A new reliable method and its convergence for nonlinear second-order fractional differential equations

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Abstract

The main goal of this article is to propose a new reliable method to solve nonlinear second-order fractional differential equations in particular, nonlinear fractional Bratu-type equation. This method called the modified Taylor fractional series method (MTFSM). The fractional derivative is defined in the Liouville-Caputo sense. Simplicity, rapid convergence, and high accuracy are the advantages of this method. In addition, the MTFSM reduces the size of calculations by not requiring linearization, discretization, perturbation or any other restriction. Three numerical examples are exhibited to demonstrate the reliability and efficiency of the proposed method, and the solutions are considered as an infinite series that converge rapidly to the exact solutions. The results display that the MTFSM is very effective and accurate to solve this type of nonlinear fractional problems.

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1 Introduction

The fractional differential equations obtained from the standard differential equations by replacing the ordinary order derivative by a fractional derivative of order $\alpha > 0$, has been recently treated by a number of authors. See for example [5],[14],[15],[17].

The study of analytical and numerical solutions of nonlinear fractional differential equations plays an important role in mathematical physics, engineering, and the other sciences. Recently, various approaches have been successfully proposed for solving nonlinear fractional differential equations such as the Adomian decomposition method (ADM) [6], Homotopy perturbation method (HPM) [7], Homotopy analysis method (HAM) [20], Variational iteration method (VIM) [16], Fractional reduced differential transform method (FRDTM) [11], Fractional residual power series method (FRPSM) [12] and so on.

The main purpose of this article is to present a new reliable method called the modified Taylor fractional series method (MTFSM) and establish its convergence to solve the nonlinaer fractional Bratu-type equation in the form

$$D^{2\alpha}u(x) + \lambda \exp\left(u(x)\right) = 0, \tag{1.1}$$

with the initial conditions

$$u(0) = a_0, D^{\alpha}u(0) = a_1, \tag{1.2}$$

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where $D^{2\alpha}$ is the Liouville-Caputo fractional derivative operator of order 2α, $1/2 < \alpha \leq 1, 0 < x < 1$, and $\lambda \in \mathbb{R}$.

When $\alpha = 1$, the equation (1.1) reduces to the classical nonlinear Bratu-type equation. The exact solution of equation (1.1) in the classical case is given by [1],[2],[3]

$$u(x) = -2 \ln \left[\frac{\cosh\left(\left(x - \frac{1}{2}\right)\frac{\theta}{2}\right)}{\cosh(\frac{\theta}{4})} \right],$$

where θ satisfies

$$\theta = \sqrt{2\lambda} \cosh\left(\frac{\theta}{4}\right).$$

The problem has zero, one or two solutions when $\lambda > \lambda_c$, $\lambda = \lambda_c$, and $\lambda < \lambda_c$, respectively, where the critical value λ_c satisfies the equation

$$1 = \frac{1}{4}\sqrt{2\lambda_c} \sinh\left(\frac{\theta_c}{4}\right),$$

where

$$\lambda_c = 3.513830719.$$

This type of equation has been studied extensively because of its mathematical and physical properties where they appear in a number of applications such as: the fuel ignition model of the thermal combustion theory, the model of thermal reaction process, the Chandrasekhar model of the expansion of the universe, questions in geometry and relativity about the Chandrasekhar model, chemical reaction theory, radiative heat transfer and nanotechnology [4],[18],[19].

The rest of the article is organized as follows. In Section 2, we give some basic definitions and properties of the fractional calculus theory. In Section 3, we present our results to solve the nonlinear fractional Bratu-type equation (1.1) with the initial conditions (1.2) using the MTFSM and establish its convergence in the form of a new theorems. In Section 4, we present three numerical examples to show the efficiency and effectiveness of the proposed method. In Section 5, we discuss our obtained results represented by figures and tables. Section 6, is devoted to the conclusions on the work.

