On uniqueness of meromorphic functions and their derivatives

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Abstract

In this paper, we deal with the uniqueness problem for the k-th derivative of power of meromorphic function and obtain some results which improve and supplement the previous theorem given by V.H. An and H.H. Khoai.

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1 Introduction and definitions and results

Let f and g be two nonconstant meromorphic functions in the open complex plane $\mathbb{C}$. If for some $a \in \mathbb{C} \cup \{\infty\}$, f and g have the same set of a-points with the same multiplicities then we say that f and g share the value a CM(counting multiplicities). If we do not take the multiplicities into account, f and g are said to share the value a IM(ignoring multiplicities). We denote by $N_{k)}(r, 1/(f-a))$ the counting function for zeros of $f-a$ with multiplicity less than or equal to k, and by $\overline{N}_{k)}(r, 1/(f-a))$ the corresponding one for which multiplicity is not counted. Let $N_{(k}(r, 1/(f-a))$ be the counting function for zeros of $f-a$ with multiplicity at least k and $\overline{N}_{(k}(r, 1/(f-a))$ the corresponding one for which multiplicity is not counted. We assume that the reader is familiar with the notations of Nevanlinna theory such as $T(r, f)$, $N(r, f)$, $S(r, f)$, that can be found, for instance, in [15][34]. The order $\rho(f)$ and the hyperorder $\rho_2(f)$ of a meromorphic function f are defined as follows:

$$\rho(f) = \lim_{r\to\infty} \frac{\log T(r,f)}{\log r}, \qquad \rho_2(f) = \limsup_{r\to\infty} \frac{\log\log T(r,f)}{\log r}.$$

Recently there has been a lot of interesting work on estimating the order, the hyperorder or the (p, q)-th order of meromorphic functions(see [8][11][12][13][32]).

Let S be a set of distinct elements of $\mathbb{C} \cup \{\infty\}$ and $E_f(S) = \cup_{a\in S}\{z : f(z) - a = 0\}$, where each zero is counted according to its multiplicity. If we do not count the multiplicity the set $E_f(S) = \cup_{a\in S}\{z : f(z) - a = 0\}$ is denoted by $\overline{E}_f(S)$. If $E_f(S) = E_g(S)$ we say that f and g share the set S CM. On the other hand, if $\overline{E}_f(S) = \overline{E}_g(S)$, we say that f and g share the set S IM. Let m be a positive integer or infinity and $a \in \mathbb{C} \cup \{\infty\}$. We denote by $E_{m)}(a, f)$ the set of distinct a-points of f with multiplicities not greater than m, where an a-point is counted according to its multiplicity. For a set S of distinct elements of $\mathbb{C}$ we define $E_{m)}(S, f) = \cup_{a\in S} E_{m)}(a, f)$. We can define $\overline{E}_{m)}(a, f)$ and $\overline{E}_{m)}(S, f)$ similarly. In 2001, I Lahiri introduced the notion of weighted sharing, which measures how close a shared value is to being shared CM or to being shared IM. The definition is as follows.

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Definition 1.1. [18][19] For a complex number $a \in C \cup \{\infty\}$, we denote by $E_k(a, f)$ the set of all a-points of f where an a-point with mutiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. For a complex number $a \in \mathbb{C} \cup \{\infty\}$, such that $E_k(a, f) = E_k(a, g)$, then we say that f and g share the value a with weight k.

The definition implies that if f, g share a value a with weight k, then z_0 is a zero of $f - a$ with multiplicity $m(\leq k)$ if and only if it is a zero of $g - a$ with multiplicity $m(\leq k)$ and z_0 is a zero of $f - a$ with multiplicity $m(> k)$ if and only if it is a zero of $g - a$ with multiplicity $n(> k)$, where m is not necessarily equal to n. We write f, g share (a, k) to mean that f, g share the value a with weight k. Clearly if f, g share (a, k) then f, g share (a, p) for all integer p, $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

Definition 1.2. [18] Let S be a set of distinct elements of $C \cup \{\infty\}$ and k a nonnegative integer or ∞. We denote by $E_f(S, k)$ the set $\cup_{a \in S} E_k(a, f)$. Clearly $E_f(S) = E_f(S, \infty)$ and $\overline{E}_f(S) = E_f(S, 0)$.

W.K. Hayman proposed the following well-known conjecture.

Hayman's Conjecture [16]. If an entire function satisties $f^n f' \neq 1$ for all positive integers $n \in N$, then f is a constant.

It has been verified by Hayman himself in [17] for the case $n > 1$ and Clunie in [10] for the case $n \geq 1$, respectively. In 1997, corresponding to the above famous conjecture of Hayman, Yang and Hua studied the unicity of differential monomials and obtained the following theorem.

