On the nonexistence of global solutions for wave equations with double damping terms and nonlinear memory

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Abstract

In this work, we consider the Cauchy problem for a wave equations with frictional and displacement dependent damping terms with nonlinear memory in multi-dimensional space $\mathbb{R}^n$, $n \geq 1$, we will prove the existence and uniqueness of the local solution and the nonexistence of global weak solutions theorems for any dimension space.

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1 Introduction

In this paper we study whether or not there exist solutions to the initial value problem:

$$\begin{cases} \partial_t^2 u(t,x) - \Delta u(t,x) + |u(t,x)|^{m-1}\partial_t u(t,x) + \partial_t u(t,x) = \displaystyle\int_0^t (t-\tau)^{-\gamma} |u(\tau,x)|^p d\tau,\ t>0,\ x \in \mathbb{R}^n, \\ u(0,x) = u_0(x),\ \ u_t(0,x) = u_1;\ \ x \in \mathbb{R}^N, \end{cases} \tag{1.1}$$

where u is the unknown real-valued function, $n \geq 1$, $m \geq 1$, $p > 1$, $\gamma \in (0,1)$ and $u_0(x)$, $u_1(x)$ are the given initial data.

It is well known that the nonlinear source term in (1.1) can be expressed it in terms of Riemann-Liouville fractional integral operator which is defined by

$$J_{a|t}^{\alpha} g(t) := \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} g(s)\, ds$$

with $a = -\infty$ by Liouville in 1832 and with $a = 0$ by Riemann in 1876 (see Chapter V in [13]), where $\Gamma(\alpha)$ is Euler Gamma function. Since the limit

$$\lim_{\gamma \to 1} \frac{1}{\Gamma(1-\gamma)} s_+^{-\gamma} = \delta(s)$$

exists in distribution sense, where $s_+ := \max(0,s)$, $\delta(s)$ is the Dirac function. The Cauchy problem (1.1) can be viewed as approximation of classical semilinear damped wave equation when γ converges to 1.

During the past few years, the question of existence and nonexistence of global solutions for nonlinear hyperbolic equations with damping term has been attracted many researchers and much progress has been made. There are many literatures concerning this type of equations and all

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researchers turn around the fact that the asymptotic behavior of solutions of semi-linear damped wave equations is similar to the one of the corresponding semi-linear heat equations, see for example, ([21],[23],[9],[19],[8],[2]), and the references therein. In order to motivate our results, it must be recall some facts concern the Cauchy problem for the dissipative nonlinear wave equation.

Recently, Todorova and Yordanov studied the global existence, blow-up and the asymptotic behavior as $t \to \infty$ to the following problem

$$\begin{cases} \partial_t^2 u - \Delta u + u_t = |u|^p, \ \ (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n, \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \ \ x \in \mathbb{R}^n, \end{cases} \tag{1.2}$$

and proved that the critical exponent of (1.2) is $p_c(n) = 1 + \frac{2}{n}$. More precisely, if $p > p_c(n)$ there exists a unique global solution of (1.2) for sufficiently small initial data in the weighted energy space while, if $1 < p < p_c(n)$ every solution with initial data having positive average must blow up in finite time. This term has its origin in the study of the nonlinear heat equation.

Later on Zhang [23], proved that the exponent $p_c(n)$ belongs to the blow-up region.

Our article is motivated by the recent paper of Cazenave et *al.* [4] which deals with the global existence and blow-up for the parabolic equation with non-local in time non-linearity

$$\partial_t u(t,x) - \Delta u(t,x) = \int_0^t (t-\tau)^{-\gamma} |u|^{p-1} u(\tau,x) d\tau, \tag{1.3}$$

where $0 \leq \gamma < 1$, $p > 1$ and $u_0 \in C_0(\mathbb{R}^n)$. They proved that, if

$$p_\gamma = 1 + \frac{2(2-\gamma)}{(n-2+2\gamma)_+}$$

and $p_* = \max\left\{\frac{1}{\gamma}, p_\gamma\right\} \in (0, +\infty]$, where $(u)_+ = \max(u, 0)$, then:

(i) If $\gamma \neq 0$, $p \leq p_*$, and $u_0 \geq 0$, $u_0 \not\equiv 0$, then u blows up in finite time.

(ii) If $\gamma \neq 0$, $p > p_*$, and $u_0 \in L_{q_{sc}}(\mathbb{R}^n)$ (where $q_{sc} = n(p-1)/(4-2\gamma)$) with $\|u_0\|_{L_{q_{sc}}}$ sufficiently small, then u exists globally.

Their study reveals surprising the fact that for Eq. (1.3) the critical exponent in Fujita's sense p_* is not the one predicted by scaling argument, as it is well known scaling technique is efficient for detecting the Fujita exponent for equations of nonlinear parabolic type one. Needless to say that the equation considered by Cazenave et *al.* [4] is a genuine extension of the one considered by Fujita in his pioneering work [7].

In the case eq. (1.1) only with linear damping, it reads

$$\partial_{tt} u(t,x) - \Delta u(t,x) + \partial_t u(t,x) = J_{0|t}^{\alpha}(|u|^p)(t). \tag{1.4}$$

Fino [6], addressed the global small data solutions and their asymptotic behavior as $t \to \infty$ to (1.4) when $n = 1, 2, 3$. If $p > 1 + 2(1+\alpha)/(n-2\alpha)$, the blow-up result was also derived under some positive average data in any dimensional space.of Eq.(1.4).

Berbiche [2], studied the global existence of solution for (1.4), he proved that if $p > 2$, $p > p_*$ for $n = 1, 2, 3$, then the mild solution with small initial data with low regularity and not necessarily

in $L^1(\mathbb{R}^n)$ exists globally in time. In addition, if $2 < p < 5$, then some of these solutions have the same behavior of the self-similar solutions of the corresponding heat equation with nonlinear memory (1.3).

A natural extension of (1.4) consists in introducing a displacement-dependent damping coefficient, thus leading to the following problem

$$u_{tt} - \Delta u + |u|^{m-1}u_t = \int_0^t (t-s)^{-\gamma} |u(s,.)|^p ds, \tag{1.5}$$

where $0 < \gamma < 1$, p,$m > 1$ and $N \geq 1$, $u_0 \in C_0(\mathbb{R}^n)$. Berbiche and Hakem [3], showed the local existence and blow-up for the problem (1.5). More precisely, they proved that, if $p > m > 1$ and the initial conditions satisfy

$$\int_{\mathbb{R}^n} u_0(x)dx > 0; \ \int_{\mathbb{R}^n} |u_0|^{m-1}u_0(x)dx > 0; \text{ and } \int_{\mathbb{R}^n} u_1(x)dx > 0, \tag{1.6}$$

and if

$$n \leq \min\left\{ \frac{2(m+(1-\gamma)p}{p-1+(1-\gamma)(m-1)}, \frac{2(1+(2-\gamma)p}{\left(\frac{(p-1)(2-\gamma)}{(p-m)} + \gamma - 1\right)(p-1)} \right\} \text{ or } p \leq \frac{1}{\gamma},$$

the solution of problem (1.6) does not exist globally in time.

It should be emphasized that the natural space where solutions are found is the one of the energy $H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$. However, such energy inequalities do not seem to fit in this setting for the following reasons: some difficulties appear due to the lack of Lipschitz continuity of function the $|u|^{m-1}v$ $(1 < m < 2)$ with respect to $(u,v) \in \mathbb{R}\times\mathbb{R}$, and the Sobolev embedding $H^1(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n)$ is true only when $n = 1$, another difficulty comes from the absence of regularity created by the singular kernel involving in the nonlinear source term.

In this paper, we will focus on the interaction between the nonlinear nonlocal source term involving fractional integral kernel and the dissipation term. In particular, we will give conditions relating the space dimension n with parameters p,γ, m, for which the solution of (1.1) with initial data having positive average does not exist globally. We also emphasize here that no effect of displacement-dependent damping on the critical exponent as unlike to what the authors saw in the article from [3].

In order to show that the problem (1.1) is well-posed, we use an idea developed by Lions and Strauss [12], Katayama et a*l.*[10] combined with energy estimates in higher order Sobolev spaces (see [17]). Moreover, our approach to derive Fujita exponent is based on the test-function method, developed by Mitidieri & Pohozaev [15], [16], Pohozaev & Tesei [14], Pohozaev & Veron [22] and Zhang [23].

The rest of the paper is divided into three sections. In Section 2, we present some definitions, notations, and lemmas which will be needed later in our proof. Section 3 is devoted to the local existence result and Section 4 contains a nonexistence of global weak solutions result.

Before closing this section, we define some notation to be used below. $L^p(\mathbb{R}^n)$ $(1 \leq p \leq \infty)$ is the usual Lebesgue space with the norm $\|.\|_{L^p(\mathbb{R}^n)}$.

