{\alpha+2}n+66\cdot 9^{\alpha+1})\equiv 0 \pmod{64}, \tag{1.32}$$

$$\overline{p}_{2,-2}(24\cdot 25^{\alpha+2}n+170\cdot 25^{\alpha+1})\equiv 0 \pmod{64}. \tag{1.33}$$

Theorem 1.5. For each $n \geq 0$ and $\alpha \geq 0$, we have

$$\overline{p}_{2,-2}(8n+5)\equiv 0 \pmod{32}, \tag{1.34}$$

$$\overline{p}_{2,-2}(200n+73)\equiv 0 \pmod{32}, \tag{1.35}$$

$$\overline{p}_{2,-2}(200n+113)\equiv 0 \pmod{32}, \tag{1.36}$$

$$\overline{p}_{2,-2}(200n+153)\equiv 0 \pmod{32}, \tag{1.37}$$

$$\overline{p}_{2,-2}(200n+193)\equiv 0 \pmod{32}, \tag{1.38}$$

$$\overline{p}_{2,-2}\left(8\cdot 25^{\alpha+2}n+\frac{220\cdot 25^{\alpha+1}-1}{3}\right)\equiv 0 \pmod{32}. \tag{1.39}$$

Theorem 1.6. For each $n \geq 0$ and $\alpha \geq 0$, we have

$$\overline{p}_{3,-2}(6n+5) \equiv 0 \pmod{27}, \tag{1.40}$$
$$\overline{p}_{3,-2}(24n+20) \equiv 0 \pmod{27}, \tag{1.41}$$
$$\overline{p}_{3,-2}(150n+20) \equiv 0 \pmod{27}, \tag{1.42}$$
$$\overline{p}_{3,-2}(150n+80) \equiv 0 \pmod{27}, \tag{1.43}$$
$$\overline{p}_{3,-2}(150n+110) \equiv 0 \pmod{27}, \tag{1.44}$$
$$\overline{p}_{3,-2}(150n+140) \equiv 0 \pmod{27}, \tag{1.45}$$
$$\overline{p}_{3,-2}(6\cdot 4^{\alpha+2}n+5\cdot 4^{\alpha+2}) \equiv 0 \pmod{27}, \tag{1.46}$$
$$\overline{p}_{3,-2}(6\cdot 25^{\alpha+2}n+20\cdot 25^{\alpha+2}) \equiv 0 \pmod{27}. \tag{1.47}$$

Theorem 1.7. For each $n \geq 0$ and $\alpha \geq 0$, we have

$$\overline{p}_{3,-2}(12n+7) \equiv 0 \pmod{16}, \tag{1.48}$$
$$\overline{p}_{3,-2}(12n+11) \equiv 0 \pmod{16}, \tag{1.49}$$
$$\overline{p}_{3,-2}(36n+21) \equiv 0 \pmod{16}, \tag{1.50}$$
$$\overline{p}_{3,-2}(36n+33) \equiv 0 \pmod{16}, \tag{1.51}$$

$$\overline{p}_{3,-2}(300n+85) \equiv 0 \pmod{16}, \tag{1.52}$$
$$\overline{p}_{3,-2}(300n+145) \equiv 0 \pmod{16}, \tag{1.53}$$
$$\overline{p}_{3,-2}(300n+205) \equiv 0 \pmod{16}, \tag{1.54}$$
$$\overline{p}_{3,-2}(300n+265) \equiv 0 \pmod{16}, \tag{1.55}$$
$$\overline{p}_{3,-2}(4\cdot 9^{\alpha+2}n+21\cdot 9^{\alpha+1}) \equiv 0 \pmod{16}, \tag{1.56}$$
$$\overline{p}_{3,-2}(12\cdot 25^{\alpha+2}n+84\cdot 25^{\alpha+1}) \equiv 0 \pmod{16}, \tag{1.57}$$
$$\overline{p}_{3,-2}(12n+9) \equiv \overline{p}_{3,-2}(4n+3) \pmod{16}. \tag{1.58}$$

2 Preliminaries

In this section, we list some lemmas which play a vital role in proving our main results.

Lemma 2.1. The following 2-dissections hold:

$$E_1^2 = \frac{E_2E_8^5}{E_4^2E_{16}^2} - 2q\frac{E_2E_{16}^2}{E_8}, \tag{2.1}$$
$$\frac{1}{E_1^2} = \frac{E_8^5}{E_2^5E_{16}^2} + 2q\frac{E_4^2E_{16}^2}{E_2^5E_8}, \tag{2.2}$$
$$\frac{1}{E_1^4} = \frac{E_4^{14}}{E_2^{14}E_8^4} + 4q\frac{E_4^2E_8^4}{E_2^{10}}. \tag{2.3}$$

Lemma (2.1) is a consequence of dissection formulas of Ramanujan, collected in Berndt's book [4, p. 40, Entry 25].

Lemma 2.2. The following 2-dissection holds:

$$\frac{E_3^3}{E_1} = \frac{E_4^3 E_6^2}{E_2^2 E_{12}} + q\frac{E_{12}^3}{E_4}. \tag{2.4}$$

Hirschhorn, Garvan and Borwein [13] proved the equation (2.4).

