Generalized multivalued integral type contraction on weak partial metric space

Özlem Acar[1,*] and Sümeyye Coşkun[2]

[1,2]Department of Mathematics, Faculty of Science, Selçuk University, 42003, Konya, Turkey
Orchid number: 0000-0001-6052-4357[1], 0000-0001-7218-8821[2]
*Corresponding author
E-mail: acarozlem@ymail.com[1], sumeyyecskn07@gmail.com[2]

Abstract

Combining the new concept of $F-$contraction and Hausdorff metric, we give some generalized results for integral type mapping on weak partial metric space.

2010 Mathematics Subject Classification. **54H25**. 47H10.
Keywords. weak partial metric space, partial Hausdorff metric, integral type mapping.

1 Introduction

Fixed point theory for multivalued map is started by Nadler [4]. In 1969, Nadler gave multivalued version of Banach Contraction Principle and proved that every multivalued contraction on complete metric space has a fixed point. Then, lots of generalizations of the result of Nadler were given (see [2, 3, 5, 6]). On the other hand, an interesting generalization of Banach Contraction Principle was given by Wardowski [7] introduced the new concept which is F-contraction. Let $\mathcal{F}$ be the set of all functions $F : (0,\infty) \to \mathbb{R}$ satisfying the following conditions:

(F1) F is strictly increasing, i.e., for all $\alpha, \beta \in (0,\infty)$ such that $\alpha < \beta, F(\alpha) < F(\beta)$,

(F2) For each sequence $\{\alpha_n\}$ of positive numbers $\lim_{n\to\infty} a_n = 0$ if and only if $\lim_{n\to\infty} F(a_n) = -\infty$,

(F3) There exists $k \in (0,1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

Definition 1.1 ([7])**.** Let (∇, d) be a metric space and $T : \nabla \to \nabla$ be a mapping. Given $F \in \mathcal{F}$, we say that T is F-contraction, if there exists $\tau > 0$ such that

$$m_1, m_2 \in \nabla,\ d(Tm_1, Tm_2) > 0 \Rightarrow \tau + F(d(Tm_1, Tm_2)) \leq F(d(m_1, m_2)). \tag{1.1}$$

Theorem 1.2 ([7])**.** Let (∇, d) be a complete metric space and let $T : \nabla \to \nabla$ be an F-contraction. Then T has a unique fixed point in ∇.

Using F-contraction, many researchers made generalizations of some important theorems known in the literature on metric or partial metric space for single or multivalued mappings. In this paper, we give some results for integral type multivalued mapping on weak partial metric space.

Tbilisi Mathematical Journal 13(3) (2020), pp. 145–151.
Tbilisi Centre for Mathematical Sciences.
Received by the editors: 18 February 2020.
Accepted for publication: 07 August 2020.

2 Preliminaries

Recently, Beg and Pathak [1] defined a weaker form of partial metric called weak partial metric space as follows:

Definition 2.1 ([1])**.** A weak partial metric on a nonempty set ϑ is a function $\omega : \vartheta \times \vartheta \to \mathbb{R}^+$ such that for all $m_1, m_2, r \in \vartheta$:

(ω_1) $m_1 = m_2 \Longleftrightarrow \omega(m_1, m_1) = \omega(m_1, m_2)$

(ω_2) $\omega(m_1, m_1) \leq \omega(m_1, m_2)$,

(ω_3) $\omega(m_1, m_2) = \omega(m_2, m_1)$,

(ω_4) $\omega(m_1, m_2) \leq \omega(m_1, r) + \omega(r, m_2)$.

Example 2.2 ([1])**.** $(\mathbb{R}^+, \omega)$, where $\omega : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is defined as

$$\omega(m_1, m_2) = |m_1 - m_2| + 1$$

for all $m_1, m_2 \in \mathbb{R}^+$.For another example, $\omega : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is defined as

$$\omega(m_1, m_2) = \frac{1}{4} |m_1 - m_2| + \max \{m_1, m_2\}.$$

Each weak partial metric ω on ϑ generates a T_0 topology τ_ω on ϑ which has as a base the family open p-balls

$$\{B_\omega(m_1, \varepsilon) : m_1 \in \vartheta, \varepsilon > 0\},$$

where

$$B_\omega(m_1, \varepsilon) = \{m_2 \in \vartheta : \omega(m_1, m_2) < \omega(m_1, m_1) + \varepsilon\}$$

for all $m_1 \in \vartheta$ and $\varepsilon > 0$.([1])