Let s be a nonnegative. Then $H^s = H^s(\mathbb{R}^n)$ denotes the Sobolev space of L^2 functions equipped with the norm $\|.\|_{H^s(\mathbb{R}^n)}$. Also, $C^k(I; H^s(\mathbb{R}^n))$ denotes the space of k-times continuously differentiable functions on the interval I with values in the Sobolev space $H^s(\mathbb{R}^n)$. Finally, positive constants will be denoted by C and will change from line to line.

2 Some preliminary results

In this section, we prepare several lemmas for the proof of Theorem 3.1 and Theorem 4.2.

If $AC[0,T]$ is the space of all functions which are absolutely continuous on $[0,T]$ with $0<T<\infty$, then, for $f\in AC[0,T]$, the left-handed and right-handed Riemann-Liouville fractional derivatives $D^{\alpha}_{0|t}f(t)$ and $D^{\alpha}_{t|T}f(t)$ of order $\alpha\in(0,1)$ are defined by

$$D^{\alpha}_{0|t}f(t)=\partial_t J^{1-\alpha}_{0|t}f(t) \text{ and } D^{\alpha}_{t|T}f(t)=\frac{1}{\Gamma(1-\alpha)}\partial_t\int_t^T (s-t)^{-\alpha}f(s)ds,\ t\in[0,T], \tag{2.1}$$

where

$$J^{\alpha}_{0|t}g(t)=\frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{\alpha-1}g(s)ds, \tag{2.2}$$

is the Riemann–Liouville fractional integral, for all $g\in L^q(0,T)(1\le q\le\infty)$. We refer the reader to [18] for the definitions above.

Furthermore, for every $f,g\in C([0,T])$ such that $D^{\alpha}_{0|t}f(t)$, $D^{\alpha}_{t|T}f(t)$ exist and are continuous, for all $t\in[0,T], 0<\alpha<1$, we have the formula of integration by parts (see (2.64) p. 46 in [18])

$$\int_0^T \left(D^{\alpha}_{0|t}f\right)(t)g(t)dt=\int_0^T f(t)\left(D^{\alpha}_{t|T}g\right)(t)dt. \tag{2.3}$$

Note also that, for all $f\in AC^{n+1}[0,T]$ and all integers $n\ge 0$, we have
(see (2.2.30) in [18])

$$(-1)^n\,\partial_t^n.D^{\alpha}_{t|T}f(t)=D^{n+\alpha}_{t|T}f(t), \tag{2.4}$$

where $AC^{n+1}[0,T]:=\{f:[0,T]\to\mathbb{R}$, and $\partial_t^n f\in AC[0,T]\}$ and ∂_t^n is the usual n times derivative. Moreover, for all $1\le q\le\infty$, the following formula
(see [11], Lemma 2.4 p. 74)

$$D^{\alpha}_{0|t}J^{\alpha}_{0|t}:=Id_{L^q(0,T)}, \tag{2.5}$$

holds almost everywhere on $[0,T]$.

In the proof of Theorem 4.2, the following results are useful: if $w_1(t)=(1-t/T)^{\sigma}_{+}, t\ge 0, T>0, \sigma>>1$, then $D^{\alpha}_{t|T}w_1(t)=CT^{-\sigma}(T-t/)^{\sigma-\alpha}_{+}$, $D^{\alpha+1}_{t|T}w_1(t)=CT^{-\sigma}(T-t)^{\sigma-\alpha-1}_{+}$, $\quad D^{\alpha+2}_{t|T}w_1(t)=CT^{-\sigma}(T-t)^{\sigma-\alpha-2}_{+}$, for all $\alpha\in(0,1)$; so

$$\left(D^{\alpha}_{t|T}w_1\right)(T)=0,\left(D^{\alpha}_{t|T}w_1\right)(0)=T^{-\alpha},\left(D^{\alpha+1}_{t|T}w_1\right)(T)=0,\text{and }\left(D^{\alpha+1}_{t|T}w_1\right)(0)=T^{-\alpha-1}. \tag{2.6}$$

For the proof of this results, see [11].

The following Lemmas will be used in the proof of Theorems 3.1.

Lemma 2.1. *(See [20], Proposition 2.4, p. 5) If $s>\frac{N}{2}$, then*

$$H^s(\mathbb{R}^n)\subset C(\mathbb{R}^n)\cap L^{\infty}(\mathbb{R}^n),$$

where the inclusion is continuous. In fact

$$\|u\|_{L^{\infty}}\le C\|u\|_{H^s}$$

The next Lemma is consequence of lemma 2.1 and Proposition 3.7, p. 10 in [20]

Lemma 2.2. *(See [20]) Assume that $s_1, s_2 \geq s > n/2$, then for $u \in H^{s_1}(\mathbb{R}^n), v \in H^{s_2}(\mathbb{R}^n)$, we have the estimates*

$$\|uv\|_{H^s(\mathbb{R}^n)} \leq C \|u\|_{H^{s_1}(\mathbb{R}^n)} \|v\|_{H^{s_2}(\mathbb{R}^n)}$$

where C is constant independent of u and v.

The last Lemma uses the equivalent norm of $\|u\|_{H^{s-1}}$ (see [1, Theorem 7.48, p. 214]). We omit their proof since it can be found in (see [5])

Lemma 2.3. *For any $s \in (1,2) \cup \mathbb{N}^*$ and $p \in (1,+\infty) \cap (s-1,+\infty)$ we have for a nonnegative function $f \in L^\infty(\mathbb{R}^n) \cap H^{s-1}(\mathbb{R}^n)$, $f^p \in H^{s-1}(\mathbb{R}^n)$ and there exists a positive constant C such that*

$$\|f^p\|_{H^{s-1}(\mathbb{R}^n)} \leq C \|f\|^{p-1}_{L^\infty(\mathbb{R}^n)} \|f\|_{H^{s-1}(\mathbb{R}^n)}.$$

Now, proceed with the following linear damped wave equation:

$$\begin{cases} u_{tt} - \Delta u + u_t = F(t,x), \ \ x \in \mathbb{R}^n, \, t > 0, \\ u(x,0) = g(x), \ u_t(x,0) = h(x), \, x \in \mathbb{R}^n. \end{cases} \tag{2.7}$$

Let us give some results which will be used in the following.

Lemma 2.4. Let $s \in \mathbb{R}$. Let $(g,h) \in H^s(\mathbb{R}^n) \times H^{s-1}(\mathbb{R}^n)$ and $F(t,x) \in L^1\left([0,T], H^{s-1}(\mathbb{R}^n)\right)$. Then for every $T > 0$, there is a unique solution $u \in C\left([0,T], H^s(\mathbb{R}^n)\right) \cap C^1\left([0,T], H^{s-1}(\mathbb{R}^n)\right)$ of cauchy problem of (2.7). Moreover, u satisfies

$$\|u\|_{H^s(\mathbb{R}^n)} + \|u_t\|_{H^{s-1}(\mathbb{R}^n)} \leq C \left(\|g\|_{H^s(\mathbb{R}^n)} + \|h\|_{H^{s-1}(\mathbb{R}^n)} + \int_0^t \|F(\tau,.)\|_{H^{s-1}(\mathbb{R}^n)} \, d\tau \right),$$

for all $0 \leq t \leq T$, where C only depends on s.

In addition, if $g \in H^{s+1}(\mathbb{R}^n)$, $h \in H^s(\mathbb{R}^n)$ and $F(t,x) \in L^1([0,T], H^s(\mathbb{R}^n)) \cap L^\infty\left([0,T], H^{s-1}(\mathbb{R}^n)\right)$, $u_{tt} \in L^\infty\left([0,T], H^{s-1}(\mathbb{R}^n)\right)$ satisfies

$$\|u_{tt}\|_{H^{s-1}(\mathbb{R}^n)} \leq C \left(\|h\|_{H^s(\mathbb{R}^n)} + \|g\|_{H^{s+1}(\mathbb{R}^n)} + \|F(t)\|_{H^{s-1}(\mathbb{R}^n)} + \int_0^t \|F(t')\|_{H^s(\mathbb{R}^n)} \, dt' \right),$$

for all $0 \leq t \leq T$, where C only depends on s.