Lemma 2.3. The following 2-dissection formula holds:

$$\frac{1}{E_1 E_3} = \frac{E_8^2 E_{12}^5}{E_2^2 E_4 E_6^4 E_{24}^2} + q\frac{E_4^5 E_{24}^2}{E_2^4 E_6^2 E_8^2 f_{12}}. \tag{2.5}$$

Equation (2.5) was proved by Baruah and Ojah [5].

Lemma 2.4. [4, p. 345, Entry 1 (iv)]. The following 3-dissection holds:

$$\begin{aligned} E_1^3 &= \frac{E_6 E_9^6}{E_3 E_{18}^3} + 4q^3\frac{E_3^2 E_{18}^6}{E_6^2 E_9^3} - 3qE_9^3 \\ &\equiv E_3 + qE_9^3 \pmod 2. \end{aligned} \tag{2.6}$$

Lemma 2.5. The following 3-dissection holds:

$$E_1 E_2 = \frac{E_6 E_9^4}{E_3 E_{18}^2} - qE_9 E_{18} - 2q^2\frac{E_3 E_{18}^4}{E_6 E_9^2}. \tag{2.7}$$

One can see this identity in [11].

Lemma 2.6. [5, Lemma 2.6] The following 3-dissection formula holds:

$$\frac{E_4}{E_1} = \frac{E_{12} E_{18}^4}{E_3^3 E_{36}^2} + q\frac{E_6^2 E_9^3 E_{36}}{E_3^4 E_{18}^2} + 2q^2\frac{E_6 E_{18} E_{36}}{E_3^3}. \tag{2.8}$$

Lemma 2.7. The following 3-dissection holds:

$$\frac{E_2}{E_1^2} = \frac{E_6^4 E_9^6}{E_3^8 E_{18}^3} + 2q\frac{E_6^3 E_9^3}{E_3^7} + 4q^2\frac{E_6^2 E_{18}^3}{E_3^6}. \tag{2.9}$$

Equation (2.9) was proved by Hirschhorn and Sellers [14].

Lemma 2.8. We have the following 5-dissection

$$E_1 = E_{25}(R(q^5)^{-1} - q - q^2 R(q^5)), \tag{2.10}$$

where $R(q) = \dfrac{E(-q, -q^4)}{E(-q^2, -q^3)}$.

For a proof of (2.10), see [12], [21].

Lemma 2.9. [4, 3, p. 303, Entry 17(v)], we have the following 7-dissection

$$E_1 = E_{49}\left(\frac{B(q^7)}{C(q^7)} - q\frac{A(q^7)}{B(q^7)} - q^2 + q^5\frac{C(q^7)}{A(q^7)}\right), \tag{2.11}$$

where $A(q) := \dfrac{E(-q^3,-q^4)}{E(-q^2)}$, $B(q) := \dfrac{E(-q^2,-q^5)}{E(-q^2)}$ and $C(q) := \dfrac{E(-q,-q^6)}{E(-q^2)}$.

Lemma 2.10. For any prime p and positive integer n,

$$E_1^{p^n} \equiv E_p^{p^{n-1}} \pmod{p^n}. \tag{2.12}$$

3 Proof of Theorem 1.1

Setting $k = 2$ in (1.5), we see that

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{p}_{2,-2}(n)q^n &= \frac{(-q;q)_\infty^2(-q^2;q^2)_\infty^2}{(q;q)_\infty^2(q^2;q^2)_\infty^2}\\ &= \frac{1}{E_1^4}\frac{E_4^2}{E_2^2}.\end{aligned} \tag{3.1}$$

Substituting (2.3) into (3.1), we find that

$$\sum_{n=0}^{\infty}\overline{p}_{2,-2}(n)q^n = \frac{E_4^{16}}{E_2^{16}E_8^4} + 4q\frac{E_4^4E_8^4}{E_2^{12}}, \tag{3.2}$$

which implies,

$$\sum_{n=0}^{\infty}\overline{p}_{2,-2}(2n)q^n = \frac{1}{(E_1^4)^4}\frac{E_2^{16}}{E_4^4}. \tag{3.3}$$

Employing (2.3) into (3.3) and extracting the terms q^{2n+1}, we have

$$\sum_{n=0}^{\infty}\overline{p}_{2,-2}(4n+2)q^n = 16\frac{E_2^{40}}{E_1^{36}E_4^8} + 256q\frac{E_2^{16}E_4^8}{E_1^{28}} \tag{3.4}$$

$$\equiv 16\frac{E_2^{40}}{E_1^{36}E_4^8} \pmod{256}. \tag{3.5}$$

Invoking (2.12) in (3.4), we get

$$\sum_{n=0}^{\infty}\overline{p}_{2,-2}(4n+2)q^n \equiv 16\frac{E_2^8}{E_1^4} \pmod{256}. \tag{3.6}$$

Substituting (2.3) into (3.6), we obtain

$$\sum_{n=0}^{\infty}\overline{p}_{2,-2}(4n+2)q^n \equiv 16\frac{E_4^{14}}{E_2^6E_8^4}+64q\frac{E_4^2E_8^4}{E_2^2} \pmod{256}, \tag{3.7}$$

which yields

$$\sum_{n=0}^{\infty}\overline{p}_{2,-2}(8n+6)q^n \equiv 64\frac{E_2^2E_4^4}{E_1^2} \pmod{256}. \tag{3.8}$$

Using (2.12) in (3.8), we get

$$\sum_{n=0}^{\infty}\overline{p}_{2,-2}(8n+6)q^n \equiv 64E_2^9 \pmod{128}. \tag{3.9}$$

Congruence (1.6) easily obtained from the above equation.