Theorem 1.3. *[33] Let f and g be two nonconstant entire functions, $n \geq 6$ a positive integer. If $f^n f'$ and $g^n g'$ share 1 CM, then either $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c_1, c_2, c are three constants satisfying $(c_1 c_2)^{n+1} c^2 = -1$, or $f = tg$ for a constant t such that $t^{n+1} = 1$.*

Since then, A lot of research works on this topic have been done by many mathematicians (see[3, 4, 5, 6, 7, 9, 14, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31]).

In 2018, V.H. An and H.H. Khoai considered the set of roots of unity of degree d and studied the relations of f and g when $E_{(f^n)^{(k)}}(S) = E_{(g^n)^{(k)}}(S)$. In fact, they proved the following result.

Theorem 1.4. *[2] Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions, and let n, d, k be positive integers with $n > 2k + \frac{2k+8}{d}$, $d \geq 2$, and $S = \{a \in C : a^d = 1\}$. If $E_{(f^n)^{(k)}}(S) = E_{(g^n)^{(k)}}(S)$, then one of the following two cases holds: (1). $f = c_1 e^{cz}$ and $g = c_2 e^{-cz}$ for three non-zero constants c_1, c_2 and c such that $(-1)^{kd}(c_1 c_2)^{nd}(nc)^{2kd} = 1$; (2). $f = tg$ with $t^{nd} = 1$, $t \in \mathbb{C}$.*

It is natural to ask whether the nature of sharing set can be reduced in Theorem 1.4. Considering this question, we prove the following results.

Theorem 1.5. *Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions, and let n, d, k be positive integers with $n > 2k + \frac{2k+6}{d}$, $d \geq 2$, and $S = \{a \in \mathbb{C} : a^d = 1\}$. If $E_{(f^n)^{(k)}}(S, 2) = E_{(g^n)^{(k)}}(S, 2)$ and $E_{\infty)}(\infty, f) = E_{\infty)}(\infty, g)$, then one of the following two cases holds: (1) $f = c_1 e^{cz}$ and $g = c_2 e^{-cz}$ for three nonzero constants c_1, c_2 and c such that $(-1)^{kd}(c_1 c_2)^{nd}(nc)^{2kd} = 1$; (2) $f = tg$ with $t^{nd} = 1$, $t \in \mathbb{C}$.*

Theorem 1.6. *Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions, and let n, d, k be positive integers with $n > 2k + \frac{2k+7}{d}$, $d \geq 2$, and $S = \{a \in \mathbb{C} : a^d = 1\}$. If $E_{(f^n)^{(k)}}(S,2) = E_{(g^n)^{(k)}}(S,2)$ and $\overline{E}_{\infty)}(\infty, f) = \overline{E}_{\infty)}(\infty, g)$, then one of the following two cases holds: (1) $f = c_1 e^{cz}$ and $g = c_2 e^{-cz}$ for three nonzero constants c_1, c_2 and c such that $(-1)^{kd}(c_1c_2)^{nd}(nc)^{2kd} = 1$; (2) $f = tg$ with $t^{nd} = 1$, $t \in \mathbb{C}$.*

Theorem 1.7. *Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions, and let n, d, k be positive integers with $n > 2k + \frac{2k+8}{d}$, $d \geq 2$, and $S = \{a \in \mathbb{C} : a^d = 1\}$. If $E_{3)}(S, (f^n)^{(k)}) = E_{3)}(S, (g^n)^{(k)})$, then one of the following two cases holds: (1) $f = c_1 e^{cz}$ and $g = c_2 e^{-cz}$ for three nonzero constants c_1, c_2 and c such that $(-1)^{kd}(c_1c_2)^{nd}(nc)^{2kd} = 1$; (2) $f = tg$ with $t^{nd} = 1$, $t \in \mathbb{C}$.*

2 Some Lemmas

In this section, we present some lemmas which will be needed in the sequel.

Lemma 2.1. *[6] Let F, G be two nonconstant meromorphic functions. If F, G share $(1,2)$ and (∞, k), where $0 \leq k \leq \infty$, then one of the following three cases holds:*

$$(1) \quad T(r,F) \leq N_2\left(r, \frac{1}{F}\right) + N_2\left(r, \frac{1}{G}\right) + \overline{N}(r,F) + \overline{N}(r,G) + \overline{N}_*(r,\infty;F,G) + S(r,F) + S(r,G)\,,$$

$$(2) \quad F \equiv G\,, \quad (3) \quad FG \equiv 1\,.$$

Here $\overline{N}_*(r,\infty;F,G)$ denotes the reduced counting function of those poles of F whose multiplicities differ from the multiplicities of the corresponding poles of G.

