Uniqueness of some differential-difference polynomials of entire functions sharing small function

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Abstract

In this article, we discuss the uniqueness problems of differential-difference polynomials of hyper order (≥ 1) meromorphic functions sharing a small function and prove some results, which generalise and improve the results due to Y. Liu, J. P. Wang and F. H. Liu.

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1 Introduction and main results

In this article, we assume that the reader is familiar with the fundamental results and the standard notation of Nevanlinna value distribution theory (see [2], [6] and [11]). Let $f(z)$ and $g(z)$ be two non constant meromorphic functions in the complex plane. By $S(r, f)$, we mean any quantity satisfying $S(r, f) = o\{T(r, f)\}$ as $r \to \infty$, possibly outside a set of finite logarithmic measure.
A meromorphic function $a(z)$ is called a small function with respect to f, provided that $T(r, a) = s(r, f)$. The order and hyper order of meromophic function f are defined respectively by

$$\sigma(f) = \lim_{r\to\infty} sup\frac{logT(r, f)}{logr}$$

and

$$\sigma_2(f) = \lim_{r\to\infty} sup\frac{log\, logT(r, f)}{logr}$$

Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions. Let and $a \in \mathbb{C} \cup \{\infty\}$. If the zeros of $f - a$ and $g - a$ coincide in locations and multiplicity, we say that f and g share the value a CM(counting multiplicities). On the other hand, if the zeros of $f - a$ and $g - a$ coincide only in their locations, then we say that f and g share the value a IM(ignoring multiplicities). For a positive integer p we denote by $N_p(r, a; f)$ the counting function of a-points of f, where an a-point of multiplicity m is counted m times if $m \leq p$ and p times if $m > p$.
Define the difference of $f(z)$ by

$$\triangle_c f(z) = f(z + c) - f(z)$$

Let k be a non-negative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m \leq k$ and $k + 1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f, g share the value a with weight k.

The definition implies that if f, g share a value a with weight k then z_0 is an a-point of f with multiplicity $m(\leq k)$ if and only if it is an a-point of g with multiplicity $m(\leq k)$ and z_0 is an a-point

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of f with multiplicity $m(> k)$ if and only if it is an a-point of g with multiplicity $n(> k)$, where m is not necessarily equal to n.

We write "f, g share (a, k)"to mean that "f, g share (a, k) then f, g share (a, p) for any integer p, $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

In 2010, Qi et al.[7] studied the uniqueness of the difference monomials and obtained the following result:

Theorem A. Let $f(z)$ and $g(z)$ be transcendental entire functions with finite order, c a nonzero complex constant, and $n \geq 6$ an integer. If $E(1, f^n(z)f(z + c)) = E(1, g^n(z)g(z + c))$, then $f(z) \equiv t_1 g(z)$ or $f(z)g(z) = t_2$, for some constants t_1 and t_2 that satisfy $t_1^{n+1} = 1$ and $t_2^{n+1} = 1$.

In 2015, Y.Liu, J.P.Wang and F.H.Liu[5] obtained the following results.

Theorem B. Let $c \in \mathbb{C} \setminus \{0\}$. Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions with finite order, and $n(\geq 14), k(\geq 3)$ be two positive integers. If $E_k(1, f^n(z)f(z + c)) = E_k(1, g^n(z)g(z + c))$, then $f(z) \equiv t_1 g(z)$ or $f(z)g(z) = t_2$, for some constants t_1 and t_2 that satisfy $t_1^{n+1} = 1$ and $t_2^{n+1} = 1$.

Theorem C. Let $c \in \mathbb{C}$ and $n \geq 16$ be an integer. Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions with finite order. If $E_2(1, f^n(z)f(z + c)) = E_2(1, g^n(z)g(z + c))$, then $f(z) \equiv t_1 g(z)$ or $f(z)g(z) = t_2$, for some constants t_1 and t_2 that satisfy $t_1^{n+1} = 1$ and $t_2^{n+1} = 1$.