From the equation (3.7), we find that

$$\sum_{n=0}^{\infty}\overline{p}_{2,-2}(8n+2)q^n \equiv 16\frac{E_2^{14}}{E_1^6E_4^4} \pmod{256}, \tag{3.10}$$

Invoking (2.12) in (3.10), we have

$$\sum_{n=0}^{\infty}\overline{p}_{2,-2}(8n+2)q^n \equiv 16E_1^2E_2^2 \pmod{128}. \tag{3.11}$$

Substituting (2.7) into (3.11), we arrive at

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{p}_{2,-2}(8n+2)q^n \equiv 16\frac{E_6^2E_9^8}{E_3^2E_{18}^4}+80q^2E_9^2E_{18}^2+96q\frac{E_6E_9^5}{E_3E_{18}}\\ +64q^3\frac{E_{18}^5E_3}{E_9E_6}+64q^4\frac{E_3^2E_{18}^8}{E_6^2E_9^4} \pmod{128}.\end{aligned} \tag{3.12}$$

From the above equation, we have

$$\sum_{n=0}^{\infty}\overline{p}_{2,-2}(24n+18)q^n \equiv 80E_3^2E_6^2 \pmod{128}. \tag{3.13}$$

Extracting the terms q^{3n+1} and q^{3n+2} from (3.13), we obtain (1.7) and (1.8).

From (3.13), we can find

$$\sum_{n=0}^{\infty}\overline{p}_{2,-2}(72n+18)q^n \equiv 80E_1^2E_2^2 \pmod{128}. \tag{3.14}$$

From (3.11) and (3.14), we arrive at

$$\overline{p}_{2,-2}(72n+18) \equiv 5\overline{p}_{2,-2}(8n+2) \pmod{128}. \tag{3.15}$$

Using the above relation and by mathematical induction on α, we have

$$\overline{p}_{2,-2}(8\cdot 9^{\alpha+1}n + 2\cdot 9^{\alpha+1}) \equiv 5^{\alpha+1}\cdot \overline{p}_{2,-2}(8n+2) \pmod{128}. \tag{3.16}$$

Using (1.7) in (3.16), we get (1.9).

4 Proof of Theorem 1.2

From the equation (3.9), we find that

$$\sum_{n=0}^{\infty} \overline{p}_{2,-2}(16n+6)q^n \equiv 64E_1^9 \pmod{128}. \tag{4.1}$$

The equation (4.1) is the $\alpha=\beta=\gamma=0$ case of (1.10).
Let us consider the case $\beta=\gamma=0$. Suppose that the congruence (1.10) holds for some integer $\alpha \geq 0$. Employing the equation (2.6) in (1.10) with $\beta=\gamma=0$, we have

$$\sum_{n=0}^{\infty} \overline{p}_{2,-2}(16\cdot 3^{4\alpha}n + 6\cdot 3^{4\alpha})q^n \equiv 64(E_3 + qE_9^3)^3 \pmod{128}, \tag{4.2}$$

which implies,

$$\sum_{n=0}^{\infty} \overline{p}_{2,-2}(16\cdot 3^{4\alpha}n + 6\cdot 3^{4\alpha})q^n \equiv 64(q^3E_9^9 + q^2E_3E_9^6 + qE_3^2E_9^3 + E_3^3) \pmod{128}. \tag{4.3}$$

Extracting the terms involving q^{3n} from (4.3), we arrive at

$$\sum_{n=0}^{\infty} \overline{p}_{2,-2}(16\cdot 3^{4\alpha+1}n + 2\cdot 3^{4\alpha+1})q^n \equiv 64(qE_3^9 + E_1^3) \pmod{128}. \tag{4.4}$$

Employing the equation (2.6) in (4.4), we can find

$$\sum_{n=0}^{\infty} \overline{p}_{2,-2}(16\cdot 3^{4\alpha+1}n + 2\cdot 3^{4\alpha+1})q^n \equiv 64(qE_3^9 + qE_9^3 + E_3) \pmod{128}. \tag{4.5}$$

Congruences (1.13) and (1.16) easily obtained from (4.5). Extracting the terms involving q^{3n+1} from (4.5), we get

$$\sum_{n=0}^{\infty} \overline{p}_{2,-2}(16\cdot 3^{4\alpha+2}n + 2\cdot 3^{4\alpha+3})q^n \equiv 64(E_1^9 + E_3^3) \pmod{128}. \tag{4.6}$$

Substituting the equation (2.6) in (4.6), we obtain

$$\sum_{n=0}^{\infty}\overline{p}_{2,-2}(16\cdot 3^{4\alpha+2}n+2\cdot 3^{4\alpha+3})q^n\equiv 64(q^3E_9^9+q^2E_3E_9^6+qE_3^2E_9^3)\pmod{128}, \tag{4.7}$$

which implies,

$$\sum_{n=0}^{\infty}\overline{p}_{2,-2}(16\cdot 3^{4\alpha+3}n+2\cdot 3^{4\alpha+3})q^n\equiv 64qE_3^9\pmod{128}. \tag{4.8}$$

