Coefficient bounds for subclasses of bi-univalent functions defined by fractional derivative operator

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Abstract

In this paper, first we introduce new subclasses of analytic and bi-univalent functions defined by fractional derivative operator in the open unit disk and then for the functions belongs to these classes obtain upper bounds for the initial coefficients.

2010 Mathematics Subject Classification. **30C45** 30C50
Keywords. Analytic functions, Coefficient bounds, Univalent functions, Fractional derivative, Bi-univalent functions..

1 Introduction

Let $\mathcal{A}$ denote the class of functions f which are *analytic* in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ and normalized by the conditions $f(0) = f'(0) - 1 = 0$. Equivalently; if $f \in \mathcal{A}$, then the Taylor-Maclaurin series representation has the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \ (z \in \mathbb{U}). \tag{1.1}$$

Further, let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ that are univalent in $\mathbb{U}$ (for details, see [5]).

In view of the results of the *Koebe One-Quarter Theorem* [5], every function $f \in \mathcal{S}$ has an inverse f^{-1} providing $f^{-1}\big(f(z)\big) = z, (z \in \mathbb{U})$ and $f\big(f^{-1}(w)\big) = w, \ \big(|w| < r_0(f); r_0(f) \geqq \frac{1}{4}\big)$. In fact, the function f^{-1} is given by

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots . \tag{1.2}$$

$$= w + \sum_{n=2}^{\infty} b_n w^n.$$

A function $f(z)$ is said to be *bi-univalent* in $\mathbb{U}$ if its inverse map $g = f^{-1}$ is also univalent in $\mathbb{U}$; $f(z)$ is belong to Σ iff (i)$f(z) \in \mathcal{S}$ and (ii) there exists a function $g(w) \in \mathcal{S}$ such that $g\big(f(z)\big) = z$ and $f\big(g(w)\big) = w$ in some neighborhood of the origin.

The problem of coefficient estimates of bi-univalent functions dates back to the 1967, when Lewin [8] investigated the class Σ. Following, Brannan and Clunie [2], Netanyahu [9], Taha [28] and Brannan and Taha [3] studied on bi-univalent functions. In fact, judging by the remarkable flood of papers on the subject, the pioneering work by Srivastava et al. [16] appears to have revived the study of analytic and bi-univalent functions in recent years. In these days a significant amount

Tbilisi Mathematical Journal 13(3) (2020), pp. 1–10.
Tbilisi Centre for Mathematical Sciences.
Received by the editors: 18 July 2019.
Accepted for publication: 14 May 2020.

of theoretical work is done by outstanding mathematicians as Srivastava et al. [16, 21, 22, 23, 17, 15, 20], Ali et al. [13], Çaglar et al. [4], Eker et al. [1, 24, 25, 29], Hamidi and Jahangiri [6], Hussain et al. [7], Şeker [26, 27], Zaprawa [30].

In recent years, one of the trendiest approaches among researchers is the use of fractional integral and fractional derivative operators. Because fractional integrals and derivatives can always be expressed using ordinary integrals and derivatives, any result obtainable through the fractional calculus may also be derived making use only of the concepts of classical calculus. In the light of the foregoing, in this paper we have defined new subclasses of bi-univalent functions using fractional derivative, which generalizes many studies. Since the fractional derivative and fractional integral are the central components of our results, we give their definitions first. (For details, see [10, 11, 18, 19]).

Definition 1. For a function f , the fractional integral of order δ is defined by

$$D_z^{-\delta} f(z) = \frac{1}{\Gamma(\delta)} \int_0^z \frac{f(\xi)}{(z-\xi)^{1-\delta}} d\xi \,;\ (\delta > 0),$$

where f is an analytic function in a simply-connected region of complex z-plane containing the origin, and the multiplicity of $(z-\xi)^{\delta-1}$ is removed by requiring, $log(z-\xi)$ to be real when $z - \xi > 0$.

Definition 2. For a function f , the fractional derivative of order δ is defined by

$$D_z^{\delta} f(z) = \frac{1}{\Gamma(1-\delta)} \frac{d}{dz} \int_0^z \frac{f(\xi)}{(z-\xi)^{\delta}} d\xi \ (0 \le \delta < 1),$$

where f is constrained, and the multiplicity of $(z-\xi)^{-\delta}$ is removed, as in Definition 1.