2 Basic definitions

In this section, we give some basic definitions and properties of the fractional calculus theory which are used further in this paper . For more details (see, [14],[15],[17]).

Definition 2.1. Let $u : [0, 1] \longrightarrow \mathbb{R}$ be a continous function. The Riemann-Liouville fractional integral of order $\alpha \geq 0$ is defined by

$$I^{\alpha}u(x) = \begin{cases} \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^x (x-\xi)^{\alpha-1} u(\xi) d\xi, \alpha > 0, \\ u(x), \qquad\qquad\qquad\qquad \alpha = 0, \end{cases} \tag{2.1}$$

where $\Gamma(.)$ is the Euler gamma function.

Definition 2.2. Let $u^{(n)} : [0,1] \longrightarrow \mathbb{R}$ be a continous function. The Liouville-Caputo fractional derivative of order $\alpha \geq 0$ is defined by

$$D^{\alpha}u(x) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \int\limits_0^x (x-\xi)^{n-\alpha-1} u^{(n)}(\xi) d\xi, n-1 < \alpha < n, \\ u^{(n)}(x), \qquad\qquad\qquad\qquad \alpha = n, \end{cases} \tag{2.2}$$

where $n = [\alpha] + 1$ with $[\alpha]$ being the integer part of α.

For this definition we have the following properties

1)

$$D^{\alpha}(c) = 0, \text{ where } c \text{ is a constant.}$$

2)

$$D^{\alpha}x^{\beta} = \begin{cases} \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} x^{\beta-\alpha}, \beta > n-1, \\ 0, \qquad\qquad\qquad \beta \leq n-1. \end{cases}$$

3 MTFSM to solve nonlinear fractional Bratu-type equation

Theorem 3.1. Consider the nonlinear fractional Bratu-type equation (1.1) with the initial conditions (1.2). Then, by MTFSM the solution of equations (1.1) and (1.2) is given in the form of infinite series as follows

$$u(x) = \sum_{i=0}^{\infty} a_i \frac{x^{i\alpha}}{\Gamma(i\alpha+1)}, 0 < \alpha \leq 1, 0 < x < R < 1, \tag{3.1}$$

where a_i the coefficients of the series (3.1) and R is the radius of convergence.

Proof. We consider the following nonlinear fractional Bratu-type equation (1.1) with the initial conditions (1.2).

Assume that the solution takes the following infinite series form

$$u(x) = \sum_{i=0}^{\infty} a_i \frac{x^{i\alpha}}{\Gamma(i\alpha+1)}. \tag{3.2}$$

Consequently, the $n^{th}-$order approximate solution to (1.1), can be written in the form of

$$u_n(x) = \sum_{i=0}^{n} a_i \frac{x^{i\alpha}}{\Gamma(i\alpha+1)} = a_0 + a_1 \frac{x^{\alpha}}{\Gamma(\alpha+1)} + \sum_{i=2}^{n} a_i \frac{x^{i\alpha}}{\Gamma(i\alpha+1)}. \tag{3.3}$$

By applying the operator $D^{2\alpha}$ on equation (3.3), we obtain the formula

$$D^{2\alpha}u_n(x) = \sum_{i=0}^{n-2} a_{i+2} \frac{x^{i\alpha}}{\Gamma(i\alpha+1)}. \tag{3.4}$$

Next, we substitute both (3.3) and (3.4) in (1.1). Therefore, we have the following recurrence relations

$$0=\sum_{i=0}^{n-2}a_{i+2}\frac{x^{i\alpha}}{\Gamma(i\alpha+1)}+\lambda\exp\left(a_0+a_1\frac{x^{\alpha}}{\Gamma(\alpha+1)}+\sum_{i=2}^{n}a_i\frac{x^{i\alpha}}{\Gamma(i\alpha+1)}\right).$$