Theorem D. Let $c \in \mathbb{C}$ and $n \geq 22$ be an integer. Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions with finite order. If $E_1(1, f^n(z)f(z + c)) = E_1(1, g^n(z)g(z + c))$, then $f(z) \equiv t_1 g(z)$ or $f(z)g(z) = t_2$, for some constants t_1 and t_2 that satisfy $t_1^{n+1} = 1$ and $t_2^{n+1} = 1$.

Let $P(z) = a_n z^n + a_{n-1} z^{n-1} + ... + a_0$ be a nonzero polynomial where $a_n(\neq 0), a_{n-1}, ..., a_0$ are complex constants. We denote Γ_1 by $\Gamma_1 = m_1 + m_2$ and $\Gamma_2 = m_1 + 2m_2$ respectively, where m_1 is the number of simple zeros of $P(z)$ and m_2 is the number of multiple zeros of $P(z)$.

Question 1. What would happen if we replaced the difference polynomials $f^n f(z+c)$ by $[P(f) \triangle_c f]^{(k)}$ in Theorems B-D, where k is any positive integer?

Regarding Theorems B-D, it is natural to ask the following question which is the motivation of the present paper.

In this paper, our aim is to find out the possible answer of the above question. We prove the following result.

Theorem 1. Let f and g be two trancendental entire functions of order $\sigma_2(f) < 1$, and let $c \in \mathbb{C}$. If $E_l(a(z), [P(f) \triangle_c f]^{(k)}) = E_l(a(z), [P(g) \triangle_c g]^{(k)})$ and l, m, n are integers satisfying one of the following conditions:
(i) $l \geq 2, n > 2\Gamma_2 + 2km_2 + 3$;

(ii) $l = 1, n > 2\Gamma_2 + \frac{5}{2}km_2 + \frac{1}{2}km_1 + 3$;
(iii) $l = 0, n > 2\Gamma_2 + 5km_2 + 3km_1 + 9$;

Then one of the following conclusions occur :
(i) $f(z) = tg(z)$ for a constant t such that $t^d = 1$, where $d = GCD(m+1, ..., m+1-i, ..., 1)$, $a_{m-i} \neq 0$ for $i = 0, 1, ..., m$
(ii) $f(z)$ and $g(z)$ satisfy the algebraic difference equation $R(f, g) \equiv 0$, where

$$R(w_1, w_2) = (a_m w_1^m + a_{m-1} w_1^{m-1} + ... + a_0)[w_1(z+c) - w_1(z)] -$$
$$(a_m w_2^m + a_{m-1} w_2^{m-1} + ... + a_0)[w_2(z+c) - w_2(z)]$$

2 Lemmas

In this section, we state some lemmas which will be needed in the sequel.

Lemma 2.1 [16, Lemma 1]. Let f be a nonconstant meromorphic function, and $P(f) = a_n f^n + a_{n-1} f^{n-1} + ... + a_0$, where $a_n (\neq 0), a_{n-1}, ..., a_0$ are complex constants. Then

$$T(r, P(f)) = nT(r, f) + S(r, f).$$

Lemma 2.2 [15]. Let f be a nonconstant meromorphic function and p, k be two positive integers. Then

$$N_p(r, 0; f^{(k)}) \leq T(r, f^{(k)}) - T(r, f) + N_{p+k}(r, 0; f) + S(r, f), \tag{2.1}$$
$$N_p(r, 0; f^{(k)}) \leq k\overline{N}(r, \infty; f) + N_{p+k}(r, 0; f) + S(r, f). \tag{2.2}$$

Lemma 2.3 [12]. Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions. Then

$$N\left(r, \infty; \frac{f}{g}\right) - N\left(r, \infty; \frac{g}{f}\right) = N(r, \infty; f) + N(r, 0; g) - N(r, \infty; g) - N(r, 0; f).$$

Lemma 2.4 [13]. Let $f(z)$ be a transcendental meromorphic function of finite order. Then

$$T(r, f(z+c)) = T(r, f) + S(r, f).$$

Lemma 2.5. By using above Lemma 2.4 and properties of $T(r, f)$, we have

$$\begin{aligned} T(r, \triangle_c f) &= T(r, f(z+c) - f(z)) \\ &\leq T(r, f(z+c)) + T(r, f(z)) + S(r, f) \\ &= 2T(r, f) + S(r, f). \end{aligned}$$