Definition 3. Considering the conditions of Definition 2, the fractional derivative of order $(n+\delta)$ is described by

$$D_z^{n+\delta} f(z) = \frac{d^n}{dz^n} D_z^{\delta} f(z) \quad (0 \le \delta < 1, n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\})$$

Considering the above definitions we obtain the following equations

$$D_z^{-\delta} z^n = \frac{\Gamma(n+1)}{\Gamma(n+\delta+1)} z^{n+\delta} \qquad (n \in \mathbb{N}, \delta > 0)$$

and

$$D_z^{\delta} z^n = \frac{\Gamma(n+1)}{\Gamma(n-\delta+1)} z^{n-\delta} \qquad (n \in \mathbb{N}, 0 \le \delta < 1)$$

The purpose of the present paper is to introduce new subclasses of Σ using by fractional derivative operator and find upper bounds for the coefficients $|a_2|$ and $|a_3|$ for functions in these classes.

2 Main results

Definition 4. A function f given by (1.1) is said to be in the class $\mathcal{F}_{\Sigma}^{\delta}(\alpha)$ if the following conditions are satisfied:

$$f \in \Sigma \text{ and } \left| arg \frac{z D_z^{\delta+1} f(z)}{(1-\delta) D_z^{\delta} f(z)} \right| < \frac{\alpha\pi}{2} \qquad (0 < \alpha \le 1, 0 \le \delta < 1, z \in \mathbb{U}) \tag{2.1}$$

and

$$\left| arg \frac{wD_w^{\delta+1}g(w)}{(1-\delta)D_w^{\delta}f(w)} \right| < \frac{\alpha\pi}{2} \qquad (0 < \alpha \leq 1, 0 \leq \delta < 1, w \in \mathbb{U}) \tag{2.2}$$

where D_z^{δ} is the fractional derivative of order δ, and $g(w)$ is given by (1.2).

From the Definition 4, it is clear that, for $\delta = 0$, this class is $S_{\Sigma}^{*}(\alpha)$ of class of strongly bi-starlike of order α $(0 < \alpha \leq 1)$, which is introduced by Brannan and Taha [3].

First we shall state and prove the estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in the class $\mathcal{F}_{\Sigma}^{\delta}(\alpha)$.

Theorem 1. Let $f \in \mathcal{F}_{\Sigma}^{\delta}(\alpha)$ be of the form (1.1). Then

$$\left|a_2\right| \leq \frac{\alpha\Gamma(3-\delta)}{\Gamma(1-\delta)} \sqrt{\frac{3-\delta}{6\alpha(1-\delta)(2-\delta) + (3-\delta)[1-\alpha(3-2\delta)]}} \tag{2.3}$$

and

$$\left|a_3\right| \leq \frac{\alpha\Gamma(3-\delta)\Gamma(4-\delta)}{2(3-2\delta)\Gamma(1-\delta)\Gamma(2-\delta)}. \tag{2.4}$$

Proof. For a function is given by (1.1), we can write from (2.1) and (2.2)

$$\frac{zD_z^{\delta+1}f(z)}{(1-\delta)D_z^{\delta}f(z)} = [p(z)]^{\alpha} \tag{2.5}$$

$$\frac{wD_w^{\delta+1}g(w)}{(1-\delta)D_w^{\delta}f(w)} = [q(w)]^{\alpha} \tag{2.6}$$

where $p(z)$ and $q(w)$ are functions in Caratheódory Class $\mathcal{P}$. Thus $p(z)$ and $q(w)$ are have the following series expansions:

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots \tag{2.7}$$

and

$$q(w) = 1 + q_1 w + q_2 w^2 + q_3 w^3 + \cdots . \tag{2.8}$$

Equating the coefficients of identical powers of z and w on the left- and right-hand sides of (2.5) and (2.6), respectively, we get the following relations:

$$\frac{2\Gamma(1-\delta)}{\Gamma(3-\delta)} a_2 = \alpha p_1, \tag{2.9}$$

$$4\Gamma(1-\delta)\left\{\frac{3}{\Gamma(4-\delta)} a_3 - \frac{\Gamma(2-\delta)}{[\Gamma(3-\delta)]^2} a_2^2\right\} = \alpha p_2 + \frac{\alpha(\alpha-1)}{2} p_1^2, \tag{2.10}$$

$$-\frac{2\Gamma(1-\delta)}{\Gamma(3-\delta)} a_2 = \alpha q_1 \tag{2.11}$$

and

$$4\Gamma(1-\delta)\left\{\left(\frac{6}{\Gamma(4-\delta)}-\frac{\Gamma(2-\delta)}{[\Gamma(3-\delta)]^2}\right)a_2^2-\frac{3}{\Gamma(4-\delta)}a_3\right\}=\alpha q_2+\frac{\alpha(\alpha-1)}{2}q_1^2. \tag{2.12}$$