Lemma 2.2. *[2] Let f be a nonconstant meromorphic function and n, k be positive integers, $n > 2k$. Then*

$$(n-2k)T(r,f) + kN(r,f) + N\left(r, \frac{f^{n-k}}{(f^n)^{(k)}}\right) \leq T\left(r, (f^n)^{(k)}\right) + S(r,f)\,.$$

Lemma 2.3. *[35] Let f be a nonconstant meromorphic function, k be a positive integer, then*

$$N_p\left(r, \frac{1}{f^{(k)}}\right) \leq N_{p+k}\left(r, \frac{1}{f}\right) + k\overline{N}(r,f) + S(r,f)\,,$$

where$N_p\left(r, \frac{1}{f^{(k)}}\right)$ denotes the counting function of the zeros of $f^{(k)}$ where a zero of multiplicity m is counted m times if $m \leq p$ and p times if $m > p$. Clearly $\overline{N}\left(r, \frac{1}{f^{(k)}}\right) = N_1\left(r, \frac{1}{f^{(k)}}\right)$.

Lemma 2.4. *[36] Let $f(z)$ and $g(z)$ be two nonconstant entire functions and n, k be positive integers, $n > k$. If $(f^n)^{(k)}(g^n)^{(k)} = h$, $h \in \mathbb{C}$, $h \neq 0$, then $f = l_1 e^{lz}$ and $g = l_2 e^{-lz}$ for three non-zero constants l_1, l_2 and l such that $(-1)^k (l_1 l_2)^n (nl)^{2k} = h$.*

Lemma 2.5. *[22] If F and G are nonconstant meromorphic functions and $E_{3)}(1,F)=E_{3)}(1,G)$, functions. Then one of the following three cases holds:*

$$(1)\quad T(r,F)+T(r,G)\le 2N_2\left(r,\frac{1}{F}\right)+2N_2\left(r,\frac{1}{G}\right)+2N_2(r,F)\\ +2N_2(r,G)+S(r,F)+S(r,G)\,,$$

$$(2)\quad F\equiv G\,,\quad (3)\quad FG\equiv 1\,.$$

3 Proof of Theorem 1.5

Let

$$F=\left((f^n)^{(k)}\right)^d,\qquad G=\left((g^n)^{(k)}\right)^d. \tag{1}$$

Since $E_{(f^n)^{(k)}}(S,2)=E_{(g^n)^{(k)}}(S,2)$ and $E_{\infty)}(\infty,f)=E_{\infty)}(\infty,g)$, we see that F and G share $(1,2)$ and (∞,∞). Then by Lemma 2.1, if possible, we suppose that

$$T(r,F)\le N_2\left(r,\frac{1}{F}\right)+N_2\left(r,\frac{1}{G}\right)+\overline{N}(r,F)\\ +\overline{N}(r,G)+S(r,F)+S(r,G)\,. \tag{2}$$

By Lemma 2.2, we obtain

$$(n-2k)T(r,f)\le T\left(r,(f^n)^{(k)}\right)+S(r,f)\le (k+1)nT(r,f)+S(r,f)\,, \tag{3}$$

$$(n-2k)T(r,g)\le T\left(r,(g^n)^{(k)}\right)+S(r,g)\le (k+1)nT(r,g)+S(r,g)\,. \tag{4}$$

Since

$$T\left(r,\left((f^n)^{(k)}\right)^d\right)=dT\left(r,(f^n)^{(k)}\right)+S\left(r,(f^n)^{(k)}\right), \tag{5}$$

$$T\left(r,\left((g^n)^{(k)}\right)^d\right)=dT\left(r,(g^n)^{(k)}\right)+S\left(r,(g^n)^{(k)}\right). \tag{6}$$

It's easy to see that

$$S\left(r,\left((f^n)^{(k)}\right)^d\right)=S\left(r,(f^n)^{(k)}\right)=S(r,f)\,, \tag{7}$$

$$S\left(r,\left((g^n)^{(k)}\right)^d\right)=S\left(r,(g^n)^{(k)}\right)=S(r,g)\,. \tag{8}$$

