Jacobi spectral discretization for nonlinear fractional generalized seventh-order KdV equations with convergence analysis

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Abstract

We developed a numerical scheme to solve the nonlinear fractional (NF) generalized seventh order KdV (sKdV) equation with time-space-fractional using the shifted Jacobi collocation method. Basically, a time-space collocation approximation for temporal and spatial discretizations is employed efficiently to tackle these equations. The convergence and stability analyses of the suggested basis functions are presented in-depth. The validity and efficiency of the proposed method are investigated and verified through numerical examples.

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1 Introduction

Fractional differential equations [1, 2] are exhibited as capable mathematical tools for factual and more precise depiction of various phenomena. They show up in different territories, counting mathematical chemistry [3, 4], viscoelasticity [5], biology [2], electrochemistry, physics [7], semiconductors, seismology, scattering theory, heat conduction, fluid flow, metallurgy, population dynamics, optimal control theory, mathematical economics and chemical reaction. As the increasing of employing fractional partial differential equations [8, 9] in many social and scientific fields.

Various works have been concentrating on the improvement of more progressed and effective techniques for KdV equation. For example, Duruflé and Israwi [10] used a finite-difference scheme for solving variable depth KdV equations. Also, Arora and Sharma [11] applied the homotopy analysis method (HAM) to obtain the approximate solutions of KdV equations of seventh order. Moreover, several methods [12, 13, 14, 15, 16] have been proposed to handle a wide variety of KdV and other differential equations equation.

The spectral methods are preferable in numerical solutions of ordinary and partial differential equations due to their high-order accuracy whenever they work [17, 18]. Collocation method is frequently used to solve various kinds of fractional differential equations, such as variable-order fractional reaction-subdiffusion equation [19], time-fractional coupled KdV equations [20], multi-term time fractional mixed diffusion and diffusion-wave equations [21], and time fractional Fokker-Planck equations [22], Singh and Srivastava [23] implemented an efficient technique for handling

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the fractional Ricatti equation, Srivastava et al. [24] suggested an efficient Ultraspherical algorithm for handling the Bagley-Trovik equation.

The basis functions are responsible for the superior approximation properties of spectral methods. The choice of different basis functions leads to different spectral approximations; for instance, trigonometric polynomials for periodic problems, Chebyshev, Legendre, ultraspherical, and Jacobi polynomials for nonperiodic problems, Laguerre polynomials for problems on the half line, and Hermite polynomials for problems on the whole line.

The framework of this paper is as per the following: In the following area, we display few revelent properties of fractional derivatives and shifted Jacobi polynomials in the coming section. Section 3 is devoted to the theoretical derivation of the shifted Jacobi collocation method (SJCM) for solving nonlinear fractional (NF) generalized sKdV equation with homogeneous conditions (HC). Section 4 is given to applying the shifted Jacobi collocation method (SJCM) with nonhomogeneous conditions (NHC). Moreover, in Section 6 the proposed technique is connected to different test problems. At last, some concluding remarks are given in Section 7.

2 Mathematical preliminaries

In this section, we will express the definition and preliminaries of fractional calculus [5, 6] and some properties of shifted Jacobi polynomials (see, e.g. [25, 26]).

Definition 1. *The Riemann-Liouville fractional derivatives of order γ is defined as*

$$D^{\gamma}f(x)=\frac{1}{\Gamma(m-\gamma)}\frac{d^m}{dx^m}(\int_0^x (x-t)^{m-\gamma-1}f(t)dt),\quad m-1<\gamma\leq m,\ x>0, \tag{1}$$

where m is the ceiling function of γ.

Definition 2. *The Liouville-Caputo fractional derivatives of order γ is defined as*

$${}^{c}D^{\gamma}f(x)=\frac{1}{\Gamma(m-\gamma)}\int_0^x (x-t)^{m-\gamma-1}\frac{d^m}{dt^m}f(t)dt,\quad m-1<\gamma\leq m,\ x>0, \tag{2}$$

where m is the ceiling function of γ.

For the Liouville-Caputo derivative we have

$$D^{\gamma}C=0,\qquad (C \text{ is a constant}), \tag{3}$$

$$D^{\gamma}x^m=\begin{cases}0, & \text{for } m\in N_0 \text{ and } m<\lceil\gamma\rceil,\\ \dfrac{\Gamma(m+1)}{\Gamma(m+1-\gamma)}\,x^{m-\gamma}, & \text{for } m\in N_0 \text{ and } m\geq\lceil\gamma\rceil \text{ or } m\notin N \text{ and } m>\lfloor\gamma\rfloor.\end{cases} \tag{4}$$

where $\lceil\gamma\rceil$ and $\lfloor\gamma\rfloor$ are the ceiling and floor functions respectively, while $N=\{1,2,\ldots\}$ and $N_0=\{0,1,2,\ldots\}$. Recall that for $\gamma\in N$, the Liouville-Caputo differential operator coincides with the usual differential operator of an integer order.

Definition 3. *For m to be the smallest integer that exceeds γ, the Liouville-Caputo time-fractional derivative operator of order γ of $u(x,t)$ is defined as*

$$\frac{\partial^\gamma u(x,t)}{\partial t^\gamma} = \begin{cases} \frac{1}{\Gamma(m-\gamma)} \int_0^t (t-\sigma)^{m-\gamma-1} \frac{\partial^m u(x,\sigma)}{\partial \sigma^m} d\sigma, & m-1 < \gamma < m, \\ \frac{\partial^m u(x,t)}{\partial t^m}, & \gamma = m \in \mathbb{N}. \end{cases} \tag{5}$$

and the space-fractional derivative of order ν of $u(x,t)$ is defined as

$$\frac{\partial^\nu u(x,t)}{\partial x^\nu} = \begin{cases} \frac{1}{\Gamma(m-\nu)} \int_0^x (x-\sigma)^{m-\nu-1} \frac{\partial^m u(\sigma,t)}{\partial \sigma^m} d\sigma, & m-1 < \nu < m, \\ \frac{\partial^m u(x,t)}{\partial x^m}, & \nu = m \in \mathbb{N}. \end{cases} \tag{6}$$

In the following, few relations related to Jacobi polynomials are listed:

$$P_k^{(\alpha,\beta)}(-x) = (-1)^k P_k^{(\alpha,\beta)}(x), \quad P_k^{(\alpha,\beta)}(-1) = \frac{(-1)^k \Gamma(k+\beta+1)}{k!\Gamma(\beta+1)}, \quad P_k^{(\alpha,\beta)}(1) = \frac{\Gamma(k+\alpha+1)}{k!\Gamma(\alpha+1)}, \tag{7}$$

where $\alpha,\ \beta > -1$, $x \in [-1,1]$.
Moreover, the rth derivative (r is positive integer) of $P_j^{(\alpha,\beta)}(x)$ is given by

$$D^r P_j^{(\alpha,\beta)}(x) = \frac{\Gamma(j+\alpha+\beta+r+1)}{2^r \Gamma(j+\alpha+\beta+1)} P_{j-r}^{(\alpha+r,\beta+r)}(x). \tag{8}$$

For the shifted Jacobi polynomial $\mathcal{P}_{\mathcal{L},k}^{(\alpha,\beta)}(x) = P_k^{(\alpha,\beta)}(\frac{2x}{\mathcal{L}} - 1),\ \mathcal{L} > 0$, the explicit analytic form is given by [27]

$$\mathcal{P}_{\mathcal{L},k}^{(\alpha,\beta)}(x) = \sum_{j=0}^{k} (-1)^{k-j} \frac{\Gamma(k+\beta+1)\Gamma(j+k+\alpha+\beta+1)}{\Gamma(j+\beta+1)\Gamma(k+\alpha+\beta+1)(k-j)!j!\mathcal{L}^j} x^j. \tag{9}$$

Thus, we can derive the following properties

$$D^r \mathcal{P}_{\mathcal{L},k}^{(\zeta,\eta)}(0) = \frac{(-1)^{k-r}\Gamma(k+\eta+1)(k+\zeta+\eta+1)_r}{L^r \Gamma(k-r+1)\Gamma(r+\eta+1)}, \tag{10}$$

$$D^r \mathcal{P}_{\mathcal{L},k}^{(\zeta,\eta)}(\mathcal{L}) = \frac{\Gamma(k+\zeta+1)(k+\zeta+\eta+1)_r}{L^r \Gamma(k-r+1)\Gamma(r+\zeta+1)}, \tag{11}$$

$$D^r \mathcal{P}_{\mathcal{L},k}^{(\zeta,\eta)}(x) = \frac{\Gamma(r+k+\zeta+\eta+1)}{\mathcal{L}^r \Gamma(k+\zeta+\eta+1)} \mathcal{P}_{\mathcal{L},k-r}^{(\zeta+r,\eta+r)}(x). \tag{12}$$

Assuming that $w_{\mathcal{L}}^{(\alpha,\beta)} = (\mathcal{L}-x)^\alpha x^\beta$, we can define the norm and inner product for the weighted space $L^2_{w_{\mathcal{L}}^{(\alpha,\beta)}}[0,\mathcal{L}]$ as

$$(u,v)_{w_{\mathcal{L}}^{(\alpha,\beta)}} = \int_0^{\mathcal{L}} u(x)\, v(x)\, w_{\mathcal{L}}^{(\alpha,\beta)}(x)\, dx, \quad \|v\|_{w_{\mathcal{L}}^{(\alpha,\beta)}} = (v,v)^{\frac{1}{2}}_{w_{\mathcal{L}}^{(\alpha,\beta)}}. \tag{13}$$

The set of shifted Jacobi polynomials forms a complete $L^2_{w_{\mathcal{L}}^{(\alpha,\beta)}}[0,\mathcal{L}]$-orthogonal system. Moreover, and due to (13), we have

$$\|\mathcal{P}_{\mathcal{L},k}^{(\alpha,\beta)}\|^2_{w_{\mathcal{L}}^{(\alpha,\beta)}} = h_{\mathcal{L},k}^{(\alpha,\beta)} = \frac{\mathcal{L}^{\alpha+\beta+1}\,\Gamma(k+\alpha+1)\,\Gamma(k+\beta+1)}{(2k+\alpha+\beta+1)\,k!\,\Gamma(k+\alpha+\beta+1)}. \tag{14}$$

3 SJCM for solving NF generalized sKdV equation with HC

In this section, we utilize the shifted Jacobi collocation method (SJCM) for solving time-fractional nonlinear (T-FN), space-fractional nonlinear (S-FN) and time-space-fractional nonlinear (T-S-FN) generalized sKdV equations respectively with the homogeneous conditions (HC).