From (2.9) and (2.11), we get

$$p_1=-q_1 \tag{2.13}$$

and

$$8\left(\frac{\Gamma(1-\delta)}{\Gamma(3-\delta)}\right)^2 a_2^2=\alpha^2(p_1^2+q_1^2). \tag{2.14}$$

Also from (2.10), (2.12) and 2.14, we get

$$a_2^2=\frac{\alpha^2(p_2+q_2)(3-\delta)[\Gamma(3-\delta)]^2}{4[\Gamma(1-\delta)]^2\left\{6\alpha(1-\delta)(2-\delta)+(3-\delta)[1-\alpha(3-2\delta)]\right\}}. \tag{2.15}$$

It is well known that from the Caratheódory Lemma, $|p_n|\leq 2$ and $|q_n|\leq 2$ for $n\in\mathbb{N}$ (see [5]). Applying the Carathéodory Lemma, we obtain

$$\left|a_2\right|\leq\frac{\alpha\Gamma(3-\delta)}{\Gamma(1-\delta)}\sqrt{\frac{3-\delta}{6\alpha(1-\delta)(2-\delta)+(3-\delta)[1-\alpha(3-2\delta)]}}.$$

Thus we get the desired result.

To find the bound on $|a_3|$, we can write

$$\frac{24(3-2\delta)\Gamma(1-\delta)\Gamma(2-\delta)}{(3-\delta)[\Gamma(3-\delta)]^2\Gamma(4-\delta)}a_3=\alpha\left\{\frac{(9-5\delta)\Gamma(2-\delta)}{(3-\delta)[\Gamma(3-\delta)]^2}p_2+\frac{\Gamma(2-\delta)}{[\Gamma(3-\delta)]^2}q_2+\frac{3(\alpha-1)}{\Gamma(4-\delta)}p_1^2\right\} \tag{2.16}$$

from (2.12), (2.10) and (2.13).

If $\alpha=1$, then

$$|a_3|\leq\frac{\Gamma(3-\delta)\Gamma(4-\delta)}{2(3-2\delta)\Gamma(1-\delta)\Gamma(2-\delta)}.$$

Now, we consider the case $0<\alpha<1$. From (2.16), we can write

$$\frac{24(3-2\delta)\Gamma(1-\delta)\Gamma(2-\delta)}{(3-\delta)[\Gamma(3-\delta)]^2\Gamma(4-\delta)}\mathrm{Re}\,(a_3)=\alpha\mathrm{Re}\left\{\frac{(9-5\delta)\Gamma(2-\delta)}{(3-\delta)[\Gamma(3-\delta)]^2}p_2+\frac{\Gamma(2-\delta)}{[\Gamma(3-\delta)]^2}q_2+\frac{3(\alpha-1)}{\Gamma(4-\delta)}p_1^2\right\} \tag{2.17}$$

From Herglotz's Representation formula [12] for the functions $p(z)$ and $q(w)$, we have

$$p(z)=\int_0^{2\pi}\frac{1+ze^{-it}}{1-ze^{-it}}d\mu_1(t),$$

and

$$q(w)=\int_0^{2\pi}\frac{1+we^{-it}}{1-we^{-it}}d\mu_2(t),$$

where $\mu_i(t)$ are increasing on $[0, 2\pi]$ and $\mu_i(2\pi) - \mu_i(0) = 1$, $i = 1, 2$.

We also have

$$p_n = 2\int_0^{2\pi} e^{-int} d\mu_1(t), \qquad n = 1, 2, \ldots$$

$$q_n = 2\int_0^{2\pi} e^{-int} d\mu_2(t), \qquad n = 1, 2, \ldots.$$

Now (2.17) can be written as follows :

$$\frac{24(3-2\delta)\Gamma(1-\delta)\Gamma(2-\delta)}{(3-\delta)[\Gamma(3-\delta)]^2\Gamma(4-\delta)}\mathrm{Re}\,(a_3) = \alpha\left\{\frac{2(9-5\delta)\Gamma(2-\delta)}{(3-\delta)[\Gamma(3-\delta)]^2}\int_0^{2\pi} cos2t d\mu_1(t) + \frac{2\Gamma(2-\delta)}{[\Gamma(3-\delta)]^2}\int_0^{2\pi} cos2t d\mu_2(t)\right\}$$

$$-\frac{12\alpha(1-\alpha)}{\Gamma(4-\delta)}\left[\left(\int_0^{2\pi} cost d\mu_1(t)\right)^2 - \left(\int_0^{2\pi} sint d\mu_1(t)\right)^2\right]$$