Lemma 2.6 [13]. Let $f(z)$ be a meromorphic function of finite order σ, and let $c \in C \setminus \{0\}$ be fixed. Then for each $\varepsilon > 0$, we have

$$m\left(r, \frac{f(z+c)}{f(z)}\right) + m\left(r, \frac{f(z)}{f(z+c)}\right) = o(r^{\sigma-1+\varepsilon}) = S(r, f).$$

Lemma 2.7 [3]. Let F and G be non-constant meromorphic functions, let n and k be two positive integers. If $E_2(1;F) = E_2(1;G)$ then one of the following cases holds:
(i) $T(r) \leq N_2(r,0;F) + N_2(r,0;G) + N_2(r,\infty;F) + N_2(r,\infty;G) + S(r)$,
(ii) $F = G$,
(iii) $FG = 1$,
where $T(r) = max\{T(r,F), T(r,G)\}$ and $S(r) = o\{T(r)\}$.

Lemma 2.8 [10, Lemma 2.3]. Let F and G be non-constant meromorphic functions, let n and k be two positive integers. If F and G share 1 CM then one of the following cases holds:
(i) $T(r,F) \leq N_2(r,0;F) + N_2(r,0;G) + N_2(r,\infty;F) + N_2(r,\infty;G) + S(r,F) + S(r,G)$,
the same inequality holding for $T(r,G)$,
(ii) $F = G$,
(iii) $FG = 1$.

Lemma 2.9. From Lemma 2.7 and Lemma 2.8, we conclude that if $E_l(1;F) = E_l(1;G)$, where $l \geq 2$, then also the following three cases of Lemma 2.7 holds:
(i) $T(r) \leq N_2(r,0;F) + N_2(r,0;G) + N_2(r,\infty;F) + N_2(r,\infty;G) + S(r)$,
(ii) $F = G$,
(iii) $FG = 1$,
where $T(r) = max\{T(r,F), T(r,G)\}$ and $S(r) = o\{T(r)\}$.

Lemma 2.10 [1, Lemma 2.14]. Let F and G be non-constant meromorphic functions, let n and k be two positive integers. If $E_1(1;F) = E_1(1;G)$ then one of the following cases holds:
(i) $T(r) \leq N_2(r,0;F) + N_2(r,0;G) + N_2(r,\infty;F) + N_2(r,\infty;G) + \frac{1}{2}\overline{N}(r,0;F) + \frac{1}{2}\overline{N}(r,\infty;F) + S(r,F) + S(r,G)$,
(ii) $F = G$,
(iii) $FG = 1$,
where $T(r) = max\{T(r,F), T(r,G)\}$ and $S(r) = o\{T(r)\}$.

Lemma 2.11 [1, Lemma 2.15]. Let F and G be non-constant meromorphic functions, let n and k be two positive integers. If $E_l(1;F) = E_l(1;G)$, where $l = 0$, then one of the following cases holds:

$$(i)\ T(r) \leq N_2(r,0;F) + N_2(r,0;G) + N_2(r,\infty;F) + N_2(r,\infty;G) + 2\overline{N}(r,0;F) + 2\overline{N}(r,\infty;F) + \overline{N}(r,0;G) + \overline{N}(r,\infty;G) + S(r,F) + S(r,G),$$

(ii) $F = G$, (iii) $FG = 1$,
where $T(r) = max\{T(r,F), T(r,G)\}$ and $S(r) = o\{T(r)\}$.