If ω is a weak partial metric on ϑ, then the functions $\omega^s : \vartheta \times \vartheta \to \mathbb{R}^+$ given by

$$\omega^s(m_1, m_2) = \omega(m_1, m_2) - \frac{1}{2} [\omega(m_1, m_1) + \omega(m_2, m_2)]$$

is a metric on ϑ.([1])

Definition 2.3 ([1])**.** A sequence $\{r_n\}$ in a WPMS (ϑ, ω) converges, with respect to τ_ω if and only if

$$\lim_{n \to \infty} \omega(r, r_n) = \omega(r, r).$$

Furthermore, a sequence $\{r_n\}$converges in (ϑ, ω^s) to a point $r \in \vartheta$ if and only if

$$\lim_{m,n \to \infty} \omega(r_n, r_m) = \lim_{n \to \infty} \omega(r, r_n) = \omega(r, r).$$

Let (ϑ, ω) be a WPMS and $CB^{\omega}(\vartheta)$ be the family of all nonempty, closed and bounded subsets of (ϑ, ω). Y is a bounded subset in (ϑ, ω) if there exists $x_0 \in \vartheta$ and $M > 0$ such that for all $y \in Y$, we have $y \in B_{\omega}(x_0, \vartheta)$, that is

$$\omega(x_0, y) < \omega(y, y) + M.$$

For $Y, Z \in CB^{\omega}(\vartheta)$ and $r \in \vartheta$, we define

$$\omega(r, Y) = \inf \{\omega(r, y) : y \in Y\},$$

$$\delta_{\omega}(Y, Z) = \sup \{\omega(y, Z) : y \in Y\}$$

and

$$\delta_{\omega}(B, Z) = \sup \{\omega(z, Z) : z \in Z\}.$$

Clearly, $\omega(r, Y) = 0 \Longrightarrow \omega^s(r, Y) = 0$.([1])

Remark 2.4 ([1])**.** Let (ϑ, ω) be a WPMS and A any nonempty set in (ϑ, ω), then

$$a \in \overline{A} \Longleftrightarrow \omega(a, A) = \omega(a, a)$$

where $\overline{A}$ denotes the closure of A with respect to weak partial metric ω.

Proposition 2.5 ([1])**.** Let (ϑ, ω) be a WPMS. For any $X, Y, Z \in CB^{\omega}(\vartheta)$, we have the following:

$$\begin{aligned} \delta_{\omega}(X, X) &= \sup \{\omega(a, a) : a \in X\}, \\ \delta_{\omega}(X, X) &\leq \delta_{\omega}(X, Y), \\ \delta_{\omega}(X, Y) &= 0 \Rightarrow X \subset Y, \\ \delta_{\omega}(X, Y) &\leq \delta_{\omega}(X, Z) + \delta_{\omega}(Z, Y). \end{aligned}$$

Definition 2.6 ([1])**.** Let (ϑ, ω) be a WPMS. For $X, Y \in CB^{\omega}(\vartheta)$ define

$$H_{\omega}^{+}(X, Y) = \frac{1}{2} \left[\delta_{\omega}(X, Y) + \delta_{\omega}(Y, X)\right].$$

The mapping $H_{\omega}^{+} : CB^{\omega}(\vartheta) \times CB^{\omega}(\vartheta) \to [0, \infty)$ is called $H_{\omega}^{+}-$type Hausdorff metric induced by ω.

Lemma 2.7 ([1])**.** Let $X, Y \in CB^{\omega}(\vartheta)$, then for any $x \in X$

$$\omega(x, Y) \leq H_{\omega}^{+}(X, Y).$$

3 Main result

Definition 3.1. Let (ϑ, ω) be a WPMS and $f : \vartheta \to K(\vartheta)$ be a mapping. Given $F \in \mathcal{F}$ we say that f is a generalized multivalued integral type F-contraction if there exists $\tau > 0$ such that

$$H_{\omega}^{+}(fm_1, fm_2) > 0 \Longrightarrow \tau + F\left(\int_0^{H_{\omega}^{+}(fm_1, fm_2)} \varkappa(s)ds\right) \leq F\left(\int_0^{M(m_1, m_2)} \varkappa(s)ds\right)$$