By Lemma 2.3, we have

$$N_2\left(r,\frac{1}{F}\right)=N_2\left(r,\frac{1}{\left((f^n)^{(k)}\right)^d}\right)=2\overline{N}\left(r,\frac{1}{(f^n)^{(k)}}\right)\\ \le 2N_{k+1}\left(r,\frac{1}{f^n}\right)+2k\overline{N}(r,f^n)+S(r,f^n)\\ =2(k+1)\overline{N}\left(r,\frac{1}{f}\right)+2k\overline{N}(r,f)+S(r,f)\,. \tag{9}$$

$$N_2\left(r,\frac{1}{G}\right)\leq 2(k+1)\overline{N}\left(r,\frac{1}{g}\right)+2k\overline{N}(r,g)+S(r,g)\,, \tag{10}$$

and

$$\overline{N}(r,F)+\overline{N}(r,G)=2\overline{N}(r,f)\,. \tag{11}$$

On the other hand,

$$\begin{aligned} N_2\left(r,\frac{1}{F}\right)=2\overline{N}\left(r,\frac{1}{(f^n)^{(k)}}\right)&\leq 2\left(\overline{N}\left(r,\frac{1}{f^{n-k}}\right)+N\left(r,\frac{f^{n-k}}{(f^n)^{(k)}}\right)\right)\\ &\leq 2\left(\overline{N}\left(r,\frac{1}{f}\right)+N\left(r,\frac{f^{n-k}}{(f^n)^{(k)}}\right)\right)\,, \end{aligned} \tag{12}$$

$$N_2\left(r,\frac{1}{G}\right)\leq 2\left(\overline{N}\left(r,\frac{1}{g}\right)+N\left(r,\frac{g^{n-k}}{(g^n)^{(k)}}\right)\right)\,. \tag{13}$$

Combine (2), (9), (11) and (13), we deduce

$$\begin{aligned} T\left(r,\left((f^n)^{(k)}\right)^d\right)&\leq 2(k+1)\overline{N}\left(r,\frac{1}{f}\right)+(2k+2)\overline{N}(r,f)\\ &+2\left(\overline{N}\left(r,\frac{1}{g}\right)+N\left(r,\frac{g^{n-k}}{(g^n)^{(k)}}\right)\right)+S(r,f)\,. \end{aligned} \tag{14}$$

Similarly we have

$$\begin{aligned} T\left(r,\left((g^n)^{(k)}\right)^d\right)&\leq 2(k+1)\overline{N}\left(r,\frac{1}{g}\right)+(2k+2)\overline{N}(r,g)\\ &+2\left(\overline{N}\left(r,\frac{1}{f}\right)+N\left(r,\frac{f^{n-k}}{(f^n)^{(k)}}\right)\right)+S(r,g)\,. \end{aligned} \tag{15}$$

Thus

$$\begin{aligned} T\left(r,\left((f^n)^{(k)}\right)^d\right)+T\left(r,\left((g^n)^{(k)}\right)^d\right)&\leq 2(k+1)\overline{N}\left(r,\frac{1}{f}\right)+(2k+2)\overline{N}(r,f)\\ &+2\left(\overline{N}\left(r,\frac{1}{g}\right)+N\left(r,\frac{g^{n-k}}{(g^n)^{(k)}}\right)\right)+2(k+1)\overline{N}\left(r,\frac{1}{g}\right)+(2k+2)\overline{N}(r,g)\\ &+2\left(\overline{N}\left(r,\frac{1}{f}\right)+N\left(r,\frac{f^{n-k}}{(f^n)^{(k)}}\right)\right)+S(r,f)+S(r,g)\,. \end{aligned} \tag{16}$$

By Lemma 2.2, we have

$$(n-2k)dT(r,f)+kdN(r,f)+dN\left(r,\frac{f^{n-k}}{(f^n)^{(k)}}\right)\leq dT\left(r,(f^n)^{(k)}\right)+S(r,f)\,, \tag{17}$$

$$(n-2k)dT(r,g)+kdN(r,g)+dN\left(r,\frac{g^{n-k}}{(g^n)^{(k)}}\right)\leq dT\left(r,(g^n)^{(k)}\right)+S(r,g)\,. \tag{18}$$

From (5), (6), (16), (17) and (18), we have

$$\begin{aligned}(n-2k)dT(r,f)+kdN(r,f)+dN\left(r,\frac{f^{n-k}}{(f^n)^{(k)}}\right)+(n-2k)dT(r,g)\\ +kdN(r,g)+dN\left(r,\frac{g^{n-k}}{(g^n)^{(k)}}\right)\\ \leq 2(k+1)\overline{N}\left(r,\frac{1}{f}\right)+(2k+2)\overline{N}(r,f)+2(k+1)\overline{N}\left(r,\frac{1}{g}\right)+(2k+2)\overline{N}(r,g)\\ +2\left(\overline{N}\left(r,\frac{1}{g}\right)+N\left(r,\frac{g^{n-k}}{(g^n)^{(k)}}\right)\right)+2\left(\overline{N}\left(r,\frac{1}{f}\right)+N\left(r,\frac{f^{n-k}}{(f^n)^{(k)}}\right)\right)\\ +S(r,f)+S(r,g)\,.\end{aligned} \tag{19}$$