Lemma 2.12. Let $f(z)$ be a transcendental meromorphic function of finite order, $c \in C \setminus \{0\}$ be finite complex constant and $n \in N$. Let $F_1(z) = P(f) \triangle_c f(z)$, where $\triangle_c f(z) \not\equiv 0$. Then we have

$$(n-2)T(r,f) \leq T(r,F_1) + S(r,f).$$

Proof. From Lemma 2.1, Lemma 2.5, Lemma 2.6 and first fundamental theorem, we obtain

$$\begin{aligned} T(r, f^{n+1}) &= T\left(r, \frac{F_1 f}{\triangle_c f}\right) \\ &\leq T(r, F_1) + T(r, \frac{f}{\triangle_c f}) + S(r, f) \\ &= T(r, F_1) + T(r, \frac{\triangle_c f}{f}) + S(r, f) \\ &= T(r, F_1) + N(r, \infty; \frac{\triangle_c f}{f}) + m(r, \infty; \frac{\triangle_c f}{f}) + S(r, f) \\ &\leq T(r, F_1) + 3T(r, f) + S(r, f) \\ \text{i.e., } (n-2)T(r, f) &\leq T(r, F_1) + S(r, f). \end{aligned}$$

Lemma 2.13 [4, Lemma 2.9]. Let $a_0(z), a_1(z), ..., a_n(z), b(z)$ be polynomials such that $a_0(z)a_n(z) \neq 0$; let c_j be constants and

$$deg\left(\sum_{dega_j=d} a_j\right) = d,$$

where $d = \max_{0\leq j\leq n} a_j$.
If $f(z)$ is a transcendental meromorphic solution of $\sum_{j=0}^{n} a_j f(z + c_j) = b(z)$, then $\sigma(f) \geq 1$.

3 Proof of the Theorem 1.

Let $F_1(z) = P(f) \triangle_c f$ and $G_1(z) = P(g) \triangle_c g, F = F_1^{(k)}$ and $G = G_1^{(k)}$. Then F and G are transcendental meromorphic functions satisfying $E_l(1; F) = E_l(1; G)$. Using (2.1) and Lemma 2.5 we get

$$\begin{aligned} N_2(r, 0; F) &\leq N_2(r, 0; (F_1)^{(k)}) + S(r, f) \\ &\leq T(r, (F_1)^{(k)}) - T(r, F_1) + N_{k+2}(r, 0; F_1) + S(r, f) \\ &= T(r, F) - (n+2)T(r, f) + N_{k+2}(r, 0; F_1) + S(r, f). \end{aligned}$$

From this we get

$$(n+2)T(r, f) \leq T(r, F) - N_2(r, 0; F) + N_{k+2}(r, 0; F_1) + S(r, f). \tag{3.1}$$

Again from (2.2) we have

$$\begin{aligned} N_2(r, 0; F) &\leq N_2(r, 0; (F_1)^{(k)}) + S(r, f) \\ &\leq k\overline{N}(r, \infty; F_1) + N_{k+2}(r, 0; F_1) + S(r, f). \end{aligned} \tag{3.2}$$

We now discuss the following three cases separately.

Case 1. Let $l \geq 2$.
Sub Case 1. Suppose, if possible, that (i) of Lemma 2.9 holds. Then using (3.2) we obtain from

(3.1)

$$\begin{aligned}(n+2)T(r,f) &\le N_2(r,0;G)+N_2(r,\infty;F)+N_2(r,\infty;G)+N_{k+2}(r,0;F_1)+S(r,f)+S(r,g)\\ &\le N_2(r,0;[P(g)\triangle_c g]^{(k)})+N_2(r,\infty;[P(f)\triangle_c f]^{(k)})+N_2(r,\infty;[P(g)\triangle_c g]^{(k)})\\ &+N_{k+2}(r,0;P(f)\triangle_c f)+S(r,f)+S(r,g)\\ &\le N_{k+2}(r,0;P(g)\triangle_c g)+k\overline{N}(r,\infty;P(g)\triangle_c g)+6T(r,f)+6T(r,g)\\ &+(m_1+(k+2)m_2)\overline{N}(r,0;f)+N(r,0;\triangle_c f)+S(r,f)+S(r,g)\\ &\le (m_1+(k+2)m_2T(r,g)+2T(r,g)+kT(r,g)+2kT(r,g)+6T(r,f)+6T(r,g)\\ &+(m_1+(k+2)m_2)T(r,f)+2T(r,f)+S(r,f)+S(r,g)\\ (n+2)T(r,f) &\le (m_1+(k+2)m_2+2)T(r,g)+(m_1+(k+2)m_2+2)T(r,f)+S(r,f)+T(r,g).\end{aligned} \tag{3.3}$$