for $m_1, m_2 \in \vartheta$ where

$$M(m_1,m_2)=\max\left\{\begin{array}{c}\omega(m_1,m_2),\omega(m_1,fm_1),\omega(m_2,fm_2),\\ \frac{\omega(m_1,fm_1)\omega(m_2,fm_2)}{1+\omega(m_1,m_2)},\frac{\omega(m_1,fm_1)\omega(m_2,fm_2)}{1+H_\omega^+(fm_1,fm_2)}\end{array}\right\} \tag{3.1}$$

and $\varkappa:\mathbb{R}^+\to\mathbb{R}^+$ is Lebesque integrable mapping which is summable on each compact subset of $\mathbb{R}^+$,nonnegative and

$$\int_0^{\varepsilon}\varkappa(s)ds>0$$

for each $\varepsilon>0$.

Theorem 3.2. Let (ϑ,ω) be a complete WPMS and $f:\vartheta\to K(\vartheta)$ be a generalized multivalued integral type F-contraction. If f or F is continuous, then f has a fixed point.

Proof. Let $y_0\in\vartheta$ be an arbitrary point and define a sequence $\{y_n\}$ by $y_{n+1}\in fy_n$ for $n=1,2,\cdots$. As fy is nonempty for all $y\in\vartheta$, we can choose $y_1\in fy_0$. If $y_1\in fy_1$, then y_1 is a fixed point of f.Let $y_1\notin fy_1$, then $\omega(y_1,fy_1)>0$ since fy_1 is compact. By using (F_1), from Lemma 2.7 and (3.1), we can write

$$\begin{aligned}
& F\left(\int_0^{\omega(y_1,fy_1)}\varkappa(s)ds\right)\\
\leq\ & F\left(\int_0^{H_\omega^+(fy_0,fy_1)}\varkappa(s)ds\right)\\
\leq\ & F\left(\int_0^{M(y_0,y_1)}\varkappa(s)ds\right)-\tau\\
=\ & F\left(\int_0^{\max\left\{\omega(y_0,y_1),\omega(y_0,fy_0),\omega(y_1,fy_1),\frac{\omega(y_0,fy_0)\omega(y_1,fy_1)}{1+\omega(y_0,y_1)},\frac{\omega(y_0,fy_0)\omega(y_1,fy_1)}{1+H_\omega^+(fy_0,fy_1)}\right\}}\varkappa(s)ds\right)-\tau\\
\leq\ & F\left(\int_0^{\max\{\omega(y_0,y_1),\omega(y_1,fy_1)\}}\varkappa(s)ds\right)-\tau\\
=\ & F\left(\int_0^{\omega(y_0,y_1)}\varkappa(s)ds\right)-\tau
\end{aligned}\tag{3.2}$$

Also, since fy_1 is compact, we obtain that $y_2\in fy_1$ such that

$$\omega(y_1,y_2)=\omega(y_1,fy_1).$$

From (3.2), we get

$$F\left(\int_0^{\omega(y_1,y_2)}\varkappa(s)ds\right)\leq F\left(\int_0^{H_\omega^+(fy_0,fy_1)}\varkappa(s)ds\right)\leq F\left(\int_0^{\omega(y_0,y_1)}\varkappa(s)ds\right)-\tau.$$

If we continue recursively, we have

$$\begin{aligned} F\left(\int_0^{\omega(y_2,y_3)} \varkappa(s)ds\right) &\leq F\left(\int_0^{\omega(y_1,y_2)} \varkappa(s)ds\right) - \tau \\ &\leq F\left(\int_0^{\omega(y_0,y_1)} \varkappa(s)ds\right) - 2\tau \\ &\vdots \\ F\left(\int_0^{\omega(y_n,y_{n+1})} \varkappa(s)ds\right) &\leq F\left(\int_0^{\omega(y_{n-1},y_n)} \varkappa(s)ds\right) - \tau \\ &\leq F\left(\int_0^{\omega(y_0,y_1)} \varkappa(s)ds\right) - n\tau. \end{aligned} \tag{3.3}$$

So, we get a sequence $\{y_n\}$ in X as $y_{n+1} \in fy_n$ for all $n \in \mathbb{N}$. If there exists $n_0 \in \mathbb{N}$ with $y_{n_0} \in fy_{n_0}$,then y_{n_0} is a fixed point of f and then the proof is finished. Let's say that for every $n \in \mathbb{N}$, $y_n \notin fy_n$. Denote

$$a_n = \int_0^{\omega(y_n,y_{n+1})} \varkappa(s)ds, \text{for } n = 0, 1, 2, \cdots.$$