Since $d\geq 2$,

$$dN\left(r,\frac{f^{n-k}}{(f^n)^{(k)}}\right)\geq 2N\left(r,\frac{f^{n-k}}{(f^n)^{(k)}}\right), \tag{20}$$

$$dN\left(r,\frac{g^{n-k}}{(g^n)^{(k)}}\right)\geq 2N\left(r,\frac{g^{n-k}}{(g^n)^{(k)}}\right), \tag{21}$$

$$kdN(r,f)\geq 2k\overline{N}(r,f)\,, \tag{22}$$

$$kdN(r,g)\geq 2k\overline{N}(r,g)\,. \tag{23}$$

Therefore,

$$(nd-2kd-2k-6)T(r,f)+(nd-2kd-2k-6)T(r,g)\leq S(r,f)+S(r,g)\,, \tag{24}$$

which contradicts with $n>2k+\frac{2k+6}{d}$. Hence $FG\equiv 1$ or $F\equiv G$.

If $FG\equiv 1$, that is, $\left((f^n)^{(k)}\right)^d\left((g^n)^{(k)}\right)^d=1$, we have $(f^n)^{(k)}(g^n)^{(k)}=h$, where $h^d=1$. Suppose z_0 is a zero of f with multiplicity p, then z_0 is a pole of g with multiplicity q such that $np-k=nq+k$. So $n(p-q)-2k=0$. Since $n>2k+\frac{2k+6}{d}$, we can deduce a contradiction. So $f(z)\neq 0$. Similarly we can prove $f(z)\neq\infty$, $g(z)\neq 0$ and $g(z)\neq\infty$. So $f(z)$ and $g(z)$ are two nonconstant entire functions. According to Lemma 2.4, we obtain $f=c_1e^{cz}$ and $g=c_2e^{-cz}$ for three nonzero constants c_1, c_2 and c such that $(-1)^{kd}(c_1c_2)^{nd}(nc)^{2kd}=1$.

If $F\equiv G$, that is, $\left((f^n)^{(k)}\right)^d=\left((g^n)^{(k)}\right)^d$, we have $(f^n)^{(k)}=h\,(g^n)^{(k)}$ with $h^d=1$. Then $(f^n)^{(k)}=((eg)^n)^{(k)}$ if we set $e^n=h$. By similar arguments as in the proof of [Theorem 1.1, 1], we can obtain $f=seg$ where $s^n=1$. Set $t=se$, we get $t^{nd}=s^{nd}e^{nd}=1$. This completes the proof of Theorem 1.5.

4 Proof of Theorem 1.6

Let

$$F=\left((f^n)^{(k)}\right)^d,\qquad G=\left((g^n)^{(k)}\right)^d. \tag{25}$$

Since $E_{(f^n)^{(k)}}(S,2) = E_{(g^n)^{(k)}}(S,2)$ and $\overline{E}_{\infty)}(\infty,f) = \overline{E}_{\infty)}(\infty,g)$, we see that F and G share $(1,2)$ and $(\infty,0)$. Then by Lemma 2.1, if possible, we suppose that

$$\begin{aligned} T(r,F) \leq N_2\left(r,\frac{1}{F}\right) + N_2\left(r,\frac{1}{G}\right) + \overline{N}(r,F) + \overline{N}(r,G) \\ + \overline{N}_*(r,\infty;F,G) + S(r,F) + S(r,G)\,. \end{aligned} \tag{26}$$

By Lemma 2.2, we obtain

$$(n-2k)T(r,f) \leq T\left(r,(f^n)^{(k)}\right) + S(r,f) \leq (k+1)nT(r,f) + S(r,f)\,, \tag{27}$$

$$(n-2k)T(r,g) \leq T\left(r,(g^n)^{(k)}\right) + S(r,g) \leq (k+1)nT(r,g) + S(r,g)\,. \tag{28}$$

Since

$$T\left(r,\left((f^n)^{(k)}\right)^d\right) = dT\left(r,(f^n)^{(k)}\right) + S\left(r,(f^n)^{(k)}\right)\,, \tag{29}$$

$$T\left(r,\left((g^n)^{(k)}\right)^d\right) = dT\left(r,(g^n)^{(k)}\right) + S\left(r,(g^n)^{(k)}\right)\,. \tag{30}$$

It's easy to see that

$$S\left(r,\left((f^n)^{(k)}\right)^d\right) = S\left(r,(f^n)^{(k)}\right) = S(r,f)\,, \tag{31}$$

$$S\left(r,\left((g^n)^{(k)}\right)^d\right) = S\left(r,(g^n)^{(k)}\right) = S(r,g)\,. \tag{32}$$