Congruences (1.14) and (1.15) follows from the above equation. Extracting the terms involving q^{3n+1} from (4.8), we find that

$$\sum_{n=0}^{\infty}\overline{p}_{2,-2}(8\cdot 3^{4\alpha+4}n+6\cdot 3^{4\alpha+4})q^n\equiv 64E_1^9\pmod{128}, \tag{4.9}$$

which implies that (1.10) is true for $\alpha+1$. Hence by induction (1.10) is true for any non-negative integer α and $\beta=\gamma=0$.

Let us consider the case $\gamma=0$, suppose that the congruence (1.10) holds for some integer $\alpha,\ \beta\geq 0$. Substituting (2.10) in (1.10),

$$\sum_{n=0}^{\infty}\overline{p}_{2,-2}(16\cdot 3^{4\alpha}\cdot 5^{2\beta}n+6\cdot 3^{4\alpha}\cdot 5^{2\beta})q^n\equiv 64E_{25}^9(R(q^5)^{-1}-q-q^2R(q^5))^9\pmod{128}. \tag{4.10}$$

Extracting the terms involving q^{5n+4} from (4.10), we arrive at

$$\sum_{n=0}^{\infty}\overline{p}_{2,-2}(16\cdot 3^{4\alpha}\cdot 5^{2\beta+1}n+14\cdot 3^{4\alpha}\cdot 5^{2\beta+1})q^n\equiv 64qE_5^9\pmod{128}, \tag{4.11}$$

which yields,

$$\sum_{n=0}^{\infty}\overline{p}_{2,-2}(16\cdot 3^{4\alpha}\cdot 5^{2\beta+2}n+6\cdot 3^{4\alpha}\cdot 5^{2\beta+2})q^n\equiv 64E_1^9\pmod{128}, \tag{4.12}$$

which implies that (1.10) is true for $\beta+1$. Hence by induction (1.10) is true for $\alpha,\beta\geq 0$ and $\gamma=0$.

Now, Suppose that the congruence (1.10) holds for some integers α, β and $\gamma\geq 0$.

Employing (2.11) in the equation (1.10), we find that

$$\begin{aligned}&\sum_{n=0}^{\infty}\overline{p}_{2,-2}\left(16\cdot 3^{4\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma}n+6\cdot 3^{4\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma}\right)q^n\\&\equiv 64E_{49}^9\left(\frac{B(q^7)}{C(q^7)}-q\frac{A(q^7)}{B(q^7)}-q^2+q^5\frac{C(q^7)}{A(q^7)}\right)^9\pmod{128}.\end{aligned} \tag{4.13}$$

Extracting the terms involving q^{7n+4} from (4.13), we get

$$\sum_{n=0}^{\infty}\overline{p}_{2,-2}\left(16\cdot 3^{4\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma+1}n+10\cdot 3^{4\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma+1}\right)q^n\equiv 16q^2E_7^9\pmod{128}, \tag{4.14}$$

which prove (1.12). Extracting the coefficient of q^{7n+2} in (4.14),

$$\sum_{n=0}^{\infty}\overline{p}_{2,-2}\left(16\cdot 3^{4\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma+2}n+6\cdot 3^{4\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma+2}\right)q^n\equiv 64E_1^9 \pmod{128}, \tag{4.15}$$

which implies that (1.10) is true for $\gamma+1$. Hence, by induction (1.10) is true for any non-negative integers α, β and γ. This completes the proof.
Employing (2.10) in (1.10), we get (1.11).

5 Proof of Theorem 1.4

From (3.12), we have

$$\sum_{n=0}^{\infty}\overline{p}_{2,-2}(24n+2)q^n\equiv 16\frac{E_2^2E_3^8}{E_1^2E_6^4}+64q\frac{E_6^5E_1}{E_3E_2} \pmod{128}, \tag{5.1}$$

which implies,

$$\sum_{n=0}^{\infty}\overline{p}_{2,-2}(24n+2)q^n\equiv 16\frac{E_2^2E_3^8}{E_1^2E_6^4} \pmod{64}. \tag{5.2}$$

Using (2.12) in (5.2), we get

$$\sum_{n=0}^{\infty}\overline{p}_{2,-2}(24n+2)q^n\equiv 16E_1^2 \pmod{64}. \tag{5.3}$$

Substituting (2.10) into (5.3), we find that

$$\sum_{n=0}^{\infty}\overline{p}_{2,-2}(24n+2)q^n\equiv 16E_{25}^2(R(q^5)^{-1}-q-q^2R(q^5))^2 \pmod{64}. \tag{5.4}$$

Extracting the coefficient of q^{5n+2} in (5.4), we have

$$\sum_{n=0}^{\infty}\overline{p}_{2,-2}(120n+50)q^n\equiv 48E_5^2 \pmod{64}. \tag{5.5}$$

Congruences (1.28) to (1.31) follow from (5.5).