Proof. The first estimate is already shown in [17]. Arguing as in [17] to show the second estimate. It is known that the first derivative of $\hat{u}$ satisfies in the domain $|\xi| < 1/2$ (see [17] Theorem 5.4),

$$\begin{aligned} \hat{u}_t(t) &= -\frac{1}{2}\hat{u}(t) + \frac{1}{2}\hat{g}\sqrt{1-4|\xi|^2}e^{-\frac{t}{2}} \sinh\left(\frac{1}{2}t\sqrt{1-4|\xi|^2}\right) \\ &\quad + \frac{1}{2}\left(\hat{g} + 2\hat{h}\right) e^{-\frac{t}{2}} \cosh\left(\frac{1}{2}t\sqrt{1-4|\xi|^2}\right) \\ &\quad + \int_0^t \hat{F}(t') e^{-\frac{1}{2}(t-t')} \cosh\left(\frac{1}{2}(t-t')\sqrt{1-4|\xi|^2}\right) dt', \end{aligned}$$

we derive this expression with respect to t, we find

$$\begin{aligned}
&\hat{u}_{tt}(t) = -\tfrac{1}{2}\hat{u}_t(t) - \tfrac{1}{4}\hat{g}\sqrt{1-4|\xi|^2}e^{-\frac{t}{2}}\sinh\left(\tfrac{1}{2}t\sqrt{1-4|\xi|^2}\right) + \tfrac{1}{4}\hat{g}\left(1-4|\xi|^2\right)e^{-\frac{t}{2}}\cosh\left(\tfrac{1}{2}t\sqrt{1-4|\xi|^2}\right)\\
&-\tfrac{1}{4}\left(\hat{g}+2\hat{h}\right)e^{-\frac{t}{2}}\cosh\left(\tfrac{1}{2}t\sqrt{1-4|\xi|^2}\right) + \tfrac{1}{4}\left(\hat{g}+2\hat{h}\right)e^{-\frac{t}{2}}\sqrt{1-4|\xi|^2}\sinh\left(\tfrac{1}{2}t\sqrt{1-4|\xi|^2}\right)\\
&+\hat{F}(t) + \int_0^t \hat{F}(t')\left[-\frac{1}{2}e^{-\frac{1}{2}(t-t')}\cosh\left(\frac{1}{2}(t-t')\sqrt{1-4|\xi|^2}\right)\right.\\
&\left.+\tfrac{1}{2}\sqrt{1-4|\xi|^2}e^{-\frac{1}{2}(t-t')}\sinh\left(\tfrac{1}{2}(t-t')\sqrt{1-4|\xi|^2}\right)\right]dt',
\end{aligned}$$

or

$$\begin{aligned}
&\hat{u}_{tt}(t) = -\tfrac{1}{2}\hat{u}_t(t) - \hat{g}|\xi|^2 e^{-\frac{t}{2}}\cosh\left(\tfrac{1}{2}t\sqrt{1-4|\xi|^2}\right) - \tfrac{1}{2}\hat{h}e^{-\frac{t}{2}}\cosh\left(\tfrac{1}{2}t\sqrt{1-4|\xi|^2}\right)\\
&+\tfrac{1}{2}\hat{h}e^{-\frac{t}{2}}\sqrt{1-4|\xi|^2}\sinh\left(\tfrac{1}{2}t\sqrt{1-4|\xi|^2}\right) + \hat{F}(t)\\
&+\int_0^t \hat{F}(t')\left[-\frac{1}{2}e^{-\frac{1}{2}(t-t')}\cosh\left(\frac{1}{2}(t-t')\sqrt{1-4|\xi|^2}\right)\right.\\
&\left.+\tfrac{1}{2}\sqrt{1-4|\xi|^2}e^{-\frac{1}{2}(t-t')}\sinh\left(\tfrac{1}{2}(t-t')\sqrt{1-4|\xi|^2}\right)\right]dt'.
\end{aligned} \tag{2.8}$$

Now we consider the case for $|\xi| > 1/2$. From (Theorem 5.4), we have

$$\begin{aligned}
\hat{u}_t(t) &= -\frac{1}{2}\hat{u}(t) - \frac{1}{2}\hat{g}\sqrt{4|\xi|^2-1}e^{-\frac{t}{2}}\sin\left(\frac{1}{2}t\sqrt{4|\xi|^2-1}\right)\\
&+\frac{1}{2}\left(\hat{g}+2\hat{h}\right)e^{-\frac{t}{2}}\cos\left(\frac{1}{2}t\sqrt{4|\xi|^2-1}\right)\\
&+\int_0^t \hat{F}(t')e^{-\frac{1}{2}(t-t')}\cos\left(\frac{1}{2}(t-t')\sqrt{4|\xi|^2-1}\right)dt'.
\end{aligned}$$

By taking the derivative of $\hat{u}_t(t)$, we get

$$\begin{aligned}
&\hat{u}_{tt}(t) = -\tfrac{1}{2}\hat{u}_t(t) + \tfrac{1}{4}\hat{g}\sqrt{4|\xi|^2-1}e^{-\frac{t}{2}}\sin\left(\tfrac{1}{2}t\sqrt{4|\xi|^2-1}\right) - \tfrac{1}{4}\hat{g}\left(4|\xi|^2-1\right)e^{-\frac{t}{2}}\cos\left(\tfrac{1}{2}t\sqrt{4|\xi|^2-1}\right)\\
&-\tfrac{1}{4}\left(\hat{g}+2\hat{h}\right)e^{-\frac{t}{2}}\cos\left(\tfrac{1}{2}t\sqrt{4|\xi|^2-1}\right) - \tfrac{1}{4}\left(\hat{g}+2\hat{h}\right)e^{-\frac{t}{2}}\sqrt{4|\xi|^2-1}\sin\left(\tfrac{1}{2}t\sqrt{4|\xi|^2-1}\right)\\
&+\hat{F}(t) + \int_0^t \hat{F}(t')\left[-\frac{1}{2}e^{-\frac{1}{2}(t-t')}\cos\left(\frac{1}{2}(t-t')\sqrt{4|\xi|^2-1}\right)\right.\\
&\left.-\tfrac{1}{2}\sqrt{4|\xi|^2-1}e^{-\frac{1}{2}(t-t')}\sin\left(\tfrac{1}{2}(t-t')\sqrt{4|\xi|^2-1}\right)\right]dt',
\end{aligned}$$

which gives

$$\begin{aligned}
&\hat{u}_{tt}(t) = -\tfrac{1}{2}\hat{u}_t(t) - \hat{g}\left|\xi\right|^2 e^{-\frac{t}{2}}\cos\left(\tfrac{1}{2}t\sqrt{4\left|\xi\right|^2-1}\right)\\
&-\tfrac{1}{2}\hat{h}e^{-\frac{t}{2}}\cos\left(\tfrac{1}{2}t\sqrt{4\left|\xi\right|^2-1}\right) - \tfrac{1}{2}\hat{h}e^{-\frac{t}{2}}\sqrt{4\left|\xi\right|^2-1}\sin\left(\tfrac{1}{2}t\sqrt{4\left|\xi\right|^2-1}\right)\\
&+\hat{F}(t) + \int_0^t \hat{F}(t')\left[-\frac{1}{2}e^{-\frac{1}{2}(t-t')}\cos\left(\frac{1}{2}(t-t')\sqrt{4\left|\xi\right|^2-1}\right)\right.\\
&\left.-\tfrac{1}{2}\sqrt{4\left|\xi\right|^2-1}e^{-\frac{1}{2}(t-t')}\sin\left(\tfrac{1}{2}(t-t')\sqrt{4\left|\xi\right|^2-1}\right)\right]dt'.
\end{aligned} \tag{2.9}$$

Now, using the definition of the H^s norm

$$\begin{aligned}
&\left\|u_{tt}\right\|_{H^{s-1}(\mathbb{R}^n)} = \left[\int_{\mathbb{R}^n}\left(1+\left|\xi\right|^2\right)^{s-1}\left|\hat{u}(t)\right|^2 d\xi\right]^{1/2}\\
&= \left[\int_{|\xi|<1/2}\left(1+\left|\xi\right|^2\right)^{s-1}\left|\hat{u}_{tt}(t)\right|^2 d\xi + \int_{|\xi|>1/2}\left(1+\left|\xi\right|^2\right)^{s-1}\left|\hat{u}_{tt}(t)\right|^2 d\xi\right]^{1/2}\\
&\leq \left[\int_{|\xi|<1/2}\left(1+\left|\xi\right|^2\right)^{s-1}\left|\hat{u}_{tt}(t)\right|^2 d\xi\right]^{1/2} + \left[\int_{|\xi|>1/2}\left(1+\left|\xi\right|^2\right)^{s-1}\left|\hat{u}_{tt}(t)\right|^2 d\xi\right]^{1/2} := I_1 + I_2
\end{aligned} \tag{2.10}$$

where we have used the Triangle Inequality.