Similarly,

$$(n+2)T(r,g)\le (m_1+(k+2)m_2+2)T(r,g)+(m_1+(k+2)m_2+2)T(r,g)+S(r,f)+T(r,g). \tag{3.4}$$

Combining (3.3) and (3.4) we have

$$\{n-2m_1-2(k+2)m_2-2\}\{T(r,f)+T(r,g)\}\le S(r,f)+S(r,g),$$

contradicting the fact that $n>2\Gamma_2+2km_2+3$. Therefore, by Lemma 2.9, we have either $FG=1$ or $F=G$.

Sub Case 2. Suppose that $FG\equiv 1$, *i.e.*,

$$[P(f)\Delta_c f]^{(k)}[P(g)(\Delta_c g)]^k\equiv 1. \tag{3.5}$$

Suppose that 'z_0' be a zero of f of order p_1. Then it is zero of $[P(f)\Delta_c f]^{(k)}$ of order $(5k+p_0)p_1-k$, where $(p_0>19)$, then z_0 is a pole of order q_1, then it is a pole of $[p(g)(\Delta_c g)]^k$ of order $(5k+p_0)q_1+k$.

$$\Rightarrow (5k+p_0)p_1-k=(5k+p_0)q_1+k$$

$$\Rightarrow (5k+p_0)(p_1-q_1)=2k$$

which is a contradiction. Since p_1, q_1 and k are integers and $p_0\ge 19$.

Hence $f\neq 0$. Similarly, we can prove $g\neq 0$. Suppose that z_1 be a pole of f of order p_2. Then it is a pole of $[P(f)\Delta_c f]^{(k)}$ of order p_2, then it is a pole of $[P(f)\Delta_c f]^{(k)}$ of order $(5k+p_0)p_2+k$, where $p_0\ge 19$, then z_1 is a zero of g of order q_2, then it is a zero of $[p(g)(\Delta_c g)]^{(k)}$ of order $(5k+p_0)q_2-k$.

$$\Rightarrow (5k+p_0)p_2+k=(5k+p_0)q_2-k$$

$$\Rightarrow (5k+p_0)(q_2-p_2)=2k$$

which is a contradiction. Since p_2, q_2 and k are integers and $p_0\ge 19$.

Hence $f\neq\infty$. Similarly, we can prove $g\neq\infty$. Now we can conclude that f and g are transcendental entire functions, having no zeros and poles.

Therefore $f(z) = e^{\alpha(z)}$ and $g(z) = e^{\beta(z)}$, where $\alpha(z)$ and $\beta(z)$ are non-zero polynomials. Substituting f and g in (3.5), we have

$$\begin{aligned}&[(a_n e^{n\alpha(z)} + a_{n-1}e^{(n-1)\alpha(z)} + ... + a_0)(e^{\alpha(z+c)} - e^{\alpha(z)})]^{(k)}\\ &[(a_n e^{n\beta(z)} + a_{n-1}e^{(n-1)\beta(z)} + ... + a_0)(e^{\beta(z+c)} - e^{\beta(z)})]^{(k)} \equiv 1\end{aligned} \tag{3.6}$$

Let

$$\begin{aligned} n\alpha(z) + \alpha(z+c) &= A_1 & n\alpha(z) + \alpha(z) &= B_1\\ (n-1)\alpha(z) + \alpha(z+c) &= A_2 & (n-1)\alpha(z) + \alpha(z) &= B_2\\ &--- & &--\\ \alpha(z+c) &= A_n & \alpha(z) &= B_n\\ n\beta(z) + \beta(z+c) &= c_1 & n\beta(z) + \beta(z) &= d_1\\ (n-1)\beta(z) + \beta(z+c) &= c_2 & (n-1)\beta(z) + \beta(z) &= d_2\\ &--- & &--\\ \beta(z+c) &= c_n & \beta(z) &= d_n. \end{aligned}$$