Then, $a_n > 0$ for all n. Using (3.3) we get

$$F(a_n) \leq F(a_{n-1}) - \tau \leq F(a_{n-2}) - 2\tau \leq \cdots \leq F(a_0) - n\tau. \tag{3.4}$$

From (3.4), we have $\lim_{n\to\infty} F(a_n) = -\infty$. Thus, from (F2)

$$\lim_{n\to\infty} a_n = 0.$$

From (F3), there exists $\kappa \in (0, 1)$ such that

$$\lim_{n\to\infty} a_n^{\kappa} F(a_n) = 0.$$

By (3.4), the following holds for all $n \in \mathbb{N}$

$$a_n^{\kappa} F(a_n) - a_n^{\kappa} F(a_0) \leq -a_n^{\kappa} n\tau \leq 0. \tag{3.5}$$

Letting $n \to \infty$ in (3.5), we obtain that

$$\lim_{n\to\infty} n a_n^{\kappa} = 0. \tag{3.6}$$

From (3.6), there exists $n_1 \in \mathbb{N}$ such that $na_n^{\kappa} \leq 1$ for all $n \geq n_1$. So we have

$$a_n \leq \frac{1}{n^{1/\kappa}}. \tag{3.7}$$

for all $n \geq n_1$. Now, consider $m, n \in \mathbb{N}$ such that $m > n \geq n_1$ and show that $\{y_n\}$ is a Cauchy sequence. From (3.7)

$$\begin{aligned}\int_0^{\omega(y_n,y_m)} \varkappa(s)ds &\leq \int_0^{\omega(y_n,y_{n+1})} \varkappa(s)ds + \int_0^{\omega(y_{n+1},y_{n+2})} \varkappa(s)ds + \cdots + \int_0^{\omega(y_{m-1},y_m)} \varkappa(s)ds \\ &= a_n + a_{n+1} + \cdots + a_{m-1} \\ &= \sum_{i=n}^{m-1} a_i \\ &\leq \sum_{i=n}^{\infty} a_i \\ &\leq \sum_{i=n}^{\infty} \frac{1}{i^{1/\kappa}}.\end{aligned}$$

is hold. By the convergence of the series $\sum_{i=1}^{\infty} \frac{1}{i^{1/\kappa}}$, we get $\int_0^{\omega(y_n,y_m)} \varkappa(s)ds \to 0$ as $n \to \infty$. This yields that $\{y_n\}$ is a Cauchy sequence in (ϑ, ω). Since (ϑ, ω) is a complete weak partial metric space, then (ϑ, ω^s) is complete metric space. So there exists $z \in \vartheta$ such that $\lim_{n\to\infty} \omega^s(z, y_n) = 0$. Moreover $\lim_{m,n\to\infty} \omega(x_n, x_m) = \lim_{n\to\infty} \omega(x, x_n) = \omega(x, x) = 0$.

If f is compact, we have $fy_n \to fz$ and

$$\omega(y_n, fz) \leq H_\omega^+(fy_n, fz),$$

so $D(z, fz) = 0$ and $z \in fz$.

If F is continuous, now we claim that $z \in fz$. Assume the contrary, that is, $z \notin fz$. So, there exist an $n_0 \in \mathbb{N}$ and a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $\omega(y_{n_k+1}, fz) > 0$ for all $n_k \geq n_0$. (Otherwise, there exists $n_1 \in \mathbb{N}$ such that $y_n \in fz$ for all $n \geq n_1$, which implies that $z \in fz$. This is a contradiction, since $z \notin fz$). Since $\omega(y_{n_k+1}, fz) > 0$ for all $n_k \geq n_0$, then

$$\begin{aligned}\tau + F\left(\int_0^{\omega(y_{n_k+1},fz)} \varkappa(s)ds\right) &\leq \tau + F\left(\int_0^{H_\omega^+(fy_{n_k},fz)} \varkappa(s)ds\right) \\ &\leq \tau + F\left(\int_0^{M(y_{n_k},z)} \varkappa(s)ds\right) \qquad (3.8) \\ &\leq \tau + F\