3.1 T-FN generalized sKdV equation

Here, we are occupied with utilizing the shifted Jacobi collocation method (SJCM) for solving time-fractional nonlinear (T-FN) generalized sKdV equation:

$$\frac{\partial^\mu u(x,t)}{\partial t^\mu} + u(x,t)\frac{\partial u(x,t)}{\partial x} + \frac{\partial^3 u(x,t)}{\partial x^3} - \frac{\partial^5 u(x,t)}{\partial x^5} + \varepsilon\frac{\partial^7 u(x,t)}{\partial x^7} = \mathcal{H}_1(x,t),\ 0 \le x \le \mathcal{L},\ 0 \le t \le \mathcal{T}, \tag{15}$$

the parameter μ refers to the fractional order of time derivative with $0 < \mu \le 1$ and the term $\mathcal{H}_1(x,t)$ denotes the field variable. We also assume the homogeneous conditions:

$$u(x,0) = u(0,t) = \frac{\partial u(\mathcal{L},t)}{\partial x} = \frac{\partial^2 u(\mathcal{L},t)}{\partial x^2} = \frac{\partial^3 u(\mathcal{L},t)}{\partial x^3} = \frac{\partial^4 u(\mathcal{L},t)}{\partial x^4} = \frac{\partial^5 u(\mathcal{L},t)}{\partial x^5} = \frac{\partial^6 u(\mathcal{L},t)}{\partial x^6} = 0, \tag{16}$$

where $\mathcal{L}$ and $\mathcal{T}$ are given and $\dfrac{\partial^\mu u(x,t)}{\partial t^\mu}$ represents the Liouville-Caputo fractional derivative. In addition, ε is given constant coefficient.

The point of our method is to get solution can be extended, using combination of basis functions of shifted Jacobi polynomials, in the form

$$u_{N,M}(x,t) \backsimeq \sum_{i=0}^{N}\sum_{j=0}^{M} c_{i,j}\varphi_{\mathcal{L},i}^{(\zeta_1,\eta_1)}(x)\psi_{\mathcal{T},j}^{(\zeta_2,\eta_2)}(t) = \varphi(x,t)C, \tag{17}$$

where, $c_{i,j},\ i = 0,1,\ldots,N,\ j = 0,1,\ldots,M$ are the unknown coefficients,

$$C = [c_{0,0}, c_{0,1}, \ldots, c_{0,M}, c_{1,0}, c_{1,1}, \ldots, c_{1,M}, c_{N,0}, c_{N,1}, \ldots, c_{N,M}]^T,$$

N and M are any arbitrary positive integers, and we choose the following set of basis functions

$$\varphi_{\mathcal{L},i}^{(\zeta_1,\eta_1)}(x) = x(\mathcal{L}-x)^7\mathcal{P}_{\mathcal{L},i}^{(\zeta_1,\eta_1)}(x), \tag{18}$$

$$\psi_{\mathcal{T},j}^{(\zeta_2,\eta_2)}(t) = t\mathcal{P}_{\mathcal{T},j}^{(\zeta_2,\eta_2)}(t). \tag{19}$$

Also $\varphi(x,t)$ is $1 \times (N+1)(M+1)$ matrix introduced as follows

$$\begin{aligned}\varphi(x,t) =&[r_{0,0}(x,t), r_{0,1}(x,t), \ldots, r_{0,M}(x,t), r_{1,0}(x,t), r_{1,1}(x,t), \ldots, r_{1,M}(x,t)\\ &, r_{N,0}(x,t), r_{N,1}(x,t), \ldots, r_{N,M}(x,t)],\end{aligned}$$

where

$$r_{i,j}(x,t) = \varphi_{\mathcal{L},i}^{(\zeta_1,\eta_1)}(x)\psi_{\mathcal{T},j}^{(\zeta_2,\eta_2)}(t),\ i = 0,1,\ldots,N,\ j = 0,1,\ldots,M.$$

Substituting Eq. (17) into Eq. (15) yields:

$$\begin{aligned}&\frac{\partial^\mu}{\partial t^\mu}(\sum_{i=0}^{N}\sum_{j=0}^{M} c_{i,j}\varphi_{\mathcal{L},i}^{(\zeta_1,\eta_1)}(x)\psi_{\mathcal{T},j}^{(\zeta_2,\eta_2)}(t)) + (\sum_{i=0}^{N}\sum_{j=0}^{M} c_{i,j}\varphi_{\mathcal{L},i}^{(\zeta_1,\eta_1)}(x)\psi_{\mathcal{T},j}^{(\zeta_2,\eta_2)}(t))\\ &\times \frac{\partial}{\partial x}(\sum_{i=0}^{N}\sum_{j=0}^{M} c_{i,j}\varphi_{\mathcal{L},i}^{(\zeta_1,\eta_1)}(x)\psi_{\mathcal{T},j}^{(\zeta_2,\eta_2)}(t)) + \frac{\partial^3}{\partial x^3}(\sum_{i=0}^{N}\sum_{j=0}^{M} c_{i,j}\varphi_{\mathcal{L},i}^{(\zeta_1,\eta_1)}(x)\psi_{\mathcal{T},j}^{(\zeta_2,\eta_2)}(t))\\ &- \frac{\partial^5}{\partial x^5}(\sum_{i=0}^{N}\sum_{j=0}^{M} c_{i,j}\varphi_{\mathcal{L},i}^{(\zeta_1,\eta_1)}(x)\psi_{\mathcal{T},j}^{(\zeta_2,\eta_2)}(t)) + \varepsilon\frac{\partial^7}{\partial x^7}(\sum_{i=0}^{N}\sum_{j=0}^{M} c_{i,j}\varphi_{\mathcal{L},i}^{(\zeta_1,\eta_1)}(x)\psi_{\mathcal{T},j}^{(\zeta_2,\eta_2)}(t)) = \mathcal{H}_1(x,t).\end{aligned} \tag{20}$$

Therefore, adopting (5), enable one to write (20) in the form:

$$\begin{aligned}&\frac{1}{\Gamma(1-\mu)}\int_0^t (t-\sigma)^{-\mu}\frac{\partial}{\partial\sigma}(\sum_{i=0}^{N}\sum_{j=0}^{M} c_{i,j}\varphi_{\mathcal{L},i}^{(\zeta_1,\eta_1)}(x)\psi_{\mathcal{T},j}^{(\zeta_2,\eta_2)}(\sigma))d\sigma + (\sum_{i=0}^{N}\sum_{j=0}^{M} c_{i,j}\varphi_{\mathcal{L},i}^{(\zeta_1,\eta_1)}(x)\psi_{\mathcal{T},j}^{(\zeta_2,\eta_2)}(t))\\ &\times \frac{\partial}{\partial x}(\sum_{i=0}^{N}\sum_{j=0}^{M} c_{i,j}\varphi_{\mathcal{L},i}^{(\zeta_1,\eta_1)}(x)\psi_{\mathcal{T},j}^{(\zeta_2,\eta_2)}(t)) + \frac{\partial^3}{\partial x^3}(\sum_{i=0}^{N}\sum_{j=0}^{M} c_{i,j}\varphi_{\mathcal{L},i}^{(\zeta_1,\eta_1)}(x)\psi_{\mathcal{T},j}^{(\zeta_2,\eta_2)}(t))\\ &- \frac{\partial^5}{\partial x^5}(\sum_{i=0}^{N}\sum_{j=0}^{M} c_{i,j}\varphi_{\mathcal{L},i}^{(\zeta_1,\eta_1)}(x)\psi_{\mathcal{T},j}^{(\zeta_2,\eta_2)}(t)) + \varepsilon\frac{\partial^7}{\partial x^7}(\sum_{i=0}^{N}\sum_{j=0}^{M} c_{i,j}\varphi_{\mathcal{L},i}^{(\zeta_1,\eta_1)}(x)\psi_{\mathcal{T},j}^{(\zeta_2,\eta_2)}(t)) = \mathcal{H}_1(x,t),\end{aligned} \tag{21}$$

using Eq. (17); then we get

$$\begin{aligned}&\frac{1}{\Gamma(1-\mu)}\int_0^t (t-\sigma)^{-\mu}\frac{\partial}{\partial\sigma}(\varphi(x,\sigma)C)d\sigma + (\varphi(x,t)C)\frac{\partial}{\partial x}(\varphi(x,t)C) + \frac{\partial^3}{\partial x^3}(\varphi(x,t)C)\\ &- \frac{\partial^5}{\partial x^5}(\varphi(x,t)C) + \varepsilon\frac{\partial^7}{\partial x^7}(\varphi(x,t)C) = \mathcal{H}_1(x,t).\end{aligned} \tag{22}$$

Now, we collocate Eq. (22) at the nodes of shifted Jacobi polynomials $(x_{\mathcal{L},n,i}^{(\zeta_1,\eta_1)}, t_{\mathcal{T},m,j}^{(\zeta_2,\eta_2)})$ *for* $n = 0,1,\ldots,N,\ m = 0,1,\ldots,M$. Hence, we have

$$\frac{1}{\Gamma(1-\mu)}\int_0^{t_{\mathcal{T},m,j}^{(\zeta_2,\eta_2)}}(t-\sigma)^{-\mu}\frac{\partial}{\partial\sigma}(\varphi(x_{\mathcal{L},n,i}^{(\zeta_1,\eta_1)},\sigma)C)d\sigma+(\varphi(x_{\mathcal{L},n,i}^{(\zeta_1,\eta_1)},t_{\mathcal{T},m,j}^{(\zeta_2,\eta_2)})C)$$
$$\times\frac{\partial}{\partial x}(\varphi(x_{\mathcal{L},n,i}^{(\zeta_1,\eta_1)},t_{\mathcal{T},m,j}^{(\zeta_2,\eta_2)})C)+\frac{\partial^3}{\partial x^3}(\varphi(x_{\mathcal{L},n,i}^{(\zeta_1,\eta_1)},t_{\mathcal{T},m,j}^{(\zeta_2,\eta_2)})C)-\frac{\partial^5}{\partial x^5}(\varphi(x_{\mathcal{L},n,i}^{(\zeta_1,\eta_1)},t_{\mathcal{T},m,j}^{(\zeta_2,\eta_2)})C) \tag{23}$$
$$+\varepsilon\frac{\partial^7}{\partial x^7}(\varphi(x_{\mathcal{L},n,i}^{(\zeta_1,\eta_1)},t_{\mathcal{T},m,j}^{(\zeta_2,\eta_2)})C)=\mathcal{H}_1(x_{\mathcal{L},n,i}^{(\zeta_1,\eta_1)},t_{\mathcal{T},m,j}^{(\zeta_2,\eta_2)}),$$

for $n=0,1,\ldots,N,\ m=0,1,\ldots,M$. Eq. (23) generates a nonlinear system of equations that can be solved numerically by Newton's iteration method to obtain unknown vector C.