Hence (3.6) becomes,

$$\begin{aligned}&[a_n(e^{A_1} - e^{B_1}) + a_{n-1}(e^{A_2} - e^{B_2}) + ... + a_n(e^{A_n} - e^{B_n})]^{(k)}\\ &[a_n(e^{c_1} - e^{d_1}) + a_{n-1}(e^{c_2} - e^{d_2}) + ... + a_n(e^{c_n} - e^{d_n})]^{(k)} \equiv 1.\end{aligned}$$

If k=1, then we have

$$\begin{aligned}&[a_n(e^{A_1} - e^{B_1}) + a_{n-1}(e^{A_2} - e^{B_2}) + ... + a_n(e^{A_n} - e^{B_n})]'\\ &[a_n(e^{c_1} - e^{d_1}) + a_{n-1}(e^{c_2} - e^{d_2}) + ... + a_n(e^{c_n} - e^{d_n})]' \equiv 1.\end{aligned} \tag{3.7}$$

From (3.7), we know that

$$(A_i' e^{A_i} - B_i' e^{B_i}) = e^{B_i}(A_i' e^{A_i - B_i} - B_i') \neq 0$$

$\forall\ i = 1, 2, ..., n$ and hence if $A_i' \neq 0$, then we have $B_i' = 0$. Thus B_i's must be a constant. By lemma 2.13, we have $\sigma(\alpha(z)) \geq 1$ and hence $\sigma_2(f) \geq 1$ which is a contradiction. Therefore $A_i' = 0$, then A_i's must be a constant, similarly as above, we deduce a contradiction.
If $k = 2$ and $i = 1, 2, ..., n$, then by calculation, we have

$$\begin{aligned}&A_i'' e^{A_i} + (A_i')^2 e^{A_i} - B_i'' e^{B_i} + (B_i')^2 e^{B_i}\\ &= e^{B_i}[e^{A_i - B_i}(Ai'' + (A_i')^2) - (B_i'' + (Bi')^2)] \neq 0.\end{aligned}$$

If $Ai'' + (A_i')^2 \neq 0$, then $B_i'' + (Bi')^2 = 0$.
If B_i's are transcendental entire, then we have

$$m(r, B_i') + m\left(r, \frac{B_i''}{B_i'}\right) = S(r, B_i')$$

which is a contradiction to B_i's being transcendental entire. If B_i's are polynomials, from lemma 2.13 which induces that $\sigma_2(f) \geq 1$, we get a contradiction. Therefore,

$$Ai'' + (A_i')^2 = 0.$$

Similarly as above, we get contradiction.

For $k \geq 3$, using the similar method as above, we deduce a contradiction. Therefore, the condition

$$[P(f)\Delta_c f]^{(k)}[P(g)(\Delta_c g)]^k \equiv 1$$

i.e., $FG \equiv 1$ is not possible.

Sub Case 3. Let $F \equiv G$

i.e.,

$$[P(f) \triangle_c f(z)]^{(k)} \equiv [P(g) \triangle_c g(z)]^{(k)}. \tag{3.8}$$

From (3.8) we get

$$P(f) \triangle_c f(z) \equiv P(g) \triangle_c g(z) + P_1(z), \tag{3.9}$$

where $P_1(z)$ is a polynomial of degree atmost $k-1$. Suppose $P_1(z) \not\equiv 0$. Then from (3.6) we have