By Lemma 2.3, we have

$$\begin{aligned} N_2\left(r,\frac{1}{F}\right) &= N_2\left(r,\frac{1}{\left((f^n)^{(k)}\right)^d}\right) = 2\overline{N}\left(r,\frac{1}{(f^n)^{(k)}}\right) \\ &\leq 2N_{k+1}\left(r,\frac{1}{f^n}\right) + 2k\overline{N}(r,f^n) + S(r,f^n) \\ &= 2(k+1)\overline{N}\left(r,\frac{1}{f}\right) + 2k\overline{N}(r,f) + S(r,f)\,. \end{aligned} \tag{33}$$

$$N_2\left(r,\frac{1}{G}\right) \leq 2(k+1)\overline{N}\left(r,\frac{1}{g}\right) + 2k\overline{N}(r,g) + S(r,g)\,, \tag{34}$$

and

$$\overline{N}(r,F) + \overline{N}(r,G) + \overline{N}_*(r,\infty;F,G) \leq 3\overline{N}(r,f)\,. \tag{35}$$

On the other hand,

$$\begin{aligned} N_2\left(r,\frac{1}{F}\right) = 2\overline{N}\left(r,\frac{1}{(f^n)^{(k)}}\right) &\leq 2\left(\overline{N}\left(r,\frac{1}{f^{n-k}}\right) + N\left(r,\frac{f^{n-k}}{(f^n)^{(k)}}\right)\right) \\ &\leq 2\left(\overline{N}\left(r,\frac{1}{f}\right) + N\left(r,\frac{f^{n-k}}{(f^n)^{(k)}}\right)\right)\,, \end{aligned} \tag{36}$$

$$N_2\left(r,\frac{1}{G}\right)\leq 2\left(\overline{N}\left(r,\frac{1}{g}\right)+N\left(r,\frac{g^{n-k}}{(g^n)^{(k)}}\right)\right). \tag{37}$$

Combine (26), (33), (35) and (37), we deduce

$$\begin{aligned}T\left(r,\left((f^n)^{(k)}\right)^d\right)&\leq 2(k+1)\overline{N}\left(r,\frac{1}{f}\right)+(2k+3)\overline{N}(r,f)\\&\quad+2\left(\overline{N}\left(r,\frac{1}{g}\right)+N\left(r,\frac{g^{n-k}}{(g^n)^{(k)}}\right)\right)+S(r,f).\end{aligned} \tag{38}$$

Similarly we have

$$\begin{aligned}T\left(r,\left((g^n)^{(k)}\right)^d\right)&\leq 2(k+1)\overline{N}\left(r,\frac{1}{g}\right)+(2k+3)\overline{N}(r,g)\\&\quad+2\left(\overline{N}\left(r,\frac{1}{f}\right)+N\left(r,\frac{f^{n-k}}{(f^n)^{(k)}}\right)\right)+S(r,g).\end{aligned} \tag{39}$$

Thus

$$\begin{aligned}T\left(r,\left((f^n)^{(k)}\right)^d\right)+T\left(r,\left((g^n)^{(k)}\right)^d\right)&\leq 2(k+1)\overline{N}\left(r,\frac{1}{f}\right)+(2k+3)\overline{N}(r,f)\\&\quad+2\left(\overline{N}\left(r,\frac{1}{g}\right)+N\left(r,\frac{g^{n-k}}{(g^n)^{(k)}}\right)\right)+2(k+1)\overline{N}\left(r,\frac{1}{g}\right)+(2k+3)\overline{N}(r,g)\\&\quad+2\left(\overline{N}\left(r,\frac{1}{f}\right)+N\left(r,\frac{f^{n-k}}{(f^n)^{(k)}}\right)\right)+S(r,f)+S(r,g).\end{aligned} \tag{40}$$

By Lemma 2.2, we have

$$(n-2k)dT(r,f)+kdN(r,f)+dN\left(r,\frac{f^{n-k}}{(f^n)^{(k)}}\right)\leq dT\left(r,(f^n)^{(k)}\right)+S(r,f), \tag{41}$$

$$(n-2k)dT(r,g)+kdN(r,g)+dN\left(r,\frac{g^{n-k}}{(g^n)^{(k)}}\right)\leq dT\left(r,(g^n)^{(k)}\right)+S(r,g). \tag{42}$$