3.2 S-FN generalized sKdV equation

Here, we consider the accompanying space-fractional nonlinear (S-FN) generalized sKdV equation:

$$\frac{\partial u(x,t)}{\partial t}+u(x,t)\frac{\partial^\nu u(x,t)}{\partial x^\nu}+\frac{\partial^3 u(x,t)}{\partial x^3}-\frac{\partial^5 u(x,t)}{\partial x^5}+\varepsilon\frac{\partial^7 u(x,t)}{\partial x^7}=\mathcal{H}_2(x,t),\ 0\le x\le\mathcal{L},\ 0\le t\le\mathcal{T}, \tag{24}$$

the parameter ν refers to the fractional order of space derivative with $0<\nu\le 1$. We also consider the homogeneous conditions in (16), where $\dfrac{\partial^\nu u(x,t)}{\partial x^\nu}$ represents the Liouville-Caputo fractional derivative.

The point of our method is to get solution in (17). Substituting Eq. (17) into Eq. (24) yields:

$$\frac{\partial}{\partial t}(\sum_{i=0}^{N}\sum_{j=0}^{M}c_{i,j}\varphi_{\mathcal{L},i}^{(\zeta_1,\eta_1)}(x)\psi_{\mathcal{T},j}^{(\zeta_2,\eta_2)}(t))+(\sum_{i=0}^{N}\sum_{j=0}^{M}c_{i,j}\varphi_{\mathcal{L},i}^{(\zeta_1,\eta_1)}(x)\psi_{\mathcal{T},j}^{(\zeta_2,\eta_2)}(t))$$
$$\times\frac{\partial^\nu}{\partial x^\nu}(\sum_{i=0}^{N}\sum_{j=0}^{M}c_{i,j}\varphi_{\mathcal{L},i}^{(\zeta_1,\eta_1)}(x)\psi_{\mathcal{T},j}^{(\zeta_2,\eta_2)}(t))+\frac{\partial^3}{\partial x^3}(\sum_{i=0}^{N}\sum_{j=0}^{M}c_{i,j}\varphi_{\mathcal{L},i}^{(\zeta_1,\eta_1)}(x)\psi_{\mathcal{T},j}^{(\zeta_2,\eta_2)}(t)) \tag{25}$$
$$-\frac{\partial^5}{\partial x^5}(\sum_{i=0}^{N}\sum_{j=0}^{M}c_{i,j}\varphi_{\mathcal{L},i}^{(\zeta_1,\eta_1)}(x)\psi_{\mathcal{T},j}^{(\zeta_2,\eta_2)}(t))+\varepsilon\frac{\partial^7}{\partial x^7}(\sum_{i=0}^{N}\sum_{j=0}^{M}c_{i,j}\varphi_{\mathcal{L},i}^{(\zeta_1,\eta_1)}(x)\psi_{\mathcal{T},j}^{(\zeta_2,\eta_2)}(t))=\mathcal{H}_2(x,t).$$

Therefore, adopting (6), enable one to write (25) in the form:

$$\begin{aligned}
&\frac{\partial}{\partial t}(\sum_{i=0}^{N}\sum_{j=0}^{M} c_{i,j}\varphi_{\mathcal{L},i}^{(\zeta_1,\eta_1)}(x)\psi_{\mathcal{T},j}^{(\zeta_2,\eta_2)}(t)) + (\sum_{i=0}^{N}\sum_{j=0}^{M} c_{i,j}\varphi_{\mathcal{L},i}^{(\zeta_1,\eta_1)}(x)\psi_{\mathcal{T},j}^{(\zeta_2,\eta_2)}(t))\\
&\times \frac{1}{\Gamma(1-\nu)}\int_0^x (x-\sigma)^{-\nu}\frac{\partial}{\partial\sigma}(\sum_{i=0}^{N}\sum_{j=0}^{M} c_{i,j}\varphi_{\mathcal{L},i}^{(\zeta_1,\eta_1)}(\sigma)\psi_{\mathcal{T},j}^{(\zeta_2,\eta_2)}(t))d\sigma\\
&+ \frac{\partial^3}{\partial x^3}(\sum_{i=0}^{N}\sum_{j=0}^{M} c_{i,j}\varphi_{\mathcal{L},i}^{(\zeta_1,\eta_1)}(x)\psi_{\mathcal{T},j}^{(\zeta_2,\eta_2)}(t)) - \frac{\partial^5}{\partial x^5}(\sum_{i=0}^{N}\sum_{j=0}^{M} c_{i,j}\varphi_{\mathcal{L},i}^{(\zeta_1,\eta_1)}(x)\psi_{\mathcal{T},j}^{(\zeta_2,\eta_2)}(t))\\
&+ \varepsilon\frac{\partial^7}{\partial x^7}(\sum_{i=0}^{N}\sum_{j=0}^{M} c_{i,j}\varphi_{\mathcal{L},i}^{(\zeta_1,\eta_1)}(x)\psi_{\mathcal{T},j}^{(\zeta_2,\eta_2)}(t)) = \mathcal{H}_2(x,t),
\end{aligned} \tag{26}$$

using Eq. (17); then we get

$$\begin{aligned}
&\frac{\partial}{\partial t}(\varphi(x,t)C) + (\varphi(x,t)C)\frac{1}{\Gamma(1-\nu)}\int_0^x (x-\sigma)^{-\nu}\frac{\partial}{\partial\sigma}(\varphi(\sigma,t)C)d\sigma + \frac{\partial^3}{\partial x^3}(\varphi(x,t)C)\\
&- \frac{\partial^5}{\partial x^5}(\varphi(x,t)C) + \varepsilon\frac{\partial^7}{\partial x^7}(\varphi(x,t)C) = \mathcal{H}_2(x,t).
\end{aligned} \tag{27}$$

Now, we collocate Eq. (27) at the nodes of shifted Jacobi polynomials $(x_{\mathcal{L},n,i}^{(\zeta_1,\eta_1)}, t_{\mathcal{T},m,j}^{(\zeta_2,\eta_2)})$ for $n = 0, 1, \ldots, N,\ m = 0, 1, \ldots, M$. Hence, we have

$$\begin{aligned}
&\frac{\partial}{\partial t}(\varphi(x_{\mathcal{L},n,i}^{(\zeta_1,\eta_1)}, t_{\mathcal{T},m,j}^{(\zeta_2,\eta_2)})C) + \frac{(\varphi(x_{\mathcal{L},n,i}^{(\zeta_1,\eta_1)}, t_{\mathcal{T},m,j}^{(\zeta_2,\eta_2)})C)}{\Gamma(1-\nu)}\int_0^{x_{\mathcal{L},n,i}^{(\zeta_1,\eta_1)}} (x-\sigma)^{-\nu}\frac{\partial}{\partial\sigma}(\varphi(\sigma, t_{\mathcal{T},m,j}^{(\zeta_2,\eta_2)})C)d\sigma\\
&+ \frac{\partial^3}{\partial x^3}(\varphi(x_{\mathcal{L},n,i}^{(\zeta_1,\eta_1)}, t_{\mathcal{T},m,j}^{(\zeta_2,\eta_2)})C) - \frac{\partial^5}{\partial x^5}(\varphi(x_{\mathcal{L},n,i}^{(\zeta_1,\eta_1)}, t_{\mathcal{T},m,j}^{(\zeta_2,\eta_2)})C) + \varepsilon\frac{\partial^7}{\partial x^7}(\varphi(x_{\mathcal{L},n,i}^{(\zeta_1,\eta_1)}, t_{\mathcal{T},m,j}^{(\zeta_2,\eta_2)})C)\\
&= \mathcal{H}_2(x_{\mathcal{L},n,i}^{(\zeta_1,\eta_1)}, t_{\mathcal{T},m,j}^{(\zeta_2,\eta_2)}),
\end{aligned} \tag{28}$$

for $n = 0, 1, \ldots, N,\ m = 0, 1, \ldots, M$. Eq. (28) generates a nonlinear system of equations that can be solved numerically by Newton's iteration method to obtain unknown vector C.

3.3 T-S-FN generalized sKdV equation

Here, we consider the accompanying timespace-fractional nonlinear (T-S-FN) generalized sKdV equation:

$$\frac{\partial^\mu u(x,t)}{\partial t^\mu} + u(x,t)\frac{\partial^\nu u(x,t)}{\partial x^\nu} + \frac{\partial^3 u(x,t)}{\partial x^3} - \frac{\partial^5 u(x,t)}{\partial x^5} + \varepsilon\frac{\partial^7 u(x,t)}{\partial x^7} = \mathcal{H}_3(x,t),\ 0 \le x \le \mathcal{L},\ 0 \le t \le \mathcal{T}, \tag{29}$$

the parameters $\mu,\ \nu$ refers to the fractional order of time and space derivatives with $0 < \mu \le 1,\ 0 < \nu \le 1$. We also consider the homogeneous conditions in (16), where $\frac{\partial^\mu u(x,t)}{\partial t^\mu}$, $\frac{\partial^\nu u(x,t)}{\partial x^\nu}$ represents the Liouville-Caputo fractional derivative.