From the equation (5.5), we arrive at

$$\sum_{n=0}^{\infty}\overline{p}_{2,-2}(600n+50)q^n\equiv 48E_1^2 \pmod{64}. \tag{5.6}$$

From (5.3) and (5.6), we get

$$\overline{p}_{2,-2}(600n+50)q^n\equiv 3\overline{p}_{2,-2}(24n+2) \pmod{64}. \tag{5.7}$$

Using the above relation and by mathematical induction on α, we can find

$$\overline{p}_{2,-2}(24\cdot 25^{\alpha+1}n+2\cdot 25^{\alpha+1})\equiv 3^{\alpha+1}\cdot\overline{p}_{2,-2}(24n+2)\pmod{64}. \tag{5.8}$$

Using (1.28) in (5.8), we obtain (1.33).

Employing (2.1) into (3.11), we find that

$$\sum_{n=0}^{\infty}\overline{p}_{2,-2}(8n+2)q^n\equiv 16\frac{E_2^3E_8^5}{E_4^2E_{16}^2}+96q\frac{E_2^3E_{16}^2}{E_8}\pmod{128}, \tag{5.9}$$

which yields

$$\sum_{n=0}^{\infty}\overline{p}_{2,-2}(16n+2)q^n\equiv 16\frac{E_1^3E_4^5}{E_2^2E_8^2}\pmod{128}. \tag{5.10}$$

Invoking (2.12) in (5.10), we have

$$\sum_{n=0}^{\infty}\overline{p}_{2,-2}(16n+2)q^n\equiv 16\frac{E_4}{E_1}\pmod{64}. \tag{5.11}$$

Substituting (2.8) into (5.11), we arrive at

$$\sum_{n=0}^{\infty}\overline{p}_{2,-2}(16n+2)q^n\equiv 16\frac{E_{12}E_{18}^4}{E_3^3E_{36}^2}+16q\frac{E_6^2E_9^3E_{36}}{E_3^4E_{18}^2}+32q^2\frac{E_6E_{18}E_{36}}{E_3^3}\pmod{64}, \tag{5.12}$$

which implies,

$$\sum_{n=0}^{\infty}\overline{p}_{2,-2}(48n+18)q^n\equiv 16\frac{E_2^2E_3^3E_{12}}{E_1^4E_6^2}\pmod{64}. \tag{5.13}$$

Using (2.12) in (5.13), we get

$$\sum_{n=0}^{\infty}\overline{p}_{2,-2}(48n+18)q^n\equiv 16\frac{E_{12}}{E_3}\pmod{64}. \tag{5.14}$$

Congruences (1.26) and (1.27) easily follow from (5.14).

From the equation (5.14), we have

$$\sum_{n=0}^{\infty}\overline{p}_{2,-2}(144n+18)q^n\equiv 16\frac{E_4}{E_1}\pmod{64}. \tag{5.15}$$

From (5.11) and (5.15), we get

$$\overline{p}_{2,-2}(144n+18)q^n\equiv\overline{p}_{2,-2}(16n+2)\pmod{64}. \tag{5.16}$$

Using the above relation and by mathematical induction on α, we find that

$$\overline{p}_{2,-2}(16\cdot 9^{\alpha+1}n+2\cdot 9^{\alpha+1})\equiv\overline{p}_{2,-2}(16n+2)\pmod{64}. \tag{5.17}$$

Using (1.26) in (5.17), we arrive at (1.32).

6 Proof of Theorem 1.5

From (3.2), we can written as

$$\sum_{n=0}^{\infty}\overline{p}_{2,-2}(2n+1)q^n = 4q\frac{1}{(E_1^4)^3}E_2^4E_4^4. \tag{6.1}$$

Employing (2.3) into (6.1), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{p}_{2,-2}(2n+1)q^n &= 4\frac{E_4^{46}}{E_2^{38}E_8^{12}} + 48q\frac{E_4^{34}}{E_2^{34}E_8^4} + 192q^2\frac{E_4^{22}E_8^4}{E_2^{30}}\\ &\quad + 256q^3\frac{E_4^{10}E_8^{12}}{E_2^{26}},\end{aligned} \tag{6.2}$$

which implies,

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{p}_{2,-2}(4n+1)q^n &= 4\frac{E_2^{46}}{E_1^{38}E_4^{12}} + 192q\frac{E_2^{22}E_4^4}{E_1^{30}}\\ &\equiv 4\frac{E_2^{46}}{E_1^{38}E_4^{12}} \pmod{64}.\end{aligned} \tag{6.3}$$

Invoking (2.12) in (6.3), we find that

$$\sum_{n=0}^{\infty}\overline{p}_{2,-2}(4n+1)q^n = 4E_2^4 \pmod{32}. \tag{6.4}$$

Congruence (1.34) easily follows from (6.4).

From (6.4), we have

$$\sum_{n=0}^{\infty}\overline{p}_{2,-2}(8n+1)q^n = 4E_1^4 \pmod{32}. \tag{6.5}$$

Employing (2.10) into (6.5), we arrive at

$$\sum_{n=0}^{\infty}\overline{p}_{2,-2}(8n+1)q^n \equiv 4E_{25}^4(R(q^5)^{-1} - q - q^2R(q^5))^4 \pmod{32}. \tag{6.6}$$

Extracting the coefficient of q^{5n+2} in (6.6), we get

$$\sum_{n=0}^{\infty}\overline{p}_{2,-2}(40n+33)q^n \equiv 12E_5^4 \pmod{32}. \tag{6.7}$$

Congruences (1.35) to (1.38) obtained from (6.7).