Let's, estimate the first integrand. From (2.8), we have

$$\begin{aligned}
&I_1 \leq \left\|u_t\right\|_{H^{s-1}(\mathbb{R}^n)} + \left[\int_{|\xi|<1/2}\left(1+\left|\xi\right|^2\right)^{s-1}\frac{\left(1+\left|\xi\right|^2\right)}{1+\left|\xi\right|^2}\left|\hat{g}\right|^2 d\xi\right]^{1/2}\\
&+C\left[\int_{|\xi|<1/2}\left(1+\left|\xi\right|^2\right)^{s-1}\left|\hat{h}(\xi)\right|^2 d\xi\right]^{1/2} + \left\|F(t)\right\|_{H^{s-1}(\mathbb{R}^n)} + \int_0^t \left\|F(t')\right\|_{H^{s-1}(\mathbb{R}^n)} dt'\\
&\leq C\left(\left\|u_t\right\|_{H^{s-1}(\mathbb{R}^n)} + \left\|h\right\|_{H^{s-1}(\mathbb{R}^n)} + \left\|g\right\|_{H^s(\mathbb{R}^n)} + \left\|F(t)\right\|_{H^{s-1}(\mathbb{R}^n)} + \int_0^t \left\|F(t')\right\|_{H^{s-1}(\mathbb{R}^n)} dt'\right).
\end{aligned}$$

Where we have used that $\left(1-y^2\right)e^{-\frac{t}{2}}\cosh\left(\frac{1}{2}ty\right) \leq 1$, $e^{-\frac{t}{2}}y\sinh\left(\frac{1}{2}ty\right) \leq 1$ for $y \in [0,1]$, $t>0$.

Inserting the estimate of $\left\|u_t\right\|_{H^{s-1}(\mathbb{R}^n)}$ in the last inequality it yields

$$I_1 \leq C\left(\left\|h\right\|_{H^{s-1}(\mathbb{R}^n)} + \left\|g\right\|_{H^s(\mathbb{R}^n)} + \left\|F(t)\right\|_{H^{s-1}(\mathbb{R}^n)} + \int_0^t \left\|F(t')\right\|_{H^{s-1}(\mathbb{R}^n)} dt'\right). \tag{2.11}$$

Estimate for the second integrand in (2.10). We have

$$\begin{aligned}
&\left[\int_{|\xi|>1/2}\left(1+|\xi|^2\right)^{s-1}|\hat{u}_{tt}(t)|^2\,d\xi\right]^{1/2} \leq \|u_t\|_{H^{s-1}(\mathbb{R}^n)} + \\
&\left[\int_{|\xi|>1/2}\left(1+|\xi|^2\right)^{s-1}\left(1+|\xi|^2\right)^2\left|\hat{g}\frac{|\xi|^2}{1+|\xi|^2}e^{-\frac{t}{2}}\cos\left(\frac{1}{2}t\sqrt{4|\xi|^2-1}\right)\right|^2 d\xi\right]^{1/2} \\
&+\left[\int_{|\xi|>1/2}\left(1+|\xi|^2\right)^{s-1}\frac{\left(1+|\xi|^2\right)}{1+|\xi|^2}\left|\hat{h}\left(1+\sqrt{4|\xi|^2-1}\sin\left(\frac{1}{2}t\sqrt{4|\xi|^2-1}\right)\right)\right|^2 d\xi\right]^{1/2} + \|F(t)\|_{H^{s-1}(\mathbb{R}^n)} \\
&+\left[\int_0^t\int_{|\xi|>1/2}\left(1+|\xi|^2\right)^{s-1}\left(1+|\xi|^2\right)\left|\hat{F}(t')\left(1+\frac{\sqrt{4|\xi|^2-1}}{\sqrt{1+|\xi|^2}}\sin\left(\frac{1}{2}t\sqrt{4|\xi|^2-1}\right)\right)\right|^2 dt'd\xi\right]^{1/2} \\
&\leq \|u_t\|_{H^{s-1}(\mathbb{R}^n)} + \|h\|_{H^s(\mathbb{R}^n)} + \|g\|_{H^{s+1}(\mathbb{R}^n)} + \|F(t)\|_{H^{s-1}(\mathbb{R}^n)} + \int_0^t \|F(t')\|_{H^s(\mathbb{R}^n)}\,dt'.
\end{aligned} \tag{2.12}$$

Using the estimate of $\|u_t\|_{H^{s-1}(\mathbb{R}^n)}$ into (2.12) and by Sobolev's embedding theorem, we find that

$$I_2 \leq C\left(\|h\|_{H^s(\mathbb{R}^n)} + \|g\|_{H^{s+1}(\mathbb{R}^n)} + \|F(t)\|_{H^{s-1}(\mathbb{R}^n)} + \int_0^t \|F(t')\|_{H^s(\mathbb{R}^n)}\,dt\right). \tag{2.13}$$

From (2.11) and (2.13), we conclude the proof of Lemma 2.4 . ∎

3 Local-existence

In this section, we first present the following Theorem concerning the existence and uniqueness of the local solution to the problem (1.1). Next, we will prove this result.

Theorem 3.1. Let $n \geq 1$, $s > n/2+1$ and $m, p \in (1,+\infty)\cap(s-1,+\infty)$. Then for any $u_0 \in H^s(\mathbb{R}^n)$ and $u_1 \in H^{s-1}(\mathbb{R}^n)$, (1.1) admits a unique solution

$$u \in C\left([0,T];H^{s-1}(\mathbb{R}^n)\right) \cap C^1\left([0,T];H^{s-1}(\mathbb{R}^n)\right)$$

with some positive T, which depends only on $\|u_0\|_{H^s} + \|u_1\|_{H^{s-1}}$.

Proof. The idea of the proof is based on the fact that we have $m|u|^{m-1}u_t = \partial_t\left(|u|^{m-1}u\right)$. By introducing a new unknown v satisfying $u = v_t$, the problem (1.1) is reduced so to the following problem of the known case where the nonlinear term is locally lipschitz

$$\begin{cases} v_{tt}(t,x) - \Delta v(t,x) + v_t(t,x) = -\frac{1}{m}|v_t|^{m-1}v_t(t,x) + \frac{1}{1-\gamma}\displaystyle\int_0^t (t-\tau)^{1-\gamma}|v_t(\tau,x)|^p\,ds \\ +\frac{1}{m}|u_0|^{m-1}u_0(x) + u_0(x) + u_1(x),\ t>0,\ x\in\mathbb{R}^n, \\ v(0,x) = 0, \quad v_t(0,x) = u_0(x),\ x\in\mathbb{R}^n. \end{cases} \tag{3.1}$$

It is not difficult to check that $u \in C\left([0,T];H^{s-1}(\mathbb{R}^n)\right) \cap C^1\left([0,T];H^{s-1}(\mathbb{R}^n)\right)$ solution to (1.1) if and only if v is solution to (3.1) in the class

$$\begin{cases} v \in C\left([0,T];H^s(\mathbb{R}^n)\right), \\ v_t \in C\left([0,T];H^{s-1}(\mathbb{R}^n)\right),\ v_{tt} \in C\left([0,T];H^{s-1}(\mathbb{R}^n)\right). \end{cases} \tag{3.2}$$

Let us define

$$X_T := C\left([0,T]; H^s\left(\mathbb{R}^n\right)\right) \cap C^1\left([0,T]; H^{s-1}\left(\mathbb{R}^n\right)\right),$$
$$Y_T := L^\infty\left([0,T]; H^s\left(\mathbb{R}^n\right)\right) \cap W^{1,\infty}\left([0,T]; H^{s-1}\left(\mathbb{R}^n\right)\right)$$
$$Y_{T,M} := \left\{ u \in Y_T;\ \sup_{0\le t\le T} \left(\|v(t,.)\|_{H^s} + \|v_t(t,.)\|_{H^{s-1}}\right) \le M \right\}.$$

Next, set $X_{T,M} = Y_{T,M} \cap X_T$. Obviously $X_T \subset Y_T$ and $X_{T,M} \subset Y_{T,M}$. Set

$$G(v_t) = -\frac{1}{m}|v_t|^{m-1} v_t(t,x) + \frac{1}{1-\gamma}\int_0^t (t-\tau)^{1-\gamma} |v_t(\tau,x)|^p \, ds + \frac{1}{m}|u_0|^{m-1} u_0(x) + u_0(x) + u_1(x).$$

For any $v \in Y_T$, define $\Phi[w] = v$, where $w \in X_T$ is a solution to

$$\begin{cases} v_{tt} - \Delta v + v_t = G(w_t) & \text{in } (0,T)\times\mathbb{R}^n, \\ v(x,0) = 0,\ v_t(x,0) = u_0(x) & x \in \mathbb{R}^n. \end{cases} \tag{3.3}$$

Since we have $G(w_t) \in L^1\left(0,T; H^{s-1}\left(\mathbb{R}^n\right)\right)$ for any $w \in Y_T$ by Sobolev's embedding theorem, existence and uniqueness of such $v \in X_T$ is guaranteed by the Cauchy problems for linear damped wave equations (Lemma 2.4).