The point of our method is to get solution in (17). Substituting Eq. (17) into Eq. (29) yields:

$$\begin{aligned}&\frac{\partial^{\mu}}{\partial t^{\mu}}(\sum_{i=0}^{N}\sum_{j=0}^{M}c_{i,j}\varphi_{\mathcal{L},i}^{(\zeta_1,\eta_1)}(x)\psi_{\mathcal{T},j}^{(\zeta_2,\eta_2)}(t))+(\sum_{i=0}^{N}\sum_{j=0}^{M}c_{i,j}\varphi_{\mathcal{L},i}^{(\zeta_1,\eta_1)}(x)\psi_{\mathcal{T},j}^{(\zeta_2,\eta_2)}(t))\\&\times\frac{\partial^{\nu}}{\partial x^{\nu}}(\sum_{i=0}^{N}\sum_{j=0}^{M}c_{i,j}\varphi_{\mathcal{L},i}^{(\zeta_1,\eta_1)}(x)\psi_{\mathcal{T},j}^{(\zeta_2,\eta_2)}(t))+\frac{\partial^{3}}{\partial x^{3}}(\sum_{i=0}^{N}\sum_{j=0}^{M}c_{i,j}\varphi_{\mathcal{L},i}^{(\zeta_1,\eta_1)}(x)\psi_{\mathcal{T},j}^{(\zeta_2,\eta_2)}(t))\\&-\frac{\partial^{5}}{\partial x^{5}}(\sum_{i=0}^{N}\sum_{j=0}^{M}c_{i,j}\varphi_{\mathcal{L},i}^{(\zeta_1,\eta_1)}(x)\psi_{\mathcal{T},j}^{(\zeta_2,\eta_2)}(t))+\varepsilon\frac{\partial^{7}}{\partial x^{7}}(\sum_{i=0}^{N}\sum_{j=0}^{M}c_{i,j}\varphi_{\mathcal{L},i}^{(\zeta_1,\eta_1)}(x)\psi_{\mathcal{T},j}^{(\zeta_2,\eta_2)}(t))=\mathcal{H}_3(x,t).\end{aligned}\tag{30}$$

Therefore, adopting (5) and (6) , enable one to write (30) in the form:

$$\begin{aligned}&\frac{1}{\Gamma(1-\mu)}\int_0^t(t-\sigma)^{-\mu}\frac{\partial}{\partial\sigma}(\sum_{i=0}^{N}\sum_{j=0}^{M}c_{i,j}\varphi_{\mathcal{L},i}^{(\zeta_1,\eta_1)}(x)\psi_{\mathcal{T},j}^{(\zeta_2,\eta_2)}(\sigma))d\sigma\\&+\frac{(\sum_{i=0}^{N}\sum_{j=0}^{M}c_{i,j}\varphi_{\mathcal{L},i}^{(\zeta_1,\eta_1)}(x)\psi_{\mathcal{T},j}^{(\zeta_2,\eta_2)}(t))}{\Gamma(1-\nu)}\int_0^x(x-\sigma)^{-\nu}\frac{\partial}{\partial\sigma}(\sum_{i=0}^{N}\sum_{j=0}^{M}c_{i,j}\varphi_{\mathcal{L},i}^{(\zeta_1,\eta_1)}(\sigma)\psi_{\mathcal{T},j}^{(\zeta_2,\eta_2)}(t))d\sigma\\&+\frac{\partial^{3}}{\partial x^{3}}(\sum_{i=0}^{N}\sum_{j=0}^{M}c_{i,j}\varphi_{\mathcal{L},i}^{(\zeta_1,\eta_1)}(x)\psi_{\mathcal{T},j}^{(\zeta_2,\eta_2)}(t))-\frac{\partial^{5}}{\partial x^{5}}(\sum_{i=0}^{N}\sum_{j=0}^{M}c_{i,j}\varphi_{\mathcal{L},i}^{(\zeta_1,\eta_1)}(x)\psi_{\mathcal{T},j}^{(\zeta_2,\eta_2)}(t))\\&+\varepsilon\frac{\partial^{7}}{\partial x^{7}}(\sum_{i=0}^{N}\sum_{j=0}^{M}c_{i,j}\varphi_{\mathcal{L},i}^{(\zeta_1,\eta_1)}(x)\psi_{\mathcal{T},j}^{(\zeta_2,\eta_2)}(t))=\mathcal{H}_3(x,t),\end{aligned}\tag{31}$$

using Eq. (17); then we get

$$\begin{aligned}&\frac{1}{\Gamma(1-\mu)}\int_0^t(t-\sigma)^{-\mu}\frac{\partial}{\partial\sigma}(\varphi(x,\sigma)C)d\sigma+(\varphi(x,t)C)\frac{1}{\Gamma(1-\nu)}\int_0^x(x-\sigma)^{-\nu}\frac{\partial}{\partial\sigma}(\varphi(\sigma,t)C)d\sigma\\&+\frac{\partial^{3}}{\partial x^{3}}(\varphi(x,t)C)-\frac{\partial^{5}}{\partial x^{5}}(\varphi(x,t)C)+\varepsilon\frac{\partial^{7}}{\partial x^{7}}(\varphi(x,t)C)=\mathcal{H}_3(x,t).\end{aligned}\tag{32}$$

Now, we collocate Eq. (32) at the nodes of shifted Jacobi polynomials $(x_{\mathcal{L},n,i}^{(\zeta_1,\eta_1)},t_{\mathcal{T},m,j}^{(\zeta_2,\eta_2)})$ for $n=0,1,\ldots,N,\ m=0,1,\ldots,M$. Hence, we have

$$\begin{aligned}&\frac{1}{\Gamma(1-\mu)}\int_0^{t_{\mathcal{T},m,j}^{(\zeta_2,\eta_2)}}(t-\sigma)^{-\mu}\frac{\partial}{\partial\sigma}(\varphi(x_{\mathcal{L},n,i}^{(\zeta_1,\eta_1)},\sigma)C)d\sigma+(\varphi(x_{\mathcal{L},n,i}^{(\zeta_1,\eta_1)},t_{\mathcal{T},m,j}^{(\zeta_2,\eta_2)})C)\\&\times\frac{1}{\Gamma(1-\nu)}\int_0^{x_{\mathcal{L},n,i}^{(\zeta_1,\eta_1)}}(x-\sigma)^{-\nu}\frac{\partial}{\partial\sigma}(\varphi(\sigma,t_{\mathcal{T},m,j}^{(\zeta_2,\eta_2)})C)d\sigma+\frac{\partial^{3}}{\partial x^{3}}(\varphi(x_{\mathcal{L},n,i}^{(\zeta_1,\eta_1)},t_{\mathcal{T},m,j}^{(\zeta_2,\eta_2)})C)\\&-\frac{\partial^{5}}{\partial x^{5}}(\varphi(x_{\mathcal{L},n,i}^{(\zeta_1,\eta_1)},t_{\mathcal{T},m,j}^{(\zeta_2,\eta_2)})C)+\varepsilon\frac{\partial^{7}}{\partial x^{7}}(\varphi(x_{\mathcal{L},n,i}^{(\zeta_1,\eta_1)},t_{\mathcal{T},m,j}^{(\zeta_2,\eta_2)})C)=\mathcal{H}_3(x_{\mathcal{L},n,i}^{(\zeta_1,\eta_1)},t_{\mathcal{T},m,j}^{(\zeta_2,\eta_2)}),\end{aligned}\tag{33}$$

for $n = 0, 1, \ldots, N,\ m = 0, 1, \ldots, M$. Eq. (33) generates a nonlinear system of equations that can be solved numerically by Newton's iteration method to obtain unknown vector C.

4 Treatment of the NHC

This section extend the previous algorithm to treat with the time-fractional nonlinear (T-FN), space-fractional nonlinear (S-FN) and time-space-fractional nonlinear (T-S-FN) generalized sKdV equations respectively with the nonhomogeneous conditions (NHC)

$$\begin{aligned} &u(x,0) = g_1(x),\ u(0,t) = g(t),\ \frac{\partial u(\mathcal{L},t)}{\partial x} = g_2(t),\ \frac{\partial^2 u(\mathcal{L},t)}{\partial x^2} = g_3(t), \frac{\partial^3 u(\mathcal{L},t)}{\partial x^3} = g_4(t),\\ &\frac{\partial^4 u(\mathcal{L},t)}{\partial x^4} = g_5(t),\ \frac{\partial^5 u(\mathcal{L},t)}{\partial x^5} = g_6(t),\ \frac{\partial^6 u(\mathcal{L},t)}{\partial x^6} = g_7(t),\ 0 \le x \le \mathcal{L},\ 0 \le x \le \mathcal{T}. \end{aligned} \tag{34}$$

where $g(t),\ g_1(x),\ g_2(t),\ g_3(t),\ g_4(t),\ g_5(t),\ g_6(t)$ and $g_7(t)$ are given functions.