From (29), (30), (40), (41) and (42), we have

$$\begin{aligned}&(n-2k)dT(r,f)+kdN(r,f)+dN\left(r,\frac{f^{n-k}}{(f^n)^{(k)}}\right)+(n-2k)dT(r,g)\\&\qquad+kdN(r,g)+dN\left(r,\frac{g^{n-k}}{(g^n)^{(k)}}\right)\\&\leq 2(k+1)\overline{N}\left(r,\frac{1}{f}\right)+(2k+3)\overline{N}(r,f)+2(k+1)\overline{N}\left(r,\frac{1}{g}\right)+(2k+3)\overline{N}(r,g)\\&\quad+2\left(\overline{N}\left(r,\frac{1}{g}\right)+N\left(r,\frac{g^{n-k}}{(g^n)^{(k)}}\right)\right)+2\left(\overline{N}\left(r,\frac{1}{f}\right)+N\left(r,\frac{f^{n-k}}{(f^n)^{(k)}}\right)\right)\\&\quad+S(r,f)+S(r,g).\end{aligned} \tag{43}$$

Since $d \geq 2$,

$$dN\left(r, \frac{f^{n-k}}{(f^n)^{(k)}}\right) \geq 2N\left(r, \frac{f^{n-k}}{(f^n)^{(k)}}\right), \tag{44}$$

$$dN\left(r, \frac{g^{n-k}}{(g^n)^{(k)}}\right) \geq 2N\left(r, \frac{g^{n-k}}{(g^n)^{(k)}}\right), \tag{45}$$

$$kdN(r,f) \geq 2k\overline{N}(r,f), \tag{46}$$

$$kdN(r,g) \geq 2k\overline{N}(r,g). \tag{47}$$

Therefore,

$$(nd-2kd-2k-7)T(r,f)+(nd-2kd-2k-7)T(r,g) \leq S(r,f)+S(r,g), \tag{48}$$

which contradicts with $n > 2k + \frac{2k+7}{d}$. Hence $FG \equiv 1$ or $F \equiv G$. Similar to the arguments in Theorem 1.5, we see that Theorem 1.6 holds.

5 Proof of Theorem 1.7

Let

$$F = \left((f^n)^{(k)}\right)^d, \qquad G = \left((g^n)^{(k)}\right)^d. \tag{49}$$

Since $E_{3)}(S,(f^n)^{(k)}) = E_{3)}(S,(g^n)^{(k)})$, we see that $E_{3)}(1,F) = E_{3)}(1,G)$. Then by Lemma 2.5, if possible, we suppose that

$$\begin{aligned} T(r,F)+T(r,G) \leq 2N_2\left(r,\frac{1}{F}\right)+2N_2\left(r,\frac{1}{G}\right)+2N_2(r,F) \\ +2N_2(r,G)+S(r,F)+S(r,G). \end{aligned} \tag{50}$$

By Lemma 2.2, we obtain

$$(n-2k)T(r,f) \leq T\left(r,(f^n)^{(k)}\right)+S(r,f) \leq (k+1)nT(r,f)+S(r,f), \tag{51}$$

$$(n-2k)T(r,g) \leq T\left(r,(g^n)^{(k)}\right)+S(r,g) \leq (k+1)nT(r,g)+S(r,g). \tag{52}$$

Since

$$T\left(r,\left((f^n)^{(k)}\right)^d\right) = dT\left(r,(f^n)^{(k)}\right)+S\left(r,(f^n)^{(k)}\right), \tag{53}$$

$$T\left(r,\left((g^n)^{(k)}\right)^d\right) = dT\left(r,(g^n)^{(k)}\right)+S\left(r,(g^n)^{(k)}\right). \tag{54}$$

It's easy to see that

$$S\left(r,\left((f^n)^{(k)}\right)^d\right) = S\left(r,(f^n)^{(k)}\right) = S(r,f), \tag{55}$$

$$S\left(r,\left((g^n)^{(k)}\right)^d\right) = S\left(r,(g^n)^{(k)}\right) = S(r,g). \tag{56}$$

$$N_2(r,F)=N_2\left(r,\left((f^n)^{(k)}\right)^d\right)=2\overline{N}(r,f)\,, \tag{57}$$

$$N_2(r,G)=N_2\left(r,\left((g^n)^{(k)}\right)^d\right)=2\overline{N}(r,g)\,. \tag{58}$$

By Lemma 2.3, we have

$$\begin{aligned} N_2\left(r,\frac{1}{F}\right)&=N_2\left(r,\frac{1}{\left((f^n)^{(k)}\right)^d}\right)=2\overline{N}\left(r,\frac{1}{(f^n)^{(k)}}\right)\\ &\le 2N_{k+1}\left(r,\frac{1}{f^n}\right)+2k\overline{N}(r,f^n)+S(r,f^n)\\ &=2(k+1)\overline{N}\left(r,\frac{1}{f}\right)+2k\overline{N}(r,f)+S(r,f)\,. \end{aligned} \tag{59}$$

$$N_2\left(r,\frac{1}{G}\right)\le 2(k+1)\overline{N}\left(r,\frac{1}{g}\right)+2k\overline{N}(r,g)+S(r,g)\,. \tag{60}$$