Let $M = 4\left(\|u_0\|_{H^s}^m + \|u_0\|_{H^s} + \|u_1\|_{H^{s-1}}\right)$. We first claim that $w \in Y_{T,M}$ implies that $\Phi[w] \in X_{T,M}$ for sufficiently
small $T > 0$. Set $\Phi[w] = v$. From the Lemma 2.4, we have

$$\|v(t,.)\|_{H^s} + \|v_t(t,.)\|_{H^{s-1}} \le C\left(\|v_0\|_{H^s} + \|v_1\|_{H^{s-1}} + \int_0^t \|G(w_t)(\tau,.)\|_{H^{s-1}} \, d\tau\right). \tag{3.4}$$

From Lemma 2.3, since $s > n/2 + 1$, we have

$$\begin{aligned} &\int_0^t \|G(w_t)(\tau,.)\|_{H^{s-1}} \, d\tau \le \int_0^t \left\| |w_t|^{m-1} w_t(\tau,.) \right\|_{H^{s-1}} d\tau \\ &+\tfrac{1}{1-\gamma}\int_0^t (t-s)^{1-\gamma} \||w_t|^p\|_{H^{s-1}}(s)\, ds + T\left\| \frac{1}{m}|u_0|^{m-1}u_0(x) + u_0(x) + u_1(x) \right\|_{H^{s-1}} \\ &\le CT\|w_t\|_{L^\infty}^{m-1} \sup_{0\le t\le Y} \|w_t\|_{H^{s-1}} + C\int_0^t (t-s)^{1-\gamma} \|w_t\|_{L^\infty}^{p-1} \|w_t\|_{H^{s-1}}(s)\, ds \\ &+CT\left(\|u_0\|_{H^{s-1}}^m + \|u_0\|_{H^{s-1}} + \|u_1\|_{H^{s-1}}\right). \end{aligned}$$

Then by Sobolev's embedding theorem and the Lemma 2.1, we find

$$\begin{aligned} &\int_0^t \|G(w_t)(\tau,.)\|_{H^{s-1}} \, d\tau \le CT \sup_{0\le t\le T} \|w_t\|_{H^{s-1}}^m + CT^{2-\gamma} \sup_{0\le t\le T} \|w_t\|_{H^{s-1}}^p \\ &+CT\left(\|u_0\|_{H^s}^m + \|u_0\|_{H^s} + \|u_1\|_{H^{s-1}}\right) \\ &\le C\left(TM^m + T^{2-\gamma}M^p + T\left(\|u_0\|_{H^s}^m + \|u_0\|_{H^s} + \|u_1\|_{H^{s-1}}\right)\right). \end{aligned} \tag{3.5}$$

From (3.4) and (3.5), we get

$$\begin{aligned} \sup_{0\le t\le T}\left(\|v(t,.)\|_{H^s} + \|v_t(t,.)\|_{H^{s-1}}\right) &\le C\left(\|u_0\|_{H^{s-1}} + TM^m + T^{2-\gamma}M^p + T\left(\|u_0\|_{H^s}^m + \|u_0\|_{H^s} + \|u_1\|_{H^{s-1}}\right)\right) \\ &\le C\left(M + TM^m + T^{2-\gamma}M^p + TM\right). \end{aligned} \tag{3.6}$$

By (3.6) we arrive at

$$\sup_{0\leq t\leq T}\left(\|v\left(t,.\right)\|_{H^{s}}+\|v_{t}\left(t,.\right)\|_{H^{s-1}}\right)\leq C_{T,M}M,$$

where $C_{T,M}=C\left(\frac{1}{4}+TM^{m-1}+CT^{2-\gamma}M^{p-1}+\frac{T}{4}\right)$. Since we can find $T_1>0$ such that $C_{T,M}\leq 1$ for any $T\in(0,T_1]$, this implies the claim.

We next prove that Φ is a contraction mapping in $X_{T,M}$ for small T by using the Lemma 2.4 and the mean value theorem. Suppose that $v_1,v_2\in Y_{T,M}$, then we have $\Phi\left[w_1\right],\Phi\left[w_2\right]\in X_{T,M}$.

Let v_i $(i=1,2)$ be solutions to the following problems

$$\begin{cases}\left(v_i\right)_{tt}-\Delta v_i+\left(v_i\right)_t=-\frac{1}{m}\left|\left(w_i\right)_t\right|^{m-1}\left(w_i\right)_t\left(t,x\right)+\frac{1}{1-\gamma}\int_0^t\left(t-\tau\right)^{1-\gamma}\left|\left(w_i\right)_t\left(\tau,x\right)\right|^{p}ds\\ +\frac{1}{m}\left|u_0\right|^{m-1}u_0\left(x\right)+u_0\left(x\right)+u_1\left(x\right),\ \text{ in } t>0,\ x\in\mathbb{R}^n,\\ w_i(0,x)=0,\ \left(w_i\right)_t\left(0,x\right)=u_0\left(x\right)\ \text{ in } \mathbb{R}^n.\end{cases}\tag{3.7}$$

Set $\tilde{v}=v_1-v_2$, we have $\tilde{v}$ verifies

$$\begin{cases}\tilde{v}_{tt}-\Delta\tilde{v}+\tilde{v}_t=-\frac{1}{m}\left|\left(w_1\right)_t\right|^{m-1}\left(w_1\right)_t\left(t,x\right)+\frac{1}{m}\left|\left(w_2\right)_t\right|^{m-1}\left(w_2\right)_t\left(t,x\right)\\ +\frac{1}{1-\gamma}\int_0^t\left(t-\tau\right)^{1-\gamma}\left(\left|\left(w_1\right)_t\left(\tau,x\right)\right|^{p}-\left|\left(w_2\right)_t\left(\tau,x\right)\right|^{p}\right)ds\quad\text{ in }\ (0,T)\times\mathbb{R}^n,\\ \tilde{v}(0,x)=\tilde{v}_t(0,x)=0\quad\text{in }\mathbb{R}^n.\end{cases}\tag{3.8}$$

for $i=1,2$, respectively.

Since $v_i\in Y_{T,M}$ implies that $\int_0^t\left(t-s\right)^{1-\gamma}\left|\left(w_i\right)_t\right|^{p}\left(s\right)ds$ and $\left|\left(w_i\right)_t\right|^{m-1}\left(w_i\right)_t$ are functions in $L^{\infty}\left(0,T;H^{s-1}\right)$ by Sobolev's embedding theorem, we have $v_i\in X_T$ and

$$\tilde{v}\in C\left(\left[0,T\right];H^{s}\left(\mathbb{R}^n\right)\right)\cap C^{1}\left(\left[0,T\right];H^{s-1}\left(\mathbb{R}^n\right)\right).\tag{3.9}$$

By (3.7), (3.8) and (3.10), the higher order energy inequality

$$\begin{aligned}&\|\tilde{v}_t\left(t,.\right)\|_{H^{s-1}}+\|\tilde{v}\left(t,.\right)\|_{H^{s}}\\ &\leq C\int_0^t\int_0^s\left(s-\tau\right)^{1-\gamma}\left\|\left(\left|\left(w_1\right)_t\right|^{p}-\left|\left(w_2\right)_t\right|^{p}\right)\left(\tau,.\right)\right\|_{H^{s-1}}d\tau ds\\ &+C\int_0^t\left\|\left|\left(w_1\right)_t\right|^{m-1}\left(w_1\right)_t\left(\tau,.\right)-\left|\left(w_2\right)_t\right|^{m-1}\left(w_2\right)_t\left(\tau,.\right)\right\|_{H^{s-1}}.\end{aligned}\tag{3.10}$$

Note that since $|v|^{l-1}v$ with $l>1$ is a C^1 function, the mean value theorem implies

$$\left||v_1|^{l-1}v_1-|v_2|^{l-1}v_2\right|\leq C\left(|v_1|^{l-1}+|v_2|^{l-1}\right)|v_1-v_2|.\tag{3.11}$$

Applying the inequality (3.11) with Sobolev's inequality $\|\tilde{v}\|_{L^{\infty}}\leq C\|\tilde{v}\|_{H^{s-1}}$ to right-hand of (3.10), it follows for $s>n/2+1$

$$\begin{aligned}&\|\tilde{v}_t\left(t,.\right)\|_{H^{s-1}}+\|\tilde{v}\left(t,.\right)\|_{H^{s}}\\ &+C\int_0^t\left(t-\tau\right)^{1-\gamma}\left(\left\|\left(w_1\right)_t\right\|_{H^{s-1}}^{p-1}+\left\|\left(w_2\right)_t\right\|_{H^{s-1}}^{p-1}\right)\left\|\left(w_1-w_2\right)_t\left(\tau,.\right)\right\|_{H^{s-1}}d\tau\\ &+C\int_0^t\left(\left\|\left(w_1\right)_t\right\|_{H^{s-1}}^{m-1}+\left\|\left(w_2\right)_t\right\|_{H^{s-1}}^{m-1}\right)\left\|\left(w_1-w_2\right)_t\left(\tau,.\right)\right\|_{H^{s-1}}d\tau\\ &\leq C\left(T^{2-\gamma}M^{p-1}+TM^{m-1}\right)\sup_{0\leq\tau\leq T}\left\|\left(w_1-w_2\right)_t\left(\tau,.\right)\right\|_{H^{s-1}}\ \text{ for } 0\leq t\leq T.\end{aligned}\tag{3.12}$$