We follow the same analogue used in obtaining the Taylor series coefficients. In particular, to determine the coefficient $a_n, n=2,3,4,..$, we have to solve the following

$$D^{(n-2)\alpha}\left\{G(x,\alpha,n)\right\}\downarrow_{x=0}=0,$$

where

$$G(x,\alpha,n)=\sum_{i=0}^{n-2}a_{i+2}\frac{x^{i\alpha}}{\Gamma(i\alpha+1)}+\lambda\exp\left(a_0+a_1\frac{x^{\alpha}}{\Gamma(\alpha+1)}+\sum_{i=2}^{n}a_i\frac{x^{i\alpha}}{\Gamma(i\alpha+1)}\right).$$

Now, we determine the first terms of the sequence $\{a_n\}_2^N$.
For $n=2$ we have

$$G(x,\alpha,2)=a_2+\lambda\exp\left(a_0+a_1\frac{x^{\alpha}}{\Gamma(\alpha+1)}+a_2\frac{x^{2\alpha}}{\Gamma(2\alpha+1)}\right).$$

Solving $G(0,\alpha,2)=0$, yields

$$a_2=-\lambda\exp\left(a_0\right).$$

To determine a_3, we consider

$$\begin{aligned}G(x,\alpha,3) &= a_2+a_3\frac{x^{\alpha}}{\Gamma(\alpha+1)}\\ &\quad+\lambda\exp\left(a_0+a_1\frac{x^{\alpha}}{\Gamma(\alpha+1)}+a_2\frac{x^{2\alpha}}{\Gamma(2\alpha+1)}+a_3\frac{x^{3\alpha}}{\Gamma(3\alpha+1)}\right).\end{aligned}$$

Then, we solve $D^{\alpha}\left\{G(x,\alpha,3)\right\}\downarrow_{x=0}=0$, to get

$$a_3=-a_1\lambda\exp\left(a_0\right).$$

To determine a_4, we consider

$$\begin{aligned}G(x,\alpha,4) &= a_2+a_3\frac{x^{\alpha}}{\Gamma(\alpha+1)}+a_4\frac{x^{2\alpha}}{\Gamma(2\alpha+1)}\\ &\quad+\lambda\exp\left(a_0+a_1\frac{x^{\alpha}}{\Gamma(\alpha+1)}+a_2\frac{x^{2\alpha}}{\Gamma(2\alpha+1)}+a_3\frac{x^{3\alpha}}{\Gamma(3\alpha+1)}\right.\\ &\quad\left.+a_4\frac{x^{4\alpha}}{\Gamma(4\alpha+1)}\right).\end{aligned}$$

Then, we solve $D^{2\alpha}\left\{G(x,\alpha,4)\right\}\downarrow_{x=0}=0$, to get

$$a_4=-\lambda a_2\exp\left(u_0\right).$$

In general, to determine a_k, we consider

$$\begin{aligned} G(x,\alpha,k) &= \sum_{i=0}^{k-2} a_{i+2}\frac{x^{i\alpha}}{\Gamma(i\alpha+1)} \\ &\quad +\lambda\exp\left(a_0+a_1\frac{x^{\alpha}}{\Gamma(\alpha+1)}+\sum_{i=2}^{k} a_i\frac{x^{i\alpha}}{\Gamma(i\alpha+1)}\right). \end{aligned}$$

Then, we solve $D^{(k-2)\alpha}\left\{G(x,\alpha,k)\right\}\downarrow_{x=0}=0$, to get

$$a_k = -\lambda a_{k-2}\exp\left(u_0\right), k\geq 3.$$

Finally, the solution of equations (1.1) and (1.2), is given by

$$\begin{aligned} u(x) &= \lim_{n\longrightarrow\infty} u_n(x) \\ &= \lim_{n\longrightarrow\infty}\sum_{i=0}^{n} a_i\frac{x^{i\alpha}}{\Gamma(i\alpha+1)} \\ &= \sum_{i=0}^{\infty} a_i\frac{x^{i\alpha}}{\Gamma(i\alpha+1)}. \end{aligned}$$

The proof is complete. Q.E.D.