From (6.7), we have

$$\sum_{n=0}^{\infty}\overline{p}_{2,-2}(200n+33)q^n \equiv 12E_1^4 \pmod{32}. \tag{6.8}$$

From (6.5) and (6.8), we find that

$$\overline{p}_{2,-2}(200n+33)q^n \equiv 3\overline{p}_{2,-2}(8n+1) \pmod{32}. \tag{6.9}$$

Using the above relation and by mathematical induction on α, we arrive at

$$\overline{p}_{2,-2}\left(8\cdot 25^{\alpha+1}n+\frac{8\cdot 25^{\alpha+1}-2}{6}\right) \equiv 3^{\alpha+1}\cdot\overline{p}_{2,-2}(8n+1) \pmod{32}. \tag{6.10}$$

Using (1.35) in (6.10), we get (1.39).

7 Proof of Theorem 1.6

Setting $k=3$ in (1.5), we arrive at

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{p}_{3,-2}(n)q^n &= \frac{(-q;q)_\infty^2(-q^3;q^3)_\infty^2}{(q;q)_\infty^2(q^3;q^3)_\infty^2}\\ &= \frac{E_2^2}{E_1^4}\frac{E_6^2}{E_3^4}.\end{aligned} \tag{7.1}$$

Employing (2.9) into (7.1), we get

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{p}_{3,-2}(n)q^n = \frac{E_6^{10}E_9^{12}}{E_3^{20}E_{18}^6} &+ 4q\frac{E_6^9E_9^9}{E_3^{19}E_{18}^3} + 12q^2\frac{E_6^8E_9^6}{E_3^{18}}\\ &+ 16q^3\frac{E_6^7E_9^3E_{18}^3}{E_3^{17}} + 16q^4\frac{E_6^6E_{18}^6}{E_3^{16}},\end{aligned} \tag{7.2}$$

which implies,

$$\sum_{n=0}^{\infty}\overline{p}_{3,-2}(3n+2)q^n \equiv 12\frac{E_2^8E_3^6}{E_1^{18}}. \tag{7.3}$$

Invoking (2.12) in (7.3), we find that

$$\sum_{n=0}^{\infty}\overline{p}_{3,-2}(3n+2)q^n \equiv 12E_2^8 \pmod{27}. \tag{7.4}$$

Congruence (1.40) easily follows from the above equation.

From the equation (7.4), we have

$$\sum_{n=0}^{\infty}\overline{p}_{3,-2}(6n+2)q^n \equiv 12E_1^8 \pmod{27}. \tag{7.5}$$

Using (2.12) in (7.5), we can expressed as

$$\sum_{n=0}^{\infty}\overline{p}_{3,-2}(6n+2)q^n \equiv 12\frac{E_3^3}{E_1} \pmod{27}. \tag{7.6}$$

Substituting (2.4) into (7.6), we obtain

$$\sum_{n=0}^{\infty}\overline{p}_{3,-2}(6n+2)q^n \equiv 12\frac{E_4^3E_6^2}{E_2^2E_{12}} + 12q\frac{E_{12}^3}{E_4} \pmod{27}, \tag{7.7}$$

which yields

$$\sum_{n=0}^{\infty}\overline{p}_{3,-2}(12n+8)q^n \equiv 12\frac{E_6^3}{E_2} \pmod{27}. \tag{7.8}$$

Extracting the coefficient of q^{2n+1} in (7.8), we get (1.41).

From (7.8), we can find that

$$\sum_{n=0}^{\infty}\overline{p}_{3,-2}(24n+8)q^n \equiv 12\frac{E_3^3}{E_1} \pmod{27}. \tag{7.9}$$

From (7.6) and (7.9), we have

$$\overline{p}_{3,-2}(24n+8) \equiv \overline{p}_{3,-2}(6n+2) \pmod{27}. \tag{7.10}$$

Using the above relation and by mathematical induction on α, we get

$$\overline{p}_{3,-2}(6\cdot 4^{\alpha+1}n + 8\cdot 4^{\alpha}) \equiv \overline{p}_{3,-2}(6n+2) \pmod{27}. \tag{7.11}$$

Using (1.41) in (7.11), we obtain (1.46).

Employing (2.10) into (7.5), we arrive at

$$\sum_{n=0}^{\infty}\overline{p}_{3,-2}(6n+2)q^n \equiv 12E_{25}^8(R(q^5)^{-1} - q - q^2R(q^5))^8 \pmod{27}. \tag{7.12}$$

Extracting the coefficient of q^{5n+3} in (7.12), we have

$$\sum_{n=0}^{\infty}\overline{p}_{3,-2}(30n+20)q^n \equiv 12qE_5^8 \pmod{27}. \tag{7.13}$$

Congruences (1.42) to (1.45) obtained from (7.13).