Presently, assume the accompanying transformation

$$V(x,t) = u(x,t) + a(t) + xb(t) + x^2c(t) + x^3d(t) + x^4e(t) + x^5\delta(t) + x^6\varepsilon(t) + x(x-\mathcal{L})^7\gamma(x). \tag{35}$$

where

$$\begin{aligned} &a(t) = -g(t),\ b(t) = \frac{1}{120}\left[-120g_2(t) + 120\mathcal{L}g_3(t) - 60\mathcal{L}^2g_4(t) + 20\mathcal{L}^3g_5(t) - 5\mathcal{L}^4g_6(t) + \mathcal{L}^5g_7(t)\right],\\ &c(t) = \frac{1}{48}\left[-24g_3(t) + 24\mathcal{L}g_4(t) - 12\mathcal{L}^2g_5(t) + 4\mathcal{L}^3g_6(t) - \mathcal{L}^4g_7(t)\right],\\ &d(t) = \frac{1}{36}\left[-6g_4(t) + 6\mathcal{L}g_5(t) - 3\mathcal{L}^2g_6(t) + \mathcal{L}^3g_7(t)\right],\ e(t) = \frac{1}{48}\left[-2g_5(t) + 2\mathcal{L}g_6(t) - \mathcal{L}^2g_7(t)\right],\\ &\delta(t) = \frac{1}{120}\left[-g_6(t) + \mathcal{L}g_7(t)\right],\ \varepsilon(t) = -\frac{1}{720}g_7(t),\\ &\gamma(x) = -\frac{a(0) + xb(0) + x^2c(0) + x^3d(0) + x^4e(0) + g_1(x) + x^5\delta(0) + x^6\varepsilon(0)}{x(x-\mathcal{L})^7}. \end{aligned}$$

The transformation (35) turns the nonhomogeneous initial-boundary conditions (34) into the homogeneous initial-boundary conditions

$$\begin{aligned} V(x,0) = V(0,t) = \frac{\partial V(\mathcal{L},t)}{\partial x} = &\frac{\partial^2 V(\mathcal{L},t)}{\partial x^2} = \frac{\partial^3 V(\mathcal{L},t)}{\partial x^3} = \frac{\partial^4 V(\mathcal{L},t)}{\partial x^4} = \frac{\partial^5 V(\mathcal{L},t)}{\partial x^5} = \frac{\partial^6 V(\mathcal{L},t)}{\partial x^6} = 0,\\ &0 \le x \le \mathcal{L},\ 0 \le x \le \mathcal{T}. \end{aligned} \tag{36}$$

Subsequently it suffices to solve the following T-FN generalized sKdV equation:

$$\begin{aligned} &\frac{\partial^\mu V(x,t)}{\partial t^\mu} + V(x,t)\frac{\partial V(x,t)}{\partial x} + \frac{\partial^3 V(x,t)}{\partial x^3} - \frac{\partial^5 V(x,t)}{\partial x^5} + \varepsilon\frac{\partial^7 V(x,t)}{\partial x^7} = \mathcal{G}_1(x,t),\\ &0 \le x \le \mathcal{L},\ 0 \le t \le \mathcal{T}, \end{aligned} \tag{37}$$

S-FN generalized sKdV equation:

$$\frac{\partial V(x,t)}{\partial t}+v(x,t)\frac{\partial^{\nu}V(x,t)}{\partial x^{\nu}}+\frac{\partial^{3}V(x,t)}{\partial x^{3}}-\frac{\partial^{5}V(x,t)}{\partial x^{5}}+\varepsilon\frac{\partial^{7}V(x,t)}{\partial x^{7}}=\mathcal{G}_{2}(x,t),\quad 0\leq x\leq\mathcal{L},\ 0\leq t\leq\mathcal{T},\tag{38}$$

and T-S-FN generalized sKdV equation:

$$\frac{\partial^{\mu}V(x,t)}{\partial t^{\mu}}+V(x,t)\frac{\partial^{\nu}V(x,t)}{\partial x^{\nu}}+\frac{\partial^{3}V(x,t)}{\partial x^{3}}-\frac{\partial^{5}V(x,t)}{\partial x^{5}}+\varepsilon\frac{\partial^{7}V(x,t)}{\partial x^{7}}=\mathcal{G}_{3}(x,t),\quad 0\leq x\leq\mathcal{L},\ 0\leq t\leq\mathcal{T}.\tag{39}$$

5 Convergence analysis

In this section we investigate the convergence analysis of the proposed polynomial series expansion.

Lemma 1. *If* $\max\{|\zeta|,|\eta|\}\leq\frac{1}{2}$, *then the following estimate holds:*

$$|\mathcal{P}_{\mathcal{L},k}^{(\zeta,\eta)}(z)|\leq\left(\frac{\mathcal{L}}{2}\right)^{\frac{\lambda}{2}}\frac{\Gamma(q+1)\binom{n+q}{n}}{\Gamma(\frac{1}{2})(n+\frac{\lambda}{2})^{q+\frac{1}{2}}}w^{(-\frac{2\zeta+1}{4},-\frac{2\eta+1}{4})}(z),$$

where

$$q=\max\{\zeta,\eta\},\lambda=\zeta+\eta+1.$$

Proof. See [29]. Q.E.D.

Lemma 2. *The following integral formula is valid:*

$$D_{x}^{-1}\mathcal{P}_{\mathcal{L},k}^{(\zeta,\eta)}(x)=\int\mathcal{P}_{\mathcal{L},k}^{(\zeta,\eta)}(x)\,dx=\frac{\mathcal{L}}{k+\lambda-1}\mathcal{P}_{\mathcal{L},k+1}^{(\zeta-1,\eta-1)}(x).$$

Proof. Integrating both sides of Eq. (12), when $r=1$, we get the result. Q.E.D.

Lemma 3. *The basis* $B_{i,j}(x,t)=\varphi_{\mathcal{L},i}^{(\zeta_1,\eta_1)}(x)\psi_{\mathcal{T},j}^{(\zeta_2,\eta_2)}(t)$ *are orthogonal in* $L_{W}^{2}([0,\mathcal{L}]\times[0,\tau])$, *where,* $W=\omega^{(\zeta_1-2,\eta_1-14)}(x)\,\omega^{(\zeta_2-2,\eta_2)}(t)$, *i.e.,*

$$\int_{0}^{\tau}\int_{0}^{\mathcal{L}}B_{i,j}(x,t)B_{m,n}(x,t)W\,dx\,dt=\Lambda_{i,j}\delta i,m\delta j,n,$$

where,

$$\Lambda_{i,j}=\frac{\mathcal{L}^{\lambda_1}\tau^{\lambda_2}\Gamma(i+\zeta_1+1)\Gamma(i+\eta_1+1)\Gamma(j+\zeta_2+1)\Gamma(j+\eta_2+1)}{i!\,j!\,(2i+\lambda_1)(2j+\lambda_2)\Gamma(i+\lambda_1)\Gamma(j+\lambda_2)}.$$

Proof. By the definition of $\varphi_{\mathcal{L},i}^{(\zeta_1,\eta_1)}(x),\psi_{\mathcal{T},j}^{(\zeta_2,\eta_2)}(t)$ and the orthogonality relation of shifted Jacobi polynomials, we get the desired result.

Q.E.D.

Lemma 4. *There exist two generic positive constants κ_1, κ_2, such that, for real constant μ, we have,*

$$\kappa_1\, n^{\mu-1}\, n! < \Gamma(n+\mu) < \kappa_2\, n^{\mu-1}\, n!.$$

Proof. See [30]. Q.E.D.

As a direct result of Lemma 4, we have the following important theorem.

Theorem 1. *The following estimates are valid:*
a) $|\Lambda_{i,j}| = \mathcal{O}(\frac{1}{ij})$.
b) *Under the assumptions of Lemma 1, there exist a generic positive constant ς such that*

$$|J_{\mathcal{L},k}^{(\zeta,\eta)}(z)| \leq \frac{\varsigma}{\sqrt{k}} w^{(-\frac{2\zeta+1}{4},-\frac{2\eta+1}{4})}(z).$$

Theorem 2. *If $u(x,t)$ satisfy the homogenous conditions* (16) *is separable, in the sense that, $u(x,t) = x(\mathcal{L}-x)^{\intercal}\, t\, f(x)\, g(t)$ is approximated by*

$$u \approx u_{M,N} = \sum_{j=0}^{N}\sum_{i=0}^{M} c_{i,j}\varphi_{\mathcal{L},i}^{(\zeta_1,\eta_2)}(x)\psi_{\tau,j}^{(\zeta_2,\eta_2)}(t),$$

and $|f^{(i)}| \leq \gamma_i$, $|g^{(i)}| \leq \varrho_i$; $i = 0,1,2$, then the unknown expansion coefficients $a_{i,j}$ is dominated by the following estimate

$$|c_{i,j}| = \mathcal{O}(i^{-\frac{3}{2}}\, j^{-\frac{3}{2}}).$$

Proof. Based on the orthogonality relation given in Lemma 3, and the assumptions of the theorem, we have,

$$c_{i,j} = \frac{1}{\Lambda_{i,j}} \left(\int_0^{\mathcal{L}} J_{\mathcal{L},i}^{(\zeta_1,\eta_1)}(x) f(x) w^{(\zeta_1+7,\eta_1+1)}(x)\, dx \right) \left(\int_0^{\tau} J_{\tau,j}^{(\zeta_2,\eta_2)}(t) g(t) w^{(\zeta_2,\eta_2+1)}(t)\, dt \right).$$

By integration by parts twice for the last two integrals, we get,

$$\begin{aligned}
c_{i,j} &= \frac{1}{\Lambda_{i,j}} \int_0^{\mathcal{L}} D_x^{-2}\left(J_{\mathcal{L},i}^{(\zeta_1,\eta_1)}(x) \right) w^{(\zeta_1+5,\eta_1-1)} \\
&\times \left[f'' x^2(\mathcal{L}-x)^2 - 2f' x(\mathcal{L}-x)(x(\zeta_1+7) - (\mathcal{L}-x)(\eta_1+1)) \right. \\
&+ \left. f(x^2(\zeta_1+6)_2 - 2(\zeta_1+7)(\eta_1+1)x(\mathcal{L}-x) + (\eta_1)_2(\mathcal{L}-x)^2) \right] dx \\
&\times \int_0^{\tau} D_t^{-2}\left(J_{\tau,j}^{(\zeta_2,\eta_2)}(t) \right) w^{(\zeta_2-2,\eta_2-1)} \\
&\times \left[g'' t^2(\tau-t)^2 - 2g' t(\tau-t)(t(\zeta_2) - (\tau-t)(\eta_2+1)) \right. \\
&+ \left. g(t^2(\zeta_2-1)_2 - 2(\zeta_2)(\eta_2+1)t(\tau-t) + (\eta_2)_2(\tau-t)^2) \right] dt.
\end{aligned}$$

Now, if $\max\{|\zeta_1-2|, |\zeta_2-2|, |\eta_1-2|, |\eta_2-2|\} \leq \frac{1}{2}$, application of Lemma 1, by the boundedness of f, g and their derivatives and by Theorem 1, then we get the desired result. Q.E.D.

Theorem 3. *If $\mathcal{E}_{M,N} = \|u - u_{M,N}\|_2$, then we have the following truncation error estimate:*

$$\mathcal{E}_{M,N} = \mathcal{O}(N^{-\frac{3}{4}}\, M^{-\frac{3}{4}}).$$

Proof. By Parseval's identity and Theorem 1 part b), we have

$$\mathcal{E}^2_{M,N} = \mathcal{O}\left(\sum_{j=N}^{\infty}\sum_{i=M}^{\infty}\frac{|a_{i,j}|^2}{i\,j}\right),$$

by the result of Theorem 2, we have

$$\mathcal{E}^2_{M,N} = \mathcal{O}\left(\sum_{j=N}^{\infty}\sum_{i=M}^{\infty}\frac{1}{i^{\frac{5}{2}}\,j^{\frac{5}{2}}}\right) = \mathcal{O}(M^{-\frac{3}{2}}\,N^{-\frac{3}{2}}),$$

which completes the proof of the theorem. Q.E.D.