Theorem 3.2. If there exists a constant $0<\gamma<1$ such that

$$\left\|u_{n+1}(x)\right\|\leq\gamma\left\|u_n(x)\right\|, n\in\mathbb{N}, 0<x<R<1,$$

then the sequence of approximate solution (3.1) converges to the exact solution.

Proof. For all $0<x<R<1$, we have

$$\begin{aligned} \left\|u(x)-u_n(x)\right\| &= \left\|\sum_{k=n+1}^{\infty} u_k(x)\right\| \leq \sum_{k=n+1}^{\infty}\left\|u_k(x)\right\| \\ &\leq \sum_{k=n+1}^{\infty}\gamma\left\|u_{k-1}(x)\right\| \leq \sum_{k=n+1}^{\infty}\gamma^2\left\|u_{k-2}(x)\right\| \\ &\leq \ldots\leq\left\|u_0\right\|\sum_{k=n+1}^{\infty}\gamma^k = \frac{\gamma^{n+1}}{1-\gamma}\left\|u_0\right\|. \end{aligned}$$

Since $0<\gamma<1$ and u_0 is bounded, then

$$\lim_{n\longrightarrow\infty}\left\|u(x)-u_n(x)\right\|=0.$$

This completes the proof. Q.E.D.

4 Numerical examples

In this section, we test the validity of the proposed method to solve three numerical examples of nonlinear fractional Bratu-type equations. We define $E_{n,\alpha}$ to be the absolute error between the exact solution u and the approximate solution $u_{n,\alpha}$, as follows

$$E_{n,\alpha}(x) = |u(x) - u_{n,\alpha}(x)|, 1/2 < \alpha \leq 1, n = 0, 1, 2, 3...$$

Example 4.1. Consider the following nonlinear fractional Bratu-type equation

$$D^{2\alpha}u(x) - 2\exp(u(x)) = 0, 0 < x < 1, \tag{4.1}$$

with the initial conditions

$$u(0) = D^{\alpha}u(0) = 0, \tag{4.2}$$

where $D^{2\alpha}$ is the Liouville-Caputo fractional derivative operator of order $2\alpha, 1/2 < \alpha \leq 1$.

For $\alpha = 1$, the exact solution of equations (4.1) and (4.2), is (see. [10])

$$u(x) = -2\ln(\cos x).$$

According the description of the MTFSM presented in Section 3, we have

$$u(x) = \sum_{i=0}^{\infty} a_i \frac{x^{i\alpha}}{\Gamma(i\alpha + 1)},$$

and

$$a_i = \begin{cases} a_{2p} = 2^p, \\ a_{2p+1} = 0, \end{cases} p \in \mathbb{N}^*.$$

Therefore, the solution of equations (4.1) and (4.2), is given by

$$u(x) = \frac{2}{\Gamma(2\alpha + 1)}x^{2\alpha} + \frac{4}{\Gamma(4\alpha + 1)}x^{4\alpha} + \frac{8}{\Gamma(6\alpha + 1)}x^{6\alpha} + ...$$

which is an exactly the same result obtained by the ADM [8], HPM [9] and FRPSM [13].

x	$\alpha = 0.8$	$\alpha = 0.9$	$\alpha = 1$
0.05	9.1265×10^{-3}	2.9348×10^{-3}	5.2136×10^{-10}
0.1	2.5451×10^{-2}	8.9658×10^{-3}	3.3469×10^{-8}
0.15	4.5845×10^{-2}	1.6967×10^{-2}	3.8317×10^{-7}
0.2	6.9288×10^{-2}	2.6485×10^{-2}	2.1683×10^{-6}
0.25	9.5299×10^{-2}	3.7273×10^{-2}	8.3482×10^{-6}

TABLE 1. Comparison of the absolute errors for the exact solution and $7th-$order approximate solution for Example 4.1.