$$\frac{P(f) \triangle_c f}{P_1} \equiv \frac{P(g) \triangle_c g}{P_1} + 1.$$

Now in view of Lemma 2.12 and the second fundamental theorem we have

$$\begin{aligned}
(n-2)T(r,f) &\leq T(r, P(f) \triangle_c f) + S(r,f) \\
&\leq T(r, \frac{P(f) \triangle_c f}{P_1(z)}) + S(r,f) \\
&\leq \overline{N}(r, \frac{P(f) \triangle_c f}{P_1(z)}) + \overline{N}(r, \frac{P_1(z)}{P(f) \triangle_c f}) + \overline{N}(r, \frac{P_1(z)}{P(g) \triangle_c g}) + S(r,f) + S(r,g) \\
&\leq \overline{N}(r, P(f)) + \overline{N}(r, \triangle_c f) + \overline{N}(r, 0; P(f)) + \overline{N}(r, 0; \triangle_c f) \\
&+ \overline{N}(r, 0; P(g)) + \overline{N}(r, 0; \triangle_c g) + S(r,f) + S(r,g) \\
&\leq T(r,f) + 2T(r,f) + \Gamma_1 T(r,f) + 2T(r,f) + \Gamma_1 T(r,g) + 2T(r,g) + S(r,f) + S(r,g) \\
(n-2)T(r,f) &\leq (\Gamma_1 + 5)T(r,f) + (\Gamma_1 + 2)T(r,g) + S(r,f) + S(r,g).
\end{aligned}$$

Similarly, we have

$$(n-2)T(r,g) \leq (\Gamma_1 + 5)T(r,g) + (\Gamma_1 + 2)T(r,f) + S(r,f) + S(r,g).$$

Therefore we get

$$\begin{aligned}
(n-2)[T(r,f) + T(r,g)] &\leq (2\Gamma_1 + 7)[T(r,f) + T(r,g)] + S(r,f) + S(r,g) \\
n[T(r,f) + T(r,g)] &\leq (2\Gamma_1 + 9)[T(r,f) + T(r,g)] + S(r,f) + S(r,g)
\end{aligned}$$

which is a contradiction since $n > 2\Gamma_1 + 9$. Hence $P_1(z) \equiv 0$ and so from (3.9) we obtain

$$P(f)\Delta_c(f) = P(g)\Delta_c(g)$$

$$\begin{aligned}&(a_m f^m + a_{m+1}f^{m+1} + ... + a_0)(f(z+c) - f(z))\\ &= (a_m g^m + a_{m+1}g^{m+1} + ... + a_0)(g(z+c) - g(z)).\end{aligned} \tag{3.10}$$

Let $h = \dfrac{f}{g}$.

⋆ If h is a constant then substituting $f = gh$ and $f(z+c) = g(z+c)h(z+c)$ in (3.10) we deduce,

$$a_m g^m[(h^m h(z+c) - 1)g(z+c) - (h^m h(z) - 1)g(z)] +$$

$$a_{m-1}g^{m-1}[(h^{m-1}h(z+c) - 1)g(z+c) - (h^{m-1}h(z) - 1)g(z)]$$

$$+ \ . \ . \ . \ + a_0[(h(z+c) - 1)g(z+c) - (h(z) - 1)g(z)] = 0$$

which implies $h^d = 1$, where

$$d = GCD(m+1, ..., m+1-i, ..., 1), a_{m-i} \neq 0$$

for $i = 0, 1, ..., m$.
Thus $f(z) = tg(z)$ for a constant t such that $t^d = 1$, where $d = GCD(m+1, ..., m+1-i, ..., 1)$, $a_{m-i} \neq 0$ for $i = 0, 1, ..., m$ which is the conclusion (1) of theorem 1.

⋆ If h is not a constant then $f(z)$ and $g(z)$ satisfy the algebraic difference equation

$$R(f, g) \equiv 0, \text{ where}$$

$$\begin{aligned}R(w_1, w_2) = &(a_m w_1^m + a_{m-1}w_1^{m-1} + ... + a_0)[w_1(z+c) - w_1(z)]\\ &- (a_m w_2^m + a_{m-1}w_2^{m-1} + ... + a_0)[w_2(z+c) - w_2(z)]\end{aligned}$$

which is conclusion(2) of theorem 1.