6 Numerical results

In this section, a few examples are outlined to demonstrate the pertinence and proficiency of the novel technique in homogeneous and nonhomogeneous conditions. The calculations are executed by utilizing Mathematica of Version 8, and all counts are completed in a PC of CPU Intel(R) Core(TM) i3-2350M 2 Duo CPU 2.30 GHz, 6.00 GB of RAM.

The distinction between the measured value of approximate solution and its actual value (absolute error), given by

$$E(x,t) = |u(x,t) - u_{N,M}(x,t)|, \tag{40}$$

where $u(x,t)$ and $u_{N,M}(x,t)$ are the exact solution and the numerical solution at the point (x,t), respectively. Moreover, the maximum absolute errors (MAEs) is given by

$$\text{MAEs} = \text{Max}\{E(x,t) : \forall (x,t) \in [0,\mathcal{L}] \times [0,\mathcal{T}]\} = L^{\infty}, \tag{41}$$

Also we can denote to the root mean square error (RMSE) by

$$RMSE = \sqrt{\frac{\sum_{i=0}^{N}(u(x^{(\zeta_1,\eta_1)}_{\mathcal{L},n,i},t) - u_{N,M}(x^{(\zeta_1,\eta_1)}_{\mathcal{L},n,i},t))^2}{N}}, \tag{42}$$

Example 1. *Consider nonlinear time-fractional generalized sKdV equation* (15) *subject to the following initial and boundary conditions,*

$$u(x,0) = 0,\ u(0,t) = \frac{t^{2\mu}}{\Gamma(1+2\mu)},\ \frac{\partial u(\mathcal{L},t)}{\partial x} = -\frac{t^{2\mu}\sin(\mathcal{L})}{\Gamma(1+2\mu)},\ \frac{\partial^2 u(\mathcal{L},t)}{\partial x^2} = -\frac{t^{2\mu}\cos(\mathcal{L})}{\Gamma(1+2\mu)},$$
$$\frac{\partial^3 u(\mathcal{L},t)}{\partial x^3} = \frac{t^{2\mu}\sin(\mathcal{L})}{\Gamma(1+2\mu)},\ \frac{\partial^4 u(\mathcal{L},t)}{\partial x^4} = \frac{t^{2\mu}\cos(\mathcal{L})}{\Gamma(1+2\mu)},\ \frac{\partial^5 u(\mathcal{L},t)}{\partial x^5} = -\frac{t^{2\mu}\sin(\mathcal{L})}{\Gamma(1+2\mu)},\ \frac{\partial^6 u(\mathcal{L},t)}{\partial x^6} = -\frac{t^{2\mu}\cos(\mathcal{L})}{\Gamma(1+2\mu)}.$$

The exact solution is given by

$$u(x,t) = \frac{t^{2\mu}\cos(x)}{\Gamma(1+2\mu)},$$

and

$$\mathcal{H}_1(x,t) = \frac{t^{\mu}\cos(x)}{\Gamma(1+\mu)} + \frac{t^{4\mu}\sin(x)\cos(x)}{(\Gamma(1+2\mu))^2} + \frac{3t^{2\mu}\sin(x)}{\Gamma(1+2\mu)}.$$

The numerical arrangements of Example 1 are displayed for $\varepsilon = 1$ in Tables 1,2 and 3. In [28], Saha and Gupta applied Legendre wavelet method (LWM) based on the Legendre wavelets expansion together with operational matrices of fractional integration and derivative of these basis functions for solving time-fractional seventh order KdV (sKdV). In Tables 1,2 and 3, we compare the L^2, L^∞ and absolute errors using our scheme at different values of $\zeta,\ \eta,\ N,\ M,\ x,\ t,\ \mu$ and those achieved using the (LWM) [28]. While in Figs. 1 and 2, we graph

TABLE 1. Comparison of the L^2 and L^∞ error for Example 1.

t	L^2 error in [28]	L^2 error by our method ($N=10,\ M=4,\ \mu=1$)		L^∞ error in [28]	L^∞ error by our method ($N=10,\ M=4,\ \mu=1$)	
		$\zeta_1=\eta_1=-\frac{1}{2}$ $\zeta_2=\eta_2=-\frac{1}{2}$	$\zeta_1=\eta_1=\frac{1}{2}$ $\zeta_2=\eta_2=\frac{1}{2}$		$\zeta_1=\eta_1=-\frac{1}{2}$ $\zeta_2=\eta_2=-\frac{1}{2}$	$\zeta_1=\eta_1=\frac{1}{2}$ $\zeta_2=\eta_2=\frac{1}{2}$
0.05	$9.1643.10^{-6}$	$3.1424.10^{-9}$	$1.1817.10^{-9}$	$5.3919.10^{-6}$	$3.7696.10^{-8}$	$2.1607.10^{-8}$
0.10	$1.0150.10^{-4}$	$1.6710.10^{-9}$	$7.0221.10^{-10}$	$5.5779.10^{-5}$	$2.0625.10^{-8}$	$1.3250.10^{-8}$
0.15	$4.6858.10^{-4}$	$6.7193.10^{-10}$	$3.6178.10^{-10}$	$2.3893.10^{-4}$	$8.6557.10^{-9}$	$7.0452.10^{-9}$
0.20	$1.4399.10^{-3}$	$4.9316.10^{-11}$	$1.3453.10^{-10}$	$6.9125.10^{-4}$	$8.2325.10^{-10}$	$2.6788.10^{-9}$
0.25	$3.5030.10^{-3}$	$2.8557.10^{-10}$	$3.7116.10^{-12}$	$1.6015.10^{-3}$	$3.7465.10^{-9}$	$1.4767.10^{-10}$
0.30	$7.3440.10^{-3}$	$4.0629.10^{-10}$	$7.3775.10^{-11}$	$3.2189.10^{-3}$	$5.8395.10^{-9}$	$1.7201.10^{-9}$
0.35	$1.3917.10^{-2}$	$3.8018.10^{-10}$	$9.7033.10^{-11}$	$5.8669.10^{-3}$	$6.1531.10^{-9}$	$1.7201.10^{-9}$
0.40	$2.4560.10^{-2}$	$2.6401.10^{-10}$	$9.1556.10^{-11}$	$9.9599.10^{-3}$	$5.2958.10^{-9}$	$2.1778.10^{-9}$
0.45	$4.1165.10^{-2}$	$1.1057.10^{-10}$	$7.4296.10^{-11}$	$1.6021.10^{-2}$	$3.7872.10^{-9}$	$1.5670.10^{-9}$
0.50	$4.8629.10^{-2}$	$1.0294.10^{-10}$	$6.0916.10^{-11}$	$2.6175.10^{-2}$	$2.0582.10^{-9}$	$7.1006.10^{-10}$

Example 2. *Consider nonlinear space-fractional generalized sKdV equation* (24) *subject to the following initial and boundary conditions,*

$$u(x,0)=0,\ u(0,t)=0,\ \frac{\partial u(\mathcal{L},t)}{\partial x} = t^3\cos(\mathcal{L}),\ \frac{\partial^2 u(\mathcal{L},t)}{\partial x^2} = -t^3\sin(\mathcal{L}), \frac{\partial^3 u(\mathcal{L},t)}{\partial x^3} = -t^3\cos(\mathcal{L}),$$
$$\frac{\partial^4 u(\mathcal{L},t)}{\partial x^4} = t^3\sin(\mathcal{L}),\ \frac{\partial^5 u(\mathcal{L},t)}{\partial x^5} = -t^3\cos(\mathcal{L}),\ \frac{\partial^6 u(\mathcal{L},t)}{\partial x^6} = -t^3\sin(\mathcal{L}).$$

The exact solution is given by

$$u(x,t) = t^3\sin(x),$$

and

$$\mathcal{H}_2(x,t) = -\frac{t^6 x^{1-\nu}\sin(x)\,{}_1F_2\left(1;1-\frac{\nu}{2},\frac{3}{2}-\frac{\nu}{2};-\frac{x^2}{4}\right)}{(1-\nu)\Gamma(1-\nu)} - 2.003t^3\cos(x) + 3t^2\sin(x).$$

TABLE 2. Comparison of the absolute errors with various choices of $x,\ t$ and $\mu = 0.75$ for Example 1.

(x,t)	Absolute error in [28]	Absolute error by our method $(N = M = 8)$		
		$\zeta_1 = \eta_1 = -\frac{1}{2}$ $\zeta_2 = \eta_2 = -\frac{1}{2}$	$\zeta_1 = \eta_1 = 0$ $\zeta_2 = \eta_2 = 0$	$\zeta_1 = \eta_1 = \frac{1}{2}$ $\zeta_2 = \eta_2 = \frac{1}{2}$
(0.1,0.05)	$1.4884.10^{-3}$	$6.9498.10^{-5}$	$8.5442.10^{-5}$	$7.4114.10^{-5}$
(0.2,0.10)	$8.0493.10^{-4}$	$6.6487.10^{-5}$	$3.0252.10^{-5}$	$5.6699.10^{-6}$
(0.3,0.15)	$2.3898.10^{-3}$	$3.5901.10^{-5}$	$2.6357.10^{-5}$	$2.3692.10^{-5}$
(0.4,0.20)	$9.7063.10^{-3}$	$2.0714.10^{-5}$	$2.1486.10^{-6}$	$2.5450.10^{-6}$
(0.5,0.25)	$2.7738.10^{-2}$	$3.5282.10^{-5}$	$2.8043.10^{-5}$	$1.0908.10^{-5}$
(0.6,0.3)	$5.4660.10^{-2}$	$1.5318.10^{-5}$	$4.0102.10^{-5}$	$8.0647.10^{-6}$

TABLE 3. Comparison of the absolute errors with various choices of $x,\ t,\ \mu$ for Example 1.