$$\leq \frac{2\alpha(9-5\delta)\Gamma(2-\delta)}{(3-\delta)[\Gamma(3-\delta)]^2}\int_0^{2\pi} cos2t d\mu_1(t) + \frac{2\alpha\Gamma(2-\delta)}{[\Gamma(3-\delta)]^2}\int_0^{2\pi} cos2t d\mu_2(t) + \frac{12\alpha(1-\alpha)}{\Gamma(4-\delta)}\left(\int_0^{2\pi} sint d\mu_1(t)\right)^2$$

$$= \frac{2\alpha(9-5\delta)\Gamma(2-\delta)}{(3-\delta)[\Gamma(3-\delta)]^2} - \frac{4\alpha(9-5\delta)\Gamma(2-\delta)}{(3-\delta)[\Gamma(3-\delta)]^2}\int_0^{2\pi} sin^2 t d\mu_1(t)$$

$$+\frac{2\alpha\Gamma(2-\delta)}{[\Gamma(3-\delta)]^2} - \frac{4\alpha\Gamma(2-\delta)}{[\Gamma(3-\delta)]^2}\int_0^{2\pi} sin^2 t d\mu_2(t) + \frac{12\alpha(1-\alpha)}{\Gamma(4-\delta)}\left(\int_0^{2\pi} sint d\mu_1(t)\right)^2.$$

By Jensen's inequality ([14]), we have

$$\left(\int_0^{2\pi} |sint| d\mu(t)\right)^2 \leq \left(\int_0^{2\pi} sin^2 t d\mu(t)\right).$$

Hence

$$\frac{24(3-2\delta)\Gamma(1-\delta)\Gamma(2-\delta)}{(3-\delta)[\Gamma(3-\delta)]^2\Gamma(4-\delta)}\mathrm{Re}\,(a_3)$$

$$\leq \frac{12\alpha\Gamma(3-\delta)}{(3-\delta)[\Gamma(3-\delta)]^2} - \left[\frac{4\alpha(9-5\delta)\Gamma(2-\delta)}{(3-\delta)[\Gamma(3-\delta)]^2} - \frac{12\alpha(1-\alpha)}{\Gamma(4-\delta)}\right]\int_0^{2\pi} sin^2 t d\mu_1(t) - \frac{4\alpha\Gamma(2-\delta)}{[\Gamma(3-\delta)]^2}\int_0^{2\pi} sin^2 t d\mu_2(t),$$

and thus

$$\mathrm{Re}\,(a_3) \leq \frac{\alpha\Gamma(3-\delta)\Gamma(4-\delta)}{2(3-2\delta)\Gamma(1-\delta)\Gamma(2-\delta)},$$

which implies

$$|a_3| \leq \frac{\alpha\Gamma(3-\delta)\Gamma(4-\delta)}{2(3-2\delta)\Gamma(1-\delta)\Gamma(2-\delta)}.$$

This completes the proof of theorem. Q.E.D.

The value $\delta = 0$ gives following corollary:

Corollary 1. [3] Let $f(z)$ given by (1.1) belong to $S_{\Sigma}^{*}(\alpha)$ $(0 < \alpha \leq 1)$. Then

$$|a_2| \leq \frac{2\alpha}{\sqrt{1+\alpha}} \qquad \text{and} \qquad |a_3| \leq 2\alpha.$$

3 Coefficient bounds for the class $\mathcal{F}_{\Sigma}^{\delta}(\beta)$

Definition 5. A function f given by (1.1) is said to be in the class $\mathcal{F}_{\Sigma}^{\delta}(\beta)$, if the following conditions are satisfied:

$$f \in \Sigma \qquad and \qquad Re\left\{\frac{zD_z^{\delta+1}f(z)}{(1-\delta)D_z^{\delta}f(z)}\right\} > \beta \qquad (0 \leq \beta < 1, 0 \leq \delta < 1, z \in \mathbb{U}) \tag{3.1}$$

and

$$Re\left\{\frac{wD_w^{\delta+1}g(w)}{(1-\delta)D_w^{\delta}f(w)}\right\} > \beta \qquad (0 \leq \beta < 1, 0 \leq \delta < 1, w \in \mathbb{U}) \tag{3.2}$$

where D_z^{δ} is the fractional derivative of order δ and g is inverse of the function f given by (1.2).

For the case $\delta = 0$, we obtain $S_{\Sigma}^{*}(\beta)$ of bi-starlike of order β $(0 \leq \beta < 1)$, which is introduced by Brannan and Taha [3].