Case 2. Let $l = 1$.
Sub Case 1. Using Lemma 2.10 and (3.2) we obtain from (3.1)

$$\begin{aligned}(n+2)T(r,f) &\leq N_2(r,0;G) + N_2(r,\infty;F) + N_2(r,\infty;G) + \frac{1}{2}\overline{N}(r,0;F) + \frac{1}{2}\overline{N}(r,\infty;F) + N_{k+2}(r,0;F_1)\\ &\quad + S(r,f) + S(r,g)\\ &\leq (m_1 + (k+2)m_2)T(r,g) + 2T(r,g) + kT(r,g) + 2kT(r,g) + 6T(r,f) + 6T(r,g)\\ &\quad + (m_1 + (k+2)m_2)T(r,f) + 2T(r,f) + \frac{1}{2}k(m_1+m_2)T(r,f) + T(r,f) + \frac{1}{2}kT(r,f)\\ &\quad + T(r,f) + S(r,f) + S(r,g)\end{aligned} \tag{3.11}$$

$$\begin{aligned}(n+2)T(r,f) &\leq (m_1 + 2m_2 + km_2 + 2)T(r,g) + (m_1 + 2m_2 + \frac{3}{2}km_2 + \frac{1}{2}km_1 + 2)T(r,f)\\ &\quad + S(r,f) + S(r,g).\end{aligned}$$

Similarly,

$$(n+2)T(r,g) \leq (m_1+2m_2+km_2+2)T(r,f)+(m_1+2m_2+\frac{3}{2}km_2+\frac{1}{2}km_1+2)T(r,g) + S(r,f)+T(r,g). \tag{3.12}$$

Combining (3.11) and (3.12) we have

$$\{n-2m_1-4m_2-\frac{5}{2}km_2-\frac{1}{2}km_1-2\}\{T(r,f)+T(r,g)\} \leq S(r,f)+T(r,g),$$

contradicting the fact that $n > 2\Gamma_2+\frac{5}{2}km_2+\frac{1}{2}km_1+3$. Therefore, by Lemma 2.15[2], we have either $FG=1$ or $F=G$.

Sub Case 2. Then proceeding in a manner similar to Subcase 2 of Case 1, $FG=1$ is not possible .

Sub Case 3. Let $F \equiv G$. Then the conclusion follows from Subcase 3 of Case 1.

Case 3. Let $l=0$.
Sub Case 1. Using Lemma 2.11 and (3.2) we obtain from (3.1)

$$\begin{aligned}(n+2)T(r,f) &\leq N_2(r,0;G)+N_2(r,\infty;F)+N_2(r,\infty;G)+2\overline{N}(r,0;F)+\overline{N}(r,0;G)+N_{k+2}(r,0;F_1)\\ &\quad+2\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)+S(r,f)+S(r,g).\\ &\leq (m_1+(k+2)m_2)T(r,g)+2T(r,g)+kT(r,g)+2kT(r,g)+6T(r,f)+6T(r,g)\\ &\quad+2(k(m_1+m_2)+2)T(r,f)+(k(m_1+m_2)+2)T(r,g)+(m_1+(k+2)m_2)T(r,f)\\ &\quad+2T(r,f)+2(k+2)T(r,f)+(k+2)T(r,g)+S(r,f)+S(r,g)\\ (n+2)T(r,f) &\leq (m_1+2m_2+2km_2+km_1+4)T(r,g)+(m_1+2m_2+3km_2+2km_1+6)T(r,f)\\ &\quad+S(r,f)+S(r,g).\end{aligned} \tag{3.13}$$

Similarly,

$$(n+2)T(r,g) \leq (m_1+2m_2+2km_2+km_1+4)T(r,f)+(m_1+2m_2+3km_2+2km_1+6)T(r,g) + S(r,f)+S(r,g). \tag{3.14}$$

Combining (3.13) and (3.14) we have

$$\{n-2m_1-4m_2-5km_2-3km_1-8\}\{T(r,f)+T(r,g)\} \leq S(r,f)+S(r,g),$$

contradicting the fact that $n > 2\Gamma_2+5km_2+3km_1+9$. Therefore, by Lemma 2.14[2], we have either $FG=1$ or $F=G$.

Then proceeding in a manner similar to Subcase 2 and 3 of Case 1, the result follows.
This Completes the proof of the theorem 1.

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