Then by (3.12), we get

$$\sup_{0\leq t\leq T}\|\Phi\left[w_1\right]-\Phi\left[w_2\right]\left(t,.\right)\|_{X_{T,M}}\leq C\left(T^{2-\gamma}M^{p-1}+TM^{m-1}\right)\times\sup_{0\leq\tau\leq T}\left\|\left(w_1-w_2\right)_t\left(\tau,.\right)\right\|_{H^{s-1}}.\tag{3.13}$$

In the following, we fix $T \in (0, T_1]$ which is small enough to have

$$C\left(T^{2-\gamma}M^{p-1} + TM^{m-1}\right) < \frac{1}{2}.$$

Therefore

$$\sup_{0\le t\le T} \|\Phi[w_1] - \Phi[w_2](t,.)\|_{X_{TM}} \le \frac{1}{2} \sup_{0\le\tau\le T} \|(w_1 - w_2)(\tau,.)\|_{X_{T,M}} \qquad \text{for such } T. \tag{3.14}$$

Finally, define

$$\begin{cases} v^{(0)}(t,x) = v_t(0,x) = u_0(x), \\ v^{(k)} = \Phi\left[v^{(k-1)}\right] \quad (k = 1,2,3,...). \end{cases}$$

By (3.14), there exists some $v \in C([0,T]; H^s)$ such that $v^{(k)} \to v$ in $C([0,T]; H^s)$ as $k \to \infty$. Now, we will show that this v belongs to X_T and is solution to (3.1). Since $v^{(k)} \in X_{T,M}$, $\left\{v^{(k)}\right\}$ (resp. $\left\{v_t^{(k)}\right\}$) has a weak-* convergent subsequence in $L^\infty(0,T;H^s)$ (resp. in $L^\infty\left(0,T;H^{s-1}\right)$). Since $v^{(k)} \to v$ in $C([0,T]; H^s)$, the above subsequence of $\left\{v^{(k)}\right\}$ (resp. $\left\{v_t^{(k)}\right\}$) converges weakly-* to v (resp. to v_t) in $L^\infty(0,T;H^s)$ (resp. in $L^\infty\left(0,T;H^{s-1}\right)$), and consequently we see that $v \in L^\infty(0,T;H^s)$ and $v_t \in L^\infty\left(0,T;H^{s-1}\right)$. Then we can see that $v \in Y_{T,M}$, and then we get $\Phi[v] \in X_{T,M}$. Hence we can apply (3.14) to have

$$\sup_{0\le t\le T} \left\|\Phi[v] - \Phi\left[v^{(k)}\right](t,.)\right\|_{X_{T,M}} \le \frac{1}{2} \sup_{0\le t\le T} \left\|\left(v - v^{(k)}\right)(t,.)\right\|_{Y_{T,M}}. \tag{3.15}$$

Since the right-hand side of (3.15) tends to 0 as $k \to \infty$, we get $\Phi\left[v^{(k)}\right] \to \Phi[v]$ in $C([0,T]; H^s)$. Since we have showed that $v^{(k)} \to v$ in $C([0,T]; H^s)$, passing to the limit in $v^{(k)} = \Phi\left[v^{(k-1)}\right]$, we obtain $v = \Phi[v] \in X_{T,M}$. This v apparently the desired solution. The uniqueness of weak solutions in $X_{T,M}$ follows immediately from Gronwall's inequality. As $u = v_t$, we have $u \in C\left([0,T]; H^{s-1}\right)$. This completes the proof of Theorem 3.1. ∎

Remark 3.2. If we take $(u_0, u_1) \in H^{s+1}(\mathbb{R}^n) \times H^s(\mathbb{R}^n)$, with $s > n/2 - 1$, then from Lemma 2.4 the solution of problem (1.1) $(u, u_t) \in C([0,T]; H^s(\mathbb{R}^n)) \times C\left([0,T]; H^{s-1}(\mathbb{R}^n)\right)$.

4 Blow-up results

This section is devoted to the blow-up of solutions of the problem (1.1). We start by introducing the definition of the weak solution of (1.1).

Definition 4.1. *Let $T > 0$, $0 < \gamma < 1$ and $u_0 \in L^1_{loc}(\mathbb{R}^n) \cap L^m_{loc}(\mathbb{R}^n)$, $u_1 \in L^1_{loc}(\mathbb{R}^n)$. We say that u is a weak solution if $u \in L^p((0,T), L^p_{loc}(\mathbb{R}^n)) \cap L^m((0,T), L^m_{loc}(\mathbb{R}^n))$ and satisfies*

$$\begin{aligned} &\Gamma(\alpha) \int_0^T \int_{\mathbb{R}^n} J^\alpha_{0|t}\left(|u|^p\right)\varphi dxdt + \int_{\mathbb{R}^n} u_1(x)\varphi(0,x)\,dx \\ &- \int_{\mathbb{R}^n} u_0(x)\varphi_t(0,x)\,dx + \int_{\mathbb{R}^n} u_0(x)\varphi(0,x)\,dx \\ &+ \tfrac{1}{m}\int_{\mathbb{R}^n} |u_0|^{m-1} u_0(x)\varphi(0,x)\,dx = \int_0^T \int_{\mathbb{R}^n} u\varphi_{tt} dxdt \\ &- \tfrac{1}{m}\int_0^T \int_{\mathbb{R}^n} |u|^{m-1} u\varphi_t(t,x)\,dxdt - \int_0^T \int_{\mathbb{R}^n} u\varphi_t(t,x)\,dxdt \\ &- \int_0^T \int_{\mathbb{R}^n} u\Delta\varphi dxdt, \end{aligned} \tag{4.1}$$

for all nonnegative test function $\varphi \in C^2([0,T]\times\mathbb{R}^n)$ *such that* $\varphi(T,.)=\varphi_t(T,.)=0$, where $\alpha=1-\gamma$.

Theorem 4.2. *Let* $n\geq 1$, $0<\gamma<1$ *and let* p, m *such that* $p>m>1$. *Assume that the initial data* (u_0,u_1) *satisfy*

$$\int_{\mathbb{R}^n} u_0(x)\,dx>0,\ \int_{\mathbb{R}^n} |u_0|^{m-1}u_0(x)>0,\ \text{and}\ \int_{\mathbb{R}^n} u_1(x)\,dx>0.$$

Then if

$$p\leq \max\left\{\frac{n+2}{(n-2+2\gamma)_+},\frac{1}{\gamma}\right\}. \tag{4.2}$$

The solution of problem (1.1) does not exist globally in time.

Proof. The proof is by contradiction. Suppose that u is non trivial weak solution of the problem (1.1) which exists globally in time. Therefore, let us define

$$\varphi(t,x)=D^{\alpha}_{t|T}\tilde{\varphi}(t,x):=\varphi_1^l(x)D^{\alpha}_{t|T}\varphi_2(t).$$

with $\varphi_1^l(x):=\Phi\left(\frac{|x|^2}{T}\right)^l,\varphi_2(t):=\left(1-\frac{t}{T}\right)_+^{\eta}$, where $l,\eta\gg 1$ and $\Phi\in C^{\infty}(\mathbb{R}_+)$ be cut-off non-increasing function such that

$$\Phi(z)=\begin{cases}1 & \text{if } 0\leq z\leq 1,\\ 0 & \text{if } z\geq 2,\end{cases},\quad \text{with}\quad 0\leq\Phi\leq 1.$$

We have from the definition of weak solution

$$\begin{aligned}
&\Gamma(\alpha)\int_0^T\int_{\mathbb{R}^n} J^{\alpha}_{0|t}\left(|u|^p\right)D^{\alpha}_{t|T}\tilde{\varphi}(t,x)dxdt+\int_{\mathbb{R}^n}u_1(x)\,D^{\alpha}_{t|T}\tilde{\varphi}(0,x)\,dx\\
&-\int_{\mathbb{R}^n}u_0(x)\,\partial_t D^{\alpha}_{t|T}\tilde{\varphi}_t(0,x)\,dx\\
&+\tfrac{1}{m}\int_{\mathbb{R}^n}|u_0|^{m-1}u_0(x)\,D^{\alpha}_{t|T}\tilde{\varphi}(0,x)\,dx+\int_{\mathbb{R}^n}u_0(x)\,D^{\alpha}_{t|T}\tilde{\varphi}(0,x)\,dx\\
&=\int_0^T\int_{\mathbb{R}^n}u\partial_t^2 D^{\alpha}_{t|T}\tilde{\varphi}(t,x)\,dxdt\\
&-\tfrac{1}{m}\int_0^T\int_{\mathbb{R}^n}|u|^{m-1}u\partial_t D^{\alpha}_{t|T}\tilde{\varphi}(t,x)\,dxdt\\
&-\int_0^T\int_{\mathbb{R}^n}u\partial_t D^{\alpha}_{t|T}\tilde{\varphi}(t,x)\,dxdt-\int_0^T\int_{\mathbb{R}^n}u\Delta D^{\alpha}_{t|T}\tilde{\varphi}(t,x)dxdt.
\end{aligned}\tag{4.3}$$