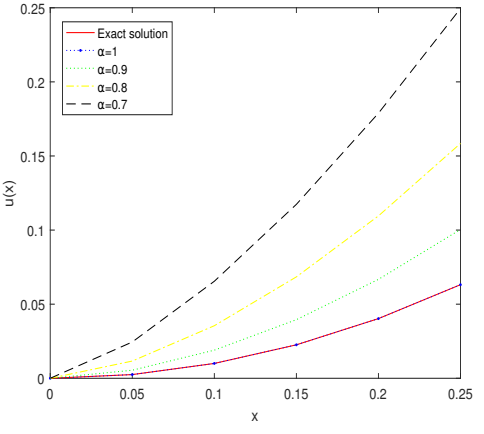


FIGURE 1. The behavior of the exact solution and approximate solution $u_7(x)$ by the MTFSM for different values of α for Example 4.1.

Example 4.2. Consider the following nonlinear fractional Bratu-type equation

$$D^{2\alpha}u(x) - \pi^2 \exp(u(x)) = 0, 0 < x < 1, \tag{4.3}$$

with the initial conditions

$$u(0) = 0, D^{\alpha}u(0) = \pi, \tag{4.4}$$

where $D^{2\alpha}$ is the Liouville-Caputo fractional derivative operator of order $2\alpha, 1/2 < \alpha \leq 1$.

For $\alpha = 1$, the exact solution of equations (4.3) and (4.4), is (see. [19])

$$u(x) = -\ln(1 - \sin \pi x).$$

According the description of the MTFSM presented in Section 3, we have

$$u(x) = \sum_{i=0}^{\infty} a_i \frac{x^{i\alpha}}{\Gamma(i\alpha + 1)},$$

and

$$a_i = \pi^i, i = 2, 3, 4...$$

Therefore, the solution of equations (4.3) and (4.4), is given by

$$u(x) = \frac{\pi}{\Gamma(\alpha + 1)}x^{\alpha} + \frac{\pi^2}{\Gamma(2\alpha + 1)}x^{2\alpha} + \frac{\pi^3}{\Gamma(3\alpha + 1)}x^{3\alpha} + ...$$

which is an exactly the same result obtained by the HPM [9] and FRPSM [13].

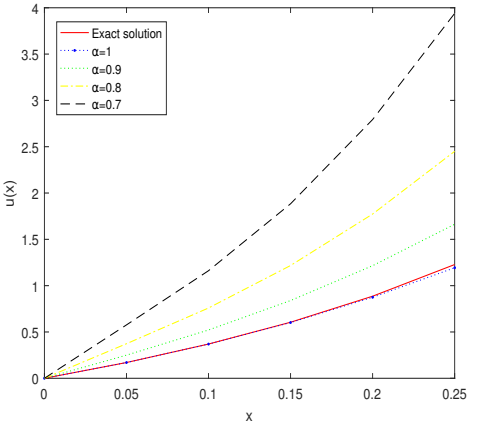


FIGURE 2. The behavior of the exact solution and approximate solution $u_7(x)$ by the MTFSM for different values of α for Example 4.2.

x	$\alpha = 0.8$	$\alpha = 0.9$	$\alpha = 1$
0.05	2.0279×10^{-1}	7.9440×10^{-2}	2.8838×10^{-5}
0.1	3.8883×10^{-1}	1.5164×10^{-1}	5.3118×10^{-4}
0.15	6.1177×10^{-1}	2.3409×10^{-1}	3.1352×10^{-3}
0.2	8.8604×10^{-1}	3.2918×10^{-1}	1.1735×10^{-2}
0.25	12.209×10^{-1}	4.3443×10^{-1}	3.4620×10^{-2}

TABLE 2. Comparison of the absolute errors for the exact solution and $7th-$order approximate solution for Example 4.2.