From (7.13), we find that

$$\sum_{n=0}^{\infty}\overline{p}_{3,-2}(15n+50)q^n \equiv 12E_1^8 \pmod{27}. \tag{7.14}$$

From (7.5) and (7.14), we arrive at

$$\overline{p}_{3,-2}(15n+50) \equiv \overline{p}_{3,-2}(6n+2) \pmod{27}. \tag{7.15}$$

Using the above relation and by mathematical induction on α, we get

$$\overline{p}_{3,-2}(6\cdot 25^{\alpha+1}n + 2\cdot 25^{\alpha+1}) \equiv \overline{p}_{3,-2}(6n+2) \pmod{27}. \tag{7.16}$$

Using (1.42) in (7.16), we obtain (1.47).

8 Proof of Theorem 1.7

Employing (2.5) into (7.1), we find that

$$\sum_{n=0}^{\infty}\overline{p}_{3,-2}(n)q^n = \frac{E_8^8E_{12}^{20}}{E_2^6E_6^{14}E_4^4E_{28}^8} + 4q\frac{E_8^4E_{12}^{14}E_4^2}{E_2^8E_6^{12}E_{24}^4} + 6q^2\frac{E_{12}^8E_4^8}{E_2^{10}E_6^{10}} + 4q^3\frac{E_{12}^2E_4^{14}E_{24}^4}{E_2^{12}E_6^8E_8^4} + q^4\frac{E_4^{20}E_{24}^8}{E_2^{14}E_6^6E_8^8E_{12}^4}, \tag{8.1}$$

which implies,

$$\sum_{n=0}^{\infty}\overline{p}_{3,-2}(2n+1)q^n = 4\frac{E_4^4E_6^{14}E_2^2}{E_1^8E_3^{12}E_{12}^4} + 4q\frac{E_6^2E_2^{14}E_4^4}{E_1^{12}E_3^8E_4^4}. \tag{8.2}$$

Invoking (2.12) in (8.2), we arrive at

$$\sum_{n=0}^{\infty}\overline{p}_{3,-2}(2n+1)q^n \equiv 4\frac{E_4^4}{E_2^2} + 4q\frac{E_{12}^4}{E_6^2} \pmod{16}, \tag{8.3}$$

which yields

$$\sum_{n=0}^{\infty}\overline{p}_{3,-2}(4n+1)q^n \equiv 4\frac{E_2^4}{E_1^2} \pmod{16}. \tag{8.4}$$

But

$$\frac{E_2^2}{E_1} = \psi(q), \tag{8.5}$$

we find that

$$\frac{E_2^2}{E_1} = \frac{E_6E_9^2}{E_3E_{18}} + q\frac{E_{18}^2}{E_9}. \tag{8.6}$$

Substituting (8.6) into (8.4), we get

$$\sum_{n=0}^{\infty}\overline{p}_{3,-2}(4n+1)q^n \equiv 4\frac{E_6^2E_9^4}{E_3^2E_{18}^2} + 8q\frac{E_6E_9E_{18}}{E_3} + 4q^2\frac{E_{18}^4}{E_9^2} \pmod{16}, \tag{8.7}$$

which implies,

$$\sum_{n=0}^{\infty}\overline{p}_{3,-2}(12n+9)q^n \equiv 4\frac{E_6^4}{E_3^2} \pmod{16}. \tag{8.8}$$

Congruences (1.50) and (1.51) follow from (8.8).

From the equation (8.8), we have

$$\sum_{n=0}^{\infty}\overline{p}_{3,-2}(36n+9)q^n \equiv 4\frac{E_2^4}{E_1^2} \pmod{16}. \tag{8.9}$$

From (8.4) and (8.9), we arrive at

$$\overline{p}_{3,-2}(36n+9) \equiv \overline{p}_{3,-2}(4n+1) \pmod{16}. \tag{8.10}$$

Using the above relation and by mathematical induction on α, we can find

$$\overline{p}_{3,-2}(4\cdot 9^{\alpha+1}n+9^{\alpha+1}) \equiv \overline{p}_{3,-2}(4n+1) \pmod{16}. \tag{8.11}$$

Using (1.50) in (8.11), we get (1.56).

From (8.3), we can written as

$$\sum_{n=0}^{\infty}\overline{p}_{3,-2}(4n+3)q^n \equiv 4\frac{E_6^4}{E_3^2} \pmod{16}. \tag{8.12}$$

Congruences (1.48) and (1.49) follow from (8.12). Comparing (8.8) and (8.12), we get (1.58).

From the equation (8.7), we have

$$\sum_{n=0}^{\infty}\overline{p}_{3,-2}(12n+1)q^n \equiv 4\frac{E_2^2E_3^4}{E_1^2E_6^2} \pmod{16}. \tag{8.13}$$

Using (2.12) in (8.13), we arrive at

$$\sum_{n=0}^{\infty}\overline{p}_{3,-2}(12n+1)q^n \equiv 4E_1^2 \pmod{16}. \tag{8.14}$$

Employing (2.10) into (8.14), we obtain

$$\sum_{n=0}^{\infty}\overline{p}_{3,-2}(12n+1)q^n \equiv 4E_{25}^2(R(q^5)^{-1}-q-q^2R(q^5))^2 \pmod{16}. \tag{8.15}$$

Extracting the coefficient of q^{5n+2} in (8.15), we find that

$$\sum_{n=0}^{\infty}\overline{p}_{3,-2}(60n+25)q^n \equiv 12E_5^2 \pmod{16}. \tag{8.16}$$

Congruences (1.52) to (1.55) easily follow from (8.16).