Using the formulas(2.3),(2.4) and (2.6) in the left-hand side of (4.3), while in the right-hand side

using (2.4), we get

$$\begin{aligned}
&\Gamma(\alpha)\int_0^T\int_{\mathbb{R}^n}(|u|^p)\,\varphi_1^l(x)\varphi_2(t)dxdt+\int_{\mathbb{R}^n}u_1(x)\,D_{t|T}^{\alpha}\varphi_1^l(x)\varphi_2(0)dx\\
&+\int_{\mathbb{R}^n}u_0(x)\,D_{t|T}^{\alpha+1}\varphi_1^l(x)\varphi_2(0)dx\\
&+\tfrac{1}{m}\int_{\mathbb{R}^n}|u_0|^{m-1}u_0(x)\,D_{t|T}^{\alpha}\varphi_1^l(x)\varphi_2(0)dx+\int_{\mathbb{R}^n}u_0(x)\,D_{t|T}^{\alpha}\varphi_1^l(x)\varphi_2(0)dx,\\
&=\int_0^T\int_{\mathbb{R}^n}uD_{t|T}^{\alpha+2}\varphi_1^l(x)\varphi_2(t)dxdt\\
&+\tfrac{1}{m}\int_0^T\int_{\mathbb{R}^n}|u|^{m-1}uD_{t|T}^{\alpha+1}\varphi_1^l(x)\varphi_2(t)dxdt\\
&+\int_0^T\int_{\mathbb{R}^n}uD_{t|T}^{\alpha+1}\varphi_1^l(x)\varphi_2(t)dxdt\\
&-\int_0^T\int_{\mathbb{R}^n}u\Delta D_{t|T}^{\alpha}\varphi_1^l(x)\varphi_2(t)dxdt.
\end{aligned}\tag{4.4}$$

From the fact that $\Delta\varphi_1^l=l\varphi_1^{l-1}\Delta\varphi_1+l(l-1)|\nabla\varphi_1|^2$, and the support of φ_1 is included in

$$\Omega_T:=\left\{x\in\mathbb{R}^N:|x|\le(2T)^{1/2}\right\},$$

we may write

$$\begin{aligned}
&\int_{Q_T}(|u|^p)\,\tilde{\varphi}dxdt+CT^{\gamma-1}\int_{\mathbb{R}^n}u_1(x)\,\varphi_1^l(x)dx+C\left(T^{\gamma-2}+T^{\gamma-1}\right)\int_{\mathbb{R}^n}u_0(x)\,\varphi_1^l(x)dx+\\
&CT^{\gamma-1}\int_{\mathbb{R}^n}|u_0|^{m-1}u_0(x)\,\varphi_1^l(x)dx\le C\int_{Q_T}|u|\varphi_1^l(x)\left|D_{t|T}^{3-\gamma}\varphi_2(t)\right|dxdt\\
&+C\int_{Q_T}|u|^m\,\varphi_1^l(x)\left|D_{t|T}^{2-\gamma}\varphi_2(t)\right|dxdt+C\int_{Q_T}|u|\varphi_1^l(x)\left|D_{t|T}^{2-\gamma}\varphi_2(t)\right|dxdt\\
&-C\int_{Q_T}u\varphi_1^{l-2}(x)\left(|\Delta\varphi_1|+|\nabla\varphi_1|^2\right)\left|D_{t|T}^{1-\gamma}\varphi_2(t)\right|dxdt,
\end{aligned}\tag{4.5}$$

where $Q_T:=[0,T]\times\mathbb{R}^n$. By the Lebesgue dominated convergence theorem, we can obtain for all $i=0,1$ the following limits

$$\begin{aligned}
\lim_{T\to\infty}\int_{\Omega_T}u_i(x)\varphi_1^l(x)dx&=\int_{\mathbb{R}^n}u_i(x)dx,\\
\lim_{T\to\infty}\int_{\Omega_T}|u_0(x)|^{m-1}u_0(x)\varphi_1^l(x)dx&=\int_{\mathbb{R}^n}|u_0(x)|^{m-1}u_0(x)dx,
\end{aligned}$$

since we have $\lim_{T\to\infty}u_i(x)\varphi_1^l(x)=u_i(x)$, $\lim_{T\to\infty}|u_0(x)|^{m-1}u_0(x)\varphi_1^l(x)=|u_0(x)|^{m-1}u_0(x)$, and $(u_i,u_0)\in L^1_{loc}\left(\mathbb{R}^N\right)\times L^m_{loc}\left(\mathbb{R}^N\right)$. So

$$\int_{\mathbb{R}^n}u_i(x)dx>0\ (\text{resp.}\int_{\mathbb{R}^n}|u_0(x)|^{m-1}u_0(x)dx>0)$$

implies that

$$\int_{\mathbb{R}^n} u_i(x)\varphi_1^l(x)dx > 0 \text{ (resp. } \int_{\mathbb{R}^n} |u_0(x)|^{m-1}u_0(x)\varphi_1^l(x)dx > 0),$$

for T large. Now, By applying the following ε-Young's inequality

$$XY \leq \varepsilon X^p + C(\varepsilon)Y^{p'}, p + p' = pp', X \geq 0, Y \geq 0,$$

to the right-hand side of (4.5), we get

$$\begin{aligned}
&\int_{Q_T} |u|\varphi_1^l(x)\left|\left(D_{t|T}^{3-\gamma} + D_{t|T}^{2-\gamma}\right)\varphi_2(t)\right|dxdt = \int_{Q_T} |u|\tilde{\varphi}^{\frac{1}{p}}\tilde{\varphi}^{-\frac{1}{p}}\varphi_1^l(x)\left|\left(D_{t|T}^{3-\gamma} + D_{t|T}^{2-\gamma}\right)\varphi_2(t)\right|dxdt \\
&\leq \varepsilon\int_{Q_T} |u|^p\tilde{\varphi}dxdt + C(\varepsilon)\int_{Q_T} \varphi_1^l(x)\varphi_2^{-\frac{1}{p-1}}(t)\left|\left(D_{t|T}^{3-\gamma} + D_{t|T}^{2-\gamma}\right)\varphi_2(t)\right|^{\frac{p}{p-1}}dxdt.
\end{aligned} \tag{4.6}$$

For $\varepsilon > 0$, also, we have the estimate

$$\begin{aligned}
&\int_{Q_T} |u|^m\varphi_1^l(x)\left|D_{t|T}^{2-\gamma}\varphi_2(t)\right|dxdt = \int_{Q_T} |u|^m\tilde{\varphi}^{\frac{m}{p}}\tilde{\varphi}^{-\frac{m}{p}}\varphi_1^l(x)\left|D_{t|T}^{2-\gamma}\varphi_2(t)\right|dxdt \\
&\leq \varepsilon\int_{Q_T} |u|^p\tilde{\varphi}dxdt + C(\varepsilon)\int_{Q_T} \varphi_1^l(x)\varphi_2^{-\frac{m}{p-m}}(t)\left|D_{t|T}^{2-\gamma}\varphi_2(t)\right|^{\frac{p}{p-m}}dxdt.
\end{aligned} \tag{4.7}$$

The same is true for the third part of the right-hand side of(4.5)

$$\begin{aligned}
&\int_{Q_T} u\varphi_1^{l-2}\left(|\Delta\varphi_1| + |\nabla\varphi_1|^2\right)\left|D_{t|T}^{1-\gamma}\varphi_2(t)\right|dxdt \leq \varepsilon\int_{Q_T} |u|^p\tilde{\varphi}dxdt \\
&+C(\varepsilon)\int_{Q_T} \varphi_1^{l-2\frac{p}{p-1}}\left(|\Delta\varphi_1|^{\frac{p}{p-1}} + |\nabla\varphi_1|^{2\frac{p}{p-1}}\right)\varphi_2^{-\frac{1}{p-1}}(t)\left|D_{t|T}^{1-\gamma}\varphi_2(t)\right|^{\frac{p}{p-1}}dxdt.
\end{aligned} \tag{4.8}$$