Theorem 2. Let $f \in \mathcal{F}_{\Sigma}^{\delta}(\beta)$ be of the form (1.1). Then

$$\left|a_2\right| \leq \Gamma(3-\delta)\sqrt{\frac{(1-\beta)(3-\delta)}{2(3-2\delta)\Gamma(1-\delta)\Gamma(2-\delta)}} \tag{3.3}$$

and

$$\left|a_3\right| \leq \frac{(1-\beta)(2-\delta)\Gamma(4-\delta)}{2(3-2\delta)\Gamma(1-\delta)}. \tag{3.4}$$

Proof. The inequalities (3.1) and (3.2) can be rewritten in their equivalent forms:

$$\frac{zD_z^{\delta+1}f(z)}{(1-\delta)D_z^{\delta}f(z)} = \beta + (1-\beta)p(z) \tag{3.5}$$

and

$$\frac{wD_w^{\delta+1}g(w)}{(1-\delta)D_w^{\delta}f(w)} = \beta + (1-\beta)q(w) \tag{3.6}$$

where $p(z)$ and $q(w)$ are given by (2.7) and (2.8), respectively.

Equating the coefficients of identical powers of z and w on the left- and right-hand sides of (3.5) and (3.6), respectively, we get the following relations:

$$\frac{2\Gamma(1-\delta)}{\Gamma(3-\delta)}a_2 = (1-\beta)p_1, \tag{3.7}$$

$$4\Gamma(1-\delta)\left\{\frac{3}{\Gamma(4-\delta)}a_3-\frac{\Gamma(2-\delta)}{[\Gamma(3-\delta)]^2}a_2^2\right\}=(1-\beta)p_2, \tag{3.8}$$

$$-\frac{2\Gamma(1-\delta)}{\Gamma(3-\delta)}a_2=(1-\beta)q_1 \tag{3.9}$$

and

$$4\Gamma(1-\delta)\left\{\left(\frac{6}{\Gamma(4-\delta)}-\frac{\Gamma(2-\delta)}{[\Gamma(3-\delta)]^2}\right)a_2^2-\frac{3}{\Gamma(4-\delta)}a_3\right\}=(1-\beta)q_2. \tag{3.10}$$

From (3.7) and (3.9) we get

$$p_1=-q_1 \tag{3.11}$$

and

$$8\left(\frac{\Gamma(1-\delta)}{\Gamma(3-\delta)}\right)^2a_2^2=(1-\beta)^2(p_1^2+q_1^2). \tag{3.12}$$

Also from (3.8) and (3.10) we obtain

$$\frac{8(3-2\delta)\Gamma(1-\delta)\Gamma(2-\delta)}{(3-\delta)[\Gamma(3-\delta)]^2}a_2^2=(1-\beta)(p_2+q_2). \tag{3.13}$$

Clearly, we have

$$|a_2|^2\leq\frac{(1-\beta)(3-\delta)[\Gamma(3-\delta)]^2}{8(3-2\delta)\Gamma(1-\delta)\Gamma(2-\delta)}\left(|p_2|+|q_2|\right). \tag{3.14}$$

With the help of the Carathéodory Lemma, we obtain the upper bound on $|a_2|$ as given in (3.3).

To obtain the bound on $|a_3|$, multiplying the relations (3.8) and (3.10) by $\frac{6}{\Gamma(4-\delta)}-\frac{\Gamma(2-\delta)}{[\Gamma(3-\delta)]^2}$ and $\frac{\Gamma(2-\delta)}{[\Gamma(3-\delta)]^2}$, respectively and adding them, we obtain:

$$\frac{24(3-2\delta)\Gamma(1-\delta)\Gamma(2-\delta)}{(3-\delta)\Gamma(4-\delta)[\Gamma(3-\delta)]^2}a_3=(1-\beta)\left(\frac{(9-5\delta)\Gamma(2-\delta)}{(3-\delta)[\Gamma(3-\delta)]^2}p_2+\frac{\Gamma(2-\delta)}{[\Gamma(3-\delta)]^2}q_2\right). \tag{3.15}$$

Finally, with the help of the Carathéodory Lemma, we obtain

$$|a_3|\leq\frac{(1-\beta)(2-\delta)\Gamma(4-\delta)}{2(3-2\delta)\Gamma(1-\delta)},$$

which is asserted in (3.4). Q.E.D.

The value $\delta=0$ gives following corollary:

Corollary 2. [3] Let $f(z)$ given by (1.1) belong to $S_\Sigma^*(\beta)$ $(0\leq\beta<1)$. Then

$$|a_2|\leq\sqrt{2(1-\beta)}\qquad\text{and}\qquad|a_3|\leq 2(1-\beta).$$

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