From the equation (8.16), we can written as

$$\sum_{n=0}^{\infty} \overline{p}_{3,-2}(300n+25)q^n \equiv 12E_1^2 \pmod{16}. \tag{8.17}$$

From (8.14) and (8.17), we find that

$$\overline{p}_{3,-2}(300n+25)q^n \equiv 3\overline{p}_{3,-2}(12n+1) \pmod{16}. \tag{8.18}$$

Using the above relation and by mathematical induction on α, we get

$$\overline{p}_{3,-2}(12\cdot 25^{\alpha+1}n + 25^{\alpha+1}) \equiv 3^{\alpha+1}\cdot \overline{p}_{3,-2}(12n+1) \pmod{16}. \tag{8.19}$$

Using (1.52) in (8.19), we obtain (1.57).

9 Proof of Corollary 1.3

Proof of Corollary (1.3) follows from the Theorem (1.2).

Aknowledgement

The authors would like to thank the anonymous referee for helpful comments and suggestions which enhanced the quality of presentation of this paper.

References

[1] C. Adiga, N. A. S. Bulkhali, D. Ranganatha and H. M. Srivastava, *Some new modular relations for the Rogers-Ramanujan type functions of order eleven with applications to partitions*, J. Number Theory, 158 (2016), 281–297.

[2] C. Adiga, N. A. S. Bulkhali, Yilmaz Simsek and H. M. Srivastava, *A Continued fraction of Ramanujan and some Ramanujan-Weber class invariants*, Filomat, 31 (2017), 3975–3997.

[3] Z. Ahmed, N. D. Baruah and M. G. Dastidar, *New congruences modulo 5 for the number of 2-color partitions*, J. Number Theory, 157 (2015), 184–198.

[4] B. C. Berndt, *Ramanujan's Notebooks, Part III*, Springer-Verlag, New York, 1991.

[5] N. D. Baruah and K. K. Ojah, *Partitions with designated summands in which all parts are odd*, Integers, 15 (2015).

[6] N. D. Baruah and B. K. Sarmah, *Identities and congruences for the general partition and Ramanujan's tau functions*, Indian J. Pure Appl. Math., 44(5) (2013), 643–671.

[7] H. C. Chan, *Ramanujan's cubic continued fraction and an analog of his "most beautiful identity"*, Int. J. Number Theory, 6(3) (2010), 673–680.

[8] W. Y. C. Chen and B. L. S. Lin, *Congruences for the number of cubic partitions derived from modular forms*, preprint, arXiv:0910.1263, 15 pp.

[9] S. P. Cui and N. S. S. Gu, *Arithmetic properties of ℓ-regular partitions*, Adv. Appl. Math., 51(2013), 507–523.

[10] S. Corteel and J. Lovejoy, *Overpartitions*, Tam. Math. Soc., 356 (2004), 1623–1635.

[11] M. D. Hirschhorn and J. A. Sellers, *A congruence modulo* 3 *for partitions into distinct non-multiples of four*, Journal of Integer Sequences, Vol. 17 (2014), Article 14.9.6.

[12] M. D. Hirschhorn, *Ramanujan's "most beautiful identity"*, Amer. Math., Monthly 118 (2011), 839–845.

[13] M. D. Hirschhorn, F. Garvan and J. Borwein, *Cubic analogs of the Jacobian cubic theta function $\theta(z, q)$*, Canad. J. Math., 45 (1993), 673–694.

[14] M. D. Hirschhorn and J. A. Sellers, *Arithmetic properties of partition with odd parts distinct*, Ramanujan J., 22 (2010), 273–284.

[15] M. S. Mahadeva Naika, S. Shivaprasada Nayaka and C. Shivashankar, *Infinite families of congruences for 2-color overpartitions*, Tamsui Oxf. J. Inf. Math. Sci., 30 (1) (2016), 61–79.

[16] Pee Choon Toh, *Ramanujan type identities and congruences for partition pairs*, Discrete Math., 312(6) (2012), 1244–1250.

[17] H. M. Srivastava, M. P. Chaudhary and Sangeeta Chaudhary, *Some Theta-Function identities related to Jacobi's triple-product identity*, European J. Pure Appl. Math., 11(1) (2018), 1–9.

[18] H. M. Srivastava and G. D. Liu, *Some identities and congruences involving a certain family of numbers*, Russian J. Math. Phys., 16 (2009), 536–542.

[19] H. M. Srivastava and M. P. Chaudhary, *Some relationships between q-product identities, combinatorial partition identities and continued fractions identities*, Adv. Stud. Contemp. Math., 25 (2015), 265–272.

[20] Shane Chern, *New congruences for 2-color partitions*, J. Number Theory, 163 (2016), 474–481.

[21] G. N. Watson, *Theorems stated by Ramanujan (VII): Theorems on continued fractions*, J. London Math. Soc., 4 (1929), 39–48.