On the other hand,

$$\begin{aligned} N_2\left(r,\frac{1}{F}\right)=2\overline{N}\left(r,\frac{1}{(f^n)^{(k)}}\right)&\le 2\left(\overline{N}\left(r,\frac{1}{f^{n-k}}\right)+N\left(r,\frac{f^{n-k}}{(f^n)^{(k)}}\right)\right)\\ &\le 2\left(\overline{N}\left(r,\frac{1}{f}\right)+N\left(r,\frac{f^{n-k}}{(f^n)^{(k)}}\right)\right)\,, \end{aligned} \tag{61}$$

$$N_2\left(r,\frac{1}{G}\right)\le 2\left(\overline{N}\left(r,\frac{1}{g}\right)+N\left(r,\frac{g^{n-k}}{(g^n)^{(k)}}\right)\right)\,, \tag{62}$$

Combining (50), (57), (58), (59), (60), (61) and (62), we deduce

$$\begin{aligned} T\left(r,\left((f^n)^{(k)}\right)^d\right)+T\left(r,\left((g^n)^{(k)}\right)^d\right)&\le (2k+4)\overline{N}\left(r,\frac{1}{f}\right)+(2k+4)\overline{N}(r,f)\\ &+(2k+4)\overline{N}\left(r,\frac{1}{g}\right)+(2k+4)\overline{N}(r,g)+2N\left(r,\frac{f^{n-k}}{(f^n)^{(k)}}\right)\\ &+2N\left(r,\frac{g^{n-k}}{(g^n)^{(k)}}\right)+S(r,f)+S(r,g)\\ &\le (2k+8)T(r,f)+(2k+8)T(r,g)+2k\overline{N}(r,f)+2k\overline{N}(r,g)\\ &+2N\left(r,\frac{f^{n-k}}{(f^n)^{(k)}}\right)+2N\left(r,\frac{g^{n-k}}{(g^n)^{(k)}}\right)+S(r,f)+S(r,g)\,. \end{aligned} \tag{63}$$

By Lemma 2.2, we have

$$(n-2k)dT(r,f)+kdN(r,f)+dN\left(r,\frac{f^{n-k}}{(f^n)^{(k)}}\right)\le dT\left(r,(f^n)^{(k)}\right)+S(r,f)\,, \tag{64}$$

$$(n-2k)dT(r,g)+kdN(r,g)+dN\left(r,\frac{g^{n-k}}{(g^n)^{(k)}}\right)\leq dT\left(r,(g^n)^{(k)}\right)+S(r,g)\,. \tag{65}$$

From (53), (54), (63), (64) and (65), we have

$$\begin{aligned}(n-2k)dT(r,f)+kdN(r,f)+dN\left(r,\frac{f^{n-k}}{(f^n)^{(k)}}\right)+(n-2k)dT(r,g)&\\ +kdN(r,g)+dN\left(r,\frac{g^{n-k}}{(g^n)^{(k)}}\right)&\\ \leq(2k+8)T(r,f)+(2k+8)T(r,g)+2k\overline{N}(r,f)+2k\overline{N}(r,g)&\\ +2N\left(r,\frac{f^{n-k}}{(f^n)^{(k)}}\right)+2N\left(r,\frac{g^{n-k}}{(g^n)^{(k)}}\right)+S(r,f)+S(r,g)\,.&\end{aligned} \tag{66}$$

Since $d\geq 2$,

$$dN\left(r,\frac{f^{n-k}}{(f^n)^{(k)}}\right)\geq 2N\left(r,\frac{f^{n-k}}{(f^n)^{(k)}}\right), \tag{67}$$

$$dN\left(r,\frac{g^{n-k}}{(g^n)^{(k)}}\right)\geq 2N\left(r,\frac{g^{n-k}}{(g^n)^{(k)}}\right), \tag{68}$$

$$kdN(r,f)\geq 2k\overline{N}(r,f)\,, \tag{69}$$

$$kdN(r,g)\geq 2k\overline{N}(r,g)\,. \tag{70}$$

Therefore,

$$(nd-2kd-2k-8)T(r,f)+(nd-2kd-2k-8)T(r,g)\leq S(r,f)+S(r,g)\,, \tag{71}$$

which contradicts with $n>2k+\frac{2k+8}{d}$. Hence $FG\equiv 1$ or $F\equiv G$. Similar to the arguments in Theorem 1.5, we see that Theorem 1.7 holds.

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