(x,t)	Absolute error in [28] at $\mu = 0.85$	Absolute error by our method $(N = M = 8,\ \mu = 0.85)$		Absolute error in [28] at $\mu = 0.9$	Absolute error by our method $(N = M = 8,\ \mu = 0.9)$	
		$\zeta_1 = \eta_1 = -\frac{1}{2}$ $\zeta_2 = \eta_2 = -\frac{1}{2}$	$\zeta_1 = \eta_1 = \frac{1}{2}$ $\zeta_2 = \eta_2 = \frac{1}{2}$		$\zeta_1 = \eta_1 = -\frac{1}{2}$ $\zeta_2 = \eta_2 = -\frac{1}{2}$	$\zeta_1 = \eta_1 = \frac{1}{2}$ $\zeta_2 = \eta_2 = \frac{1}{2}$
(0.1,0.05)	$6.9542.10^{-4}$	$2.5953.10^{-5}$	$3.1766.10^{-5}$	$1.9246.10^{-2}$	$1.3277.10^{-5}$	$4.0545.10^{-6}$
(0.2,0.10)	$1.0150.10^{-2}$	$2.5953.10^{-5}$	$2.3601.10^{-6}$	$1.2199.10^{-2}$	$1.3173.10^{-5}$	$9.5936.10^{-7}$
(0.3,0.15)	$2.1255.10^{-2}$	$1.4195.10^{-5}$	$9.9107.10^{-6}$	$6.0247.10^{-3}$	$7.2507.10^{-6}$	$4.4648.10^{-6}$
(0.4,0.20)	$3.2285.10^{-2}$	$8.2580.10^{-6}$	$1.0698.10^{-6}$	$4.0212.10^{-2}$	$4.2341.10^{-6}$	$5.0299.10^{-7}$
(0.5,0.25)	$3.8269.10^{-2}$	$1.4149.10^{-5}$	$4.5949.10^{-6}$	$8.0539.10^{-2}$	$7.2760.10^{-6}$	$2.2260.10^{-6}$
(0.6,0.05)	$3.1420.10^{-4}$	$2.1958.10^{-5}$	$2.4568.10^{-5}$	$9.4467.10^{-3}$	$1.1101.10^{-5}$	$9.2542.10^{-6}$
(0.7,0.10)	$6.3441.10^{-3}$	$2.0275.10^{-5}$	$1.8081.10^{-6}$	$2.5638.10^{-3}$	$1.0319.10^{-5}$	$7.5946.10^{-7}$
(0.8,0.15)	$2.6382.10^{-2}$	$1.0371.10^{-5}$	$7.4219.10^{-6}$	$9.5203.10^{-3}$	$5.3212.10^{-6}$	$2.5477.10^{-6}$
(0.9,0.20)	$5.3601.10^{-2}$	$5.5798.10^{-6}$	$8.1291.10^{-7}$	$3.4243.10^{-2}$	$2.8689.10^{-6}$	$1.2363.10^{-7}$
(1.0,0.25)	$9.3426.10^{-2}$	$8.6962.10^{-6}$	$3.3992.10^{-6}$	$7.1281.10^{-2}$	$4.4538.10^{-6}$	$3.3303.10^{-7}$

The numerical arrangements of Example 2 are displayed for $\varepsilon = 0.003$ in Tables 4 and 5. Moreover, we plotted the exact and approximate solutions in Figs. 3 and 4, where

Example 3. *Consider nonlinear time-space-fractional generalized sKdV equation* (29) *subject to the following initial and boundary conditions,*

$$u(x,0) = 0,\ u(0,t) = t^3,\ \frac{\partial u(\mathcal{L},t)}{\partial x} = -t^3\sin(\mathcal{L}),\ \frac{\partial^2 u(\mathcal{L},t)}{\partial x^2} = -t^3\cos(\mathcal{L}), \frac{\partial^3 u(\mathcal{L},t)}{\partial x^3} = t^3\sin(\mathcal{L}),$$
$$\frac{\partial^4 u(\mathcal{L},t)}{\partial x^4} = t^3\cos(\mathcal{L}),\ \frac{\partial^5 u(\mathcal{L},t)}{\partial x^5} = -t^3\sin(\mathcal{L}),\ \frac{\partial^6 u(\mathcal{L},t)}{\partial x^6} = -t^3\cos(\mathcal{L}).$$

The exact solution is given by

$$u(x,t) = t^3\cos(x),$$

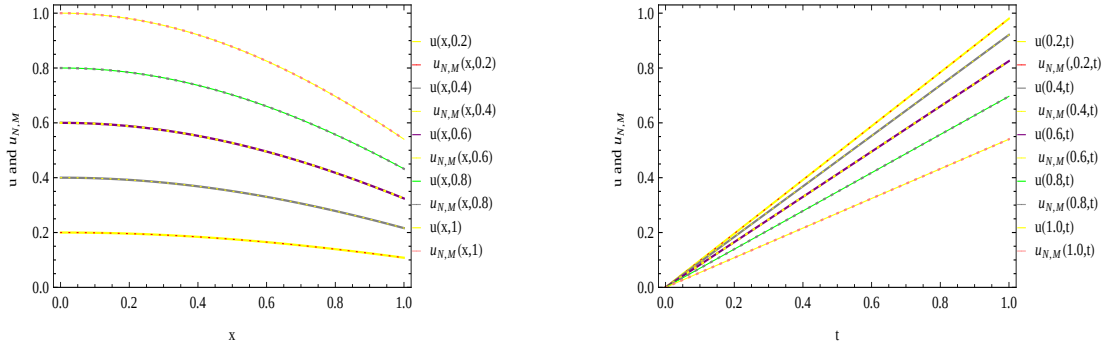


FIGURE 1. The curves-graph of exact $u(x,t)$ and numerical $u_{N,M}(x,t)$ solutions for Example 1 where $N=8,\ M=4$ at different values of t.

FIGURE 2. The curves-graph of exact $u(x,t)$ and numerical $u_{N,M}(x,t)$ solutions for Example 1 where $N=8,\ M=4$ at different values of x.

TABLE 4. Comparison of the L^2 and L^∞ error for Example 2.

t	L^2 error by our method ($N=8,\ M=4,\ \nu=1$)			L^∞ error by our method ($N=8,\ M=4,\ \nu=1$)		
	$\zeta_1=\eta_1=-\frac{1}{2}$ $\zeta_2=\eta_2=-\frac{1}{2}$	$\zeta_1=\eta_1=0$ $\zeta_2=\eta_2=0$	$\zeta_1=\eta_1=\frac{1}{2}$ $\zeta_2=\eta_2=\frac{1}{2}$	$\zeta_1=\eta_1=-\frac{1}{2}$ $\zeta_2=\eta_2=-\frac{1}{2}$	$\zeta_1=\eta_1=0$ $\zeta_2=\eta_2=0$	$\zeta_1=\eta_1=\frac{1}{2}$ $\zeta_2=\eta_2=\frac{1}{2}$
0.1	$1.6108.10^{-10}$	$2.3619.10^{-11}$	$1.5761.10^{-11}$	$2.9935.10^{-8}$	$5.8379.10^{-9}$	$2.8635.10^{-11}$
0.2	$1.7670.10^{-11}$	$1.3038.10^{-11}$	$1.3160.10^{-11}$	$8.8962.10^{-10}$	$5.3094.10^{-10}$	$1.7092.10^{-10}$
0.3	$1.2426.10^{-10}$	$1.5250.10^{-11}$	$9.6646.10^{-12}$	$7.5788.10^{-9}$	$1.7409.10^{-8}$	$5.9271.10^{-10}$
0.4	$2.2495.10^{-11}$	$1.8438.10^{-11}$	$2.3031.10^{-12}$	$5.5930.10^{-9}$	$3.5202.10^{-8}$	$1.3250.10^{-9}$
0.5	$3.2398.10^{-10}$	$2.3005.10^{-11}$	$6.1150.10^{-11}$	$6.5054.10^{-10}$	$4.9952.10^{-8}$	$2.4796.10^{-9}$
0.6	$8.1303.10^{-10}$	$1.0057.10^{-10}$	$1.8162.10^{-10}$	$2.3764.10^{-9}$	$6.3343.10^{-8}$	$4.2239.10^{-9}$
0.7	$1.2325.10^{-9}$	$1.8492.10^{-10}$	$3.3568.10^{-10}$	$2.2883.10^{-9}$	$8.2700.10^{-8}$	$6.7819.10^{-9}$
0.8	$1.2458.10^{-9}$	$2.2060.10^{-10}$	$4.7545.10^{-10}$	$4.2400.10^{-9}$	$1.2098.10^{-7}$	$1.0433.10^{-8}$
0.9	$4.0116.10^{-10}$	$1.3000.10^{-10}$	$5.2960.10^{-10}$	$9.3769.10^{-9}$	$1.9680.10^{-7}$	$1.5514.10^{-8}$
1.0	$1.9088.10^{-9}$	$2.0914.10^{-10}$	$4.0993.10^{-10}$	$1.0220.10^{-8}$	$3.3440.10^{-7}$	$2.2417.10^{-8}$

and

$$\mathcal{H}_3(x,t)=t^3\left(\frac{t^3x^{2-\nu}\cos(x)\,{}_1F_2\left(1;\frac{3}{2}-\frac{\nu}{2},2-\frac{\nu}{2};-\frac{x^2}{4}\right)}{\Gamma(3-\nu)}+\frac{6t^{-\mu}\cos(x)}{\Gamma(4-\mu)}+2.01\sin(x)\right).$$

The numerical arrangements of Example 3 are displayed for $\varepsilon=0.01$ in Tables 6 and 7, we

TABLE 5. Comparison of the absolute errors with various choices of $x,\ t$ and $\mu = 0.75$ for Example 2.