Combining (4.6), (4.7),(4.8), with ε small enough, and some positive constant C, we obtain

$$\begin{aligned}
&\int_{Q_T} |u|^p\tilde{\varphi}dxdt \leq C\int_{Q_T} \varphi_1^l(x)\varphi_2^{-\frac{1}{p-1}}(t)\left|\left(D_{t|T}^{3-\gamma} + D_{t|T}^{2-\gamma}\right)\varphi_2(t)\right|^{\frac{p}{p-1}}dxdt \\
&+C\int_{Q_T} \varphi_1^l(x)\varphi_2^{-\frac{m}{p-m}}(t)\left|D_{t|T}^{2-\gamma}\varphi_2(t)\right|^{\frac{p}{p-m}}dxdt \\
&+C\int_{Q_T} \varphi_1^{l-2\frac{p}{p-1}}(x)\left(|\Delta\varphi_1|^{\frac{p}{p-1}} + |\nabla\varphi_1|^{2\frac{p}{p-1}}\right)\varphi_2^{-\frac{1}{p-1}}(t)\left|D_{t|T}^{1-\gamma}\varphi_2(t)\right|^{\frac{p}{p-1}}dxdt.
\end{aligned} \tag{4.9}$$

Using the change of variables

$$y = T^{-\frac{1}{2}}x, s = T^{-1}t.$$

The equation (4.9) leads to

$$\int_{Q_T} |u|^p\tilde{\varphi}dxdt \leq C\left(T^{(\gamma-3)\frac{p}{p-1}+\frac{N}{2}+1} + 2T^{(\gamma-2)\frac{p}{p-1}+\frac{N}{2}+1} + T^{(\gamma-2)\frac{p}{p-m}+\frac{N}{2}+1}\right) \equiv T^{\sigma}. \tag{4.10}$$

With $\sigma = \max\left\{(\gamma-2)\frac{p}{p-1} + \frac{N}{2} + 1, (\gamma-2)\frac{p}{p-m} + \frac{N}{2} + 1\right\} = (\gamma-2)\frac{p}{p-1} + \frac{N}{2} + 1$. At this stage, we have to distinguish two cases: The case $\sigma < 0$: we pass to the limit in (4.10) as $T \to \infty$, we get

$$\lim_{T\to\infty}\int_{Q_T} |u|^p\tilde{\varphi}dxdt = 0.$$

Using the continuity in time and space of u and the fact that $\lim_{T\to\infty}\tilde{\varphi}(t,x)=1$, then we based on the Lebesgue dominated convergence theorem, we can conclude the following

$$\int_0^\infty\int_{\Omega_T}|u|^p\tilde{\varphi}dxdt=0\Rightarrow u\equiv 0.$$

which is contradiction. Now, we move to the second case : case $\sigma=0$: using inequality (4.10) with $T\to\infty$, we have

$$u\in L^p\left((0,\infty);L^p(\mathbb{R}^N)\right).$$

Which implies that

$$\lim_{T\to\infty}\int_0^T\int_{\sum_T}|u|^p\tilde{\varphi}(x,t)dxdt=0,$$

where

$$\Omega_T:=\left\{x\in\mathbb{R}^N,\ |x|^2\le 2T\right\},\quad \sum\nolimits_T:=\left\{x\in\mathbb{R}^n,\ T\le|x|^2\le 2T\right\}.$$

On the other hand, using Hölder's inequality instead of Young's one to the term

$$\int_0^T\int_{\mathbb{R}^N}|u|\varphi_1^{l-2}(|\Delta\varphi_1|+|\nabla\varphi_1|^2)\left|D^\alpha_{t|T}\varphi_2(t)\right|dxdt,$$

we find

$$\int_0^T\int_{\mathbb{R}^N}|u|\varphi_1^{l-2}(|\Delta\varphi_1|+|\nabla\varphi_1|^2)\left|D^\alpha_{t|T}\varphi_2(t)\right|dxdt\le\left(\int_0^T\int_{\sum_{TR^{-1}}}|u|^p\tilde{\varphi}(t,x)dxdt\right)^{1/p}\times$$
$$\left(\int_0^T\int_{\sum_{TR^{-1}}}\varphi_1^{l-2p'}\varphi_2^{-p'}(|\Delta\varphi_1|^{p'}+|\nabla\varphi_1|^{2p'})\left|D^\alpha_{t|T}\varphi_2(t)\right|^{p'}dxdt\right)^{1/p'}.$$

We repeat the same calculation as above by taking in this time $\varphi_1(x):=\Phi\left(\frac{|x|^2}{R^{-1}T}\right)$ where R is fixed number such that $1<R<T$. Using the change of variables $y=R^{\frac{1}{2}}T^{-\frac{1}{2}}x$, $\tau=T^{-1}t$, and the fact $\sigma=0$, we get

$$\int_0^T\int_{\Omega_{TR^{-1}}}|u|^p\tilde{\varphi}dxdt$$
$$\le C(\varepsilon)\int_0^T\int_{\Omega_{TR^{-1}}}\varphi_1^l(x)\varphi_2^{-\frac{1}{p-1}}(t)\left(\left|D^{3-\gamma}_{t|T}\varphi_2(t)\right|^{p'}+\left|D^{2-\gamma}_{t|T}\varphi_2(t)\right|^{p'}\right)dxdt$$
$$+\int_0^T\int_{\Omega_{TR^{-1}}}\varphi_1^l(x)|\varphi_2(t)|^{-\frac{m}{p-m}}\left|D^{2-\gamma}_{t|T}\varphi_2(t)\right|^{\frac{p}{p-m}}dxdt+\left(\int_0^T\int_{\sum_{TR^{-1}}}|u|^p\tilde{\varphi}(x,t)dxdt\right)^{1/p}$$
$$\times\left(\int_0^T\int_{\sum_{TR^{-1}}}\varphi_1^{l-2p'}(x)\varphi_2(t)^{-p'}(|\Delta\varphi_1|^{p'}+|\nabla\varphi_1|^{2p'})\left|D^{1-\gamma}_{t|T}\varphi_2(t)\right|^{p'}dxdt\right)^{1/p'},$$

which yields

$$\int_0^T \int_{\left\{|x|\leq\sqrt{2}R^{-\frac{1}{2}}T^{\frac{1}{2}}\right\}} |u|^p\tilde{\varphi}dxdt \leq \left(T^{(\gamma-3)\frac{p}{p-1}+\frac{N}{2}+1} + T^{(\gamma-2)\frac{p}{p-1}+\frac{N}{2}+1} + T^{(\gamma-2)\frac{p}{p-m}+\frac{N}{2}+1}\right) R^{-\frac{N}{2}}$$
$$+C\left(T^{(\gamma-2)\frac{p}{p-1}+\frac{N}{2}+1}\right)^{1/p'} R^{1-\frac{N}{2p'}} \times \left(\int_{\sum_{TR^{-1}}} |u|^p\tilde{\varphi}(x,t)dxdt\right)^{1/p'}.$$

Because of $\sigma = 0$, we can get

$$\int_0^T \int_{\left\{|x|\leq\sqrt{2}R^{-\frac{1}{2}}T^{\frac{1}{2}}\right\}} |u|^p\tilde{\varphi}dxdt \leq CR^{-\frac{N}{2}} + R^{1-\frac{N}{2p'}} \times \left(\int_0^T \int_{\sum_{TR^{-1}}} |u|^p\tilde{\varphi}(x,t)dxdt\right)^{1/p}. \quad (4.11)$$

Passing to the limit in (4.11) as $T \to \infty$, we get

$$\int_0^\infty \int_{\mathbb{R}^N} |u|^p dxdt \leq CR^{-\frac{N}{2}},$$

and then $R \to \infty$ which give a contradiction.

The case $p \leq \frac{1}{\gamma}$ we choose $\varphi_1(x) := \Phi\left(\frac{|x|^2}{R}\right), \varphi_2(t) := \left(1-\frac{t}{T}\right)_+^\eta$, then by taking the change variables $x = R^{\frac{1}{2}}y$, $t = T\tau$, it follows from (4.9) that

$$\begin{aligned}\int_0^\infty \int_{\mathbb{R}^n} |u|^p\tilde{\varphi}dxdt &\leq CR^{\frac{n}{2}}\left(T^{(\gamma-3)p'+1} + T^{(\gamma-2)p'+1} + T^{(\gamma-2)\frac{p}{p-m}+1}\right)\\ &\quad +CR^{(\frac{n}{2}-p')}T^{(\gamma-1)p'+1}.\end{aligned}$$

Now, passing to the limit as $T \to \infty$ in the last inequality by taking account that $p < 1/\gamma$, we deduce that

$$\int_0^\infty \int_{\mathbb{R}^n} |u|^p\tilde{\varphi}dxdt = 0,$$

and then, by taking $R \to \infty$, we get contradiction.

Precisely, in the case $p = \frac{1}{\gamma}$, we have to use condition $\frac{n}{2} - \frac{p}{p-1} < 0$, which is equivalent to $\gamma > \frac{n-2}{2}$ to obtain the desired convergence. This completes the proof ∎

Remark 4.3. When $m = 1$, we recover the case studied by Fino [6], Berbiche [2].

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