Example 4.3. Consider the following nnlinear fractional Bratu-type equation

$$D^{2\alpha}u(x) + \pi^2 \exp(-u(x)) = 0, 0 < x < 1, \tag{4.5}$$

with the initial conditions

$$u(0) = 0, D^{\alpha}u(0) = \pi, \tag{4.6}$$

where $D^{2\alpha}$ is the Liouville-Caputo fractional derivative operator of order $2\alpha, 1/2 < \alpha \leq 1$.

For $\alpha = 1$, the exact solution of equations (4.5) and (4.6), is (see. [19])

$$u(x) = \ln(1 + \sin \pi x).$$

According the description of the MGTFSM presented in Section 3, we have

$$u(x) = \sum_{i=0}^{\infty} a_i \frac{x^{i\alpha}}{\Gamma(i\alpha + 1)},$$

and

$$a_i = (-1)^{i+1}\pi^i, i = 2, 3, 4...$$

Therefore, the solution of equations (4.5) and (4.6), is given by

$$u(x) = \frac{\pi}{\Gamma(\alpha+1)}x^{\alpha} - \frac{\pi^2}{\Gamma(2\alpha+1)}x^{2\alpha} + \frac{\pi^3}{\Gamma(3\alpha+1)}x^{3\alpha} - ...$$

which is an exactly the same result obtained by the HPM [9] and FRPSM [13].

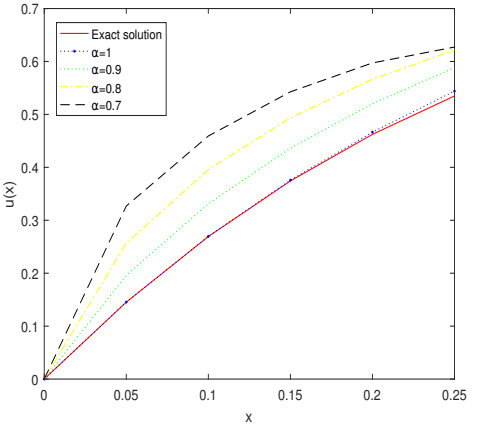


FIGURE 3. The behavior of the exact solution and approximate solution $u_7(x)$ by the MTFSM for different values of α for Example 4.3.

x	$\alpha = 0.8$	$\alpha = 0.9$	$\alpha = 1$
0.05	1.1151×10^{-1}	5.0355×10^{-2}	2.2422×10^{-5}
0.1	1.2645×10^{-1}	6.1802×10^{-2}	3.2017×10^{-4}
0.15	1.1912×10^{-1}	6.1887×10^{-2}	1.4560×10^{-3}
0.2	1.0392×10^{-1}	5.7854×10^{-2}	4.1571×10^{-3}
0.25	8.6636×10^{-2}	5.3482×10^{-2}	9.2121×10^{-3}

TABLE 3. Comparison of the absolute errors for the exact solution and $7th-$order approximate solution for Example 4.1.

5 Numerical results and discussion

In this section the numerical results for Examples 4.1–4.3 are presented. Figures 1-3 represents the behavior of the exact solution and the approximate solution $u_7(x)$ at $\alpha = 0.7, 0.8, 0.9, 1$. Tables 1–3 show the absolute errors between the exact solution and the approximate solution $u_7(x)$ at different

values of x and α. The numerical results afirm that when α approaches 1, our obtained results by MTFSM approach the exact solutions.

6 Conclusion

In this article, we successfully applied a new reliable method called modified Taylor fractional series method (MTFSM) to find an approximate solution of the nonlinear fractional Bratu-type equation. This method was applied directly without resorting to linearization or any restrictive assumptions. The results show that the MTFSM is an very powerful and easy to use technique to find an approximate solution for this problem. Moreover, the numerical results presented in the tables and graphs show that the proposed method approximate to the exact solution very well. So, we can conclude that the MTFSM is practically well suited to solve of such types of nonlinear fractional problems.

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