(x,t)	Absolute error by our method $(N=8,\ M=4)$		
	$\zeta_1=\eta_1=-\frac{1}{2}$ $\zeta_2=\eta_2=-\frac{1}{2}$	$\zeta_1=\eta_1=0$ $\zeta_2=\eta_2=0$	$\zeta_1=\eta_1=\frac{1}{2}$ $\zeta_2=\eta_2=\frac{1}{2}$
$(0.1,0.1)$	$2.4891.10^{-10}$	$3.3579.10^{-11}$	$2.2233.10^{-11}$
$(0.2,0.2)$	$3.4772.10^{-12}$	$1.8109.10^{-11}$	$1.6449.10^{-11}$
$(0.3,0.3)$	$9.5885.10^{-11}$	$2.2900.10^{-11}$	$1.1841.10^{-11}$
$(0.4,0.4)$	$2.9513.10^{-10}$	$4.0096.10^{-11}$	$6.5230.10^{-13}$
$(0.5,0.5)$	$1.4970.10^{-10}$	$1.0382.10^{-10}$	$1.1778.10^{-10}$
$(0.6,0.6)$	$1.1029.10^{-9}$	$5.8436.10^{-10}$	$4.2631.10^{-10}$
$(0.7,0.7)$	$1.1677.10^{-9}$	$3.5590.10^{-10}$	$5.1464.10^{-10}$
$(0.8,0.8)$	$3.9598.10^{-9}$	$9.0180.10^{-9}$	$2.1126.10^{-9}$
$(0.9,0.9)$	$4.7504.10^{-9}$	$6.8000.10^{-8}$	$1.2240.10^{-8}$
$(1.0,1.0)$	$3.3440.10^{-10}$	$3.3440.10^{-7}$	$2.2417.10^{-8}$

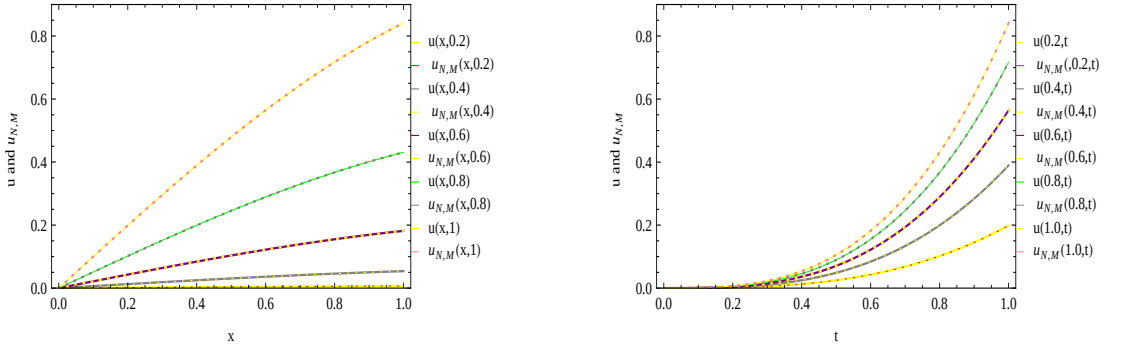


FIGURE 3. The curves-graph of exact $u(x,t)$ and numerical $u_{N,M}(x,t)$ solutions for Example 2 where $N=8,\ M=4$ at different values of t.

FIGURE 4. The curves-graph of exact $u(x,t)$ and numerical $u_{N,M}(x,t)$ solutions for Example 2 where $N=8,\ M=4$ at different values of x.

depicted

TABLE 6. Absolute error using our method at $N = 10,\ M = 4,\ \zeta_1 = \eta_1 = -\frac{1}{2},\ \zeta_2 = \eta_2 = 0,\ \mu = 0.3$ and $\nu = 0.9$ for Example 3.

x	t	$E(x,t)$	x	t	$E(x,t)$	x	t	$E(x,t)$
0.1	0.1	$2.2705.10^{-11}$	0.1	0.5	$1.0363.10^{-10}$	0.1	0.9	$4.4543.10^{-10}$
0.2		$2.4626.10^{-11}$	0.2		$1.0118.10^{-10}$	0.2		$2.7573.10^{-10}$
0.3		$1.5098.10^{-11}$	0.3		$9.9031.10^{-11}$	0.3		$1.3464.10^{-10}$
0.4		$9.0768.10^{-11}$	0.4		$9.8848.10^{-11}$	0.4		$6.5240.10^{-12}$
0.5		$1.6016.10^{-11}$	0.5		$1.0210.10^{-10}$	0.5		$1.0642.10^{-10}$
0.6		$3.2365.10^{-10}$	0.6		$1.1756.10^{-10}$	0.6		$1.9913.10^{-10}$
0.7		$1.4480.10^{-10}$	0.7		$1.7378.10^{-10}$	0.7		$3.2082.10^{-10}$
0.8		$1.3410.10^{-9}$	0.8		$2.9205.10^{-10}$	0.8		$5.2479.10^{-10}$
0.9		$1.9786.10^{-9}$	0.9		$2.7797.10^{-10}$	0.9		$4.2034.10^{-9}$
1.0		$2.7472.10^{-9}$	1.0		$9.8094.10^{-10}$	1.0		$2.1290.10^{-9}$

TABLE 7. Comparison of the L^2 and L^∞ error for Example 3.

t	L^2 error by our method ($N = 8,\ M = 4,\ \mu = 0.3,\ \nu = 0.9$)			L^∞ error by our method ($N = 8,\ M = 4,\ \mu = 0.3,\ \nu = 0.9$)		
	$\zeta_1 = \eta_1 = -\frac{1}{2}$ $\zeta_2 = \eta_2 = -\frac{1}{2}$	$\zeta_1 = \eta_1 = 0$ $\zeta_2 = \eta_2 = 0$	$\zeta_1 = \eta_1 = \frac{1}{2}$ $\zeta_2 = \eta_2 = \frac{1}{2}$	$\zeta_1 = \eta_1 = -\frac{1}{2}$ $\zeta_2 = \eta_2 = -\frac{1}{2}$	$\zeta_1 = \eta_1 = 0$ $\zeta_2 = \eta_2 = 0$	$\zeta_1 = \eta_1 = \frac{1}{2}$ $\zeta_2 = \eta_2 = \frac{1}{2}$
0.1	$9.1927.10^{-9}$	$1.2879.10^{-9}$	$9.6617.10^{-10}$	$3.6515.10^{-7}$	$4.4780.10^{-5}$	$8.0986.10^{-8}$
0.2	$3.8836.10^{-10}$	$2.1089.10^{-10}$	$1.6551.10^{-10}$	$1.2014.10^{-8}$	$5.7817.10^{-6}$	$1.5390.10^{-8}$
0.3	$2.4201.10^{-9}$	$2.0318.10^{-10}$	$3.4190.10^{-11}$	$8.8398.10^{-8}$	$6.1425.10^{-6}$	$7.3722.10^{-9}$
0.4	$1.8854.10^{-9}$	$2.2644.10^{-10}$	$4.3218.10^{-11}$	$6.1445.10^{-8}$	$4.6575.10^{-6}$	$7.6642.10^{-9}$
0.5	$8.5327.10^{-11}$	$7.9485.10^{-11}$	$1.5178.10^{-10}$	$1.4525.10^{-9}$	$4.2402.10^{-7}$	$2.5064.10^{-10}$
0.6	$1.4783.10^{-9}$	$1.3209.10^{-10}$	$1.6462.10^{-10}$	$4.2813.10^{-8}$	$3.1423.10^{-6}$	$5.6692.10^{-9}$
0.7	$1.8770.10^{-9}$	$2.7361.10^{-10}$	$4.2608.10^{-11}$	$3.9094.10^{-8}$	$1.3900.10^{-6}$	$6.4682.10^{-9}$
0.8	$7.5760.10^{-10}$	$3.2937.10^{-10}$	$1.5530.10^{-10}$	$2.1259.10^{-9}$	$3.0873.10^{-6}$	$4.0914.10^{-9}$
0.9	$1.6617.10^{-9}$	$2.5802.10^{-10}$	$2.7031.10^{-10}$	$2.8062.10^{-8}$	$4.6916.10^{-6}$	$6.0560.10^{-9}$
1.0	$4.5900.10^{-9}$	$1.3219.10^{-10}$	$5.4201.10^{-11}$	$3.1108.10^{-8}$	$6.0279.10^{-6}$	$2.5451.10^{-8}$

7 Conclusions

We have applied a shifted Jacobi collocation method to solve nonlinear fractional generalized seventh order KdV. This method used shifted Jacobi collocation approach to reduce the considered nonlinear fractional generalized seventh order KdV equations to the solution of a matrix equation. In the given examples, through selection of a relatively few shifted Jacobi collocation points, we are able to get very accurate approximations. We are thus able to demonstrate the utility of our approach over other analytical or numerical methods such as other collocation methods or perturbation methods. The comparison of the obtained results with those based on other methods shows that the present

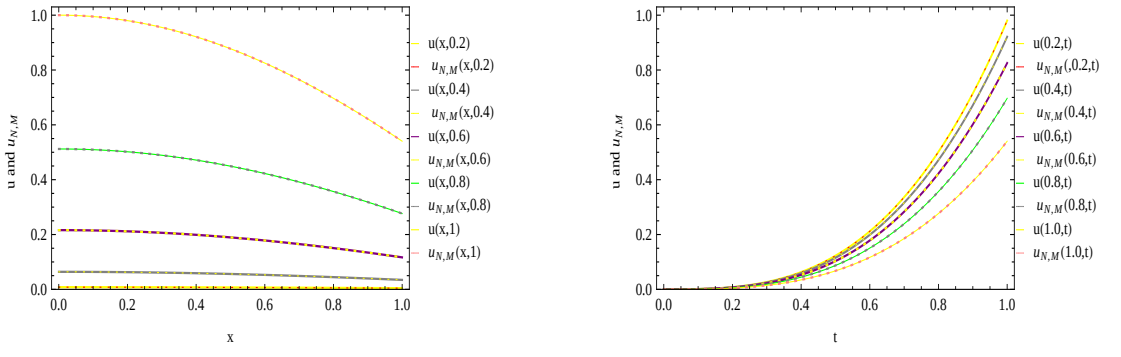


FIGURE 5. The curves-graph of exact $u(x,t)$ and numerical $u_{N,M}(x,t)$ solutions for Example 3 where $N = 8,\ M = 4$ at different values of t.

FIGURE 6. The curves-graph of exact $u(x,t)$ and numerical $u_{N,M}(x,t)$ solutions for Example 3 where $N = 8,\ M = 4$ at different values of x.

technique is an effective device for finding the numerical solutions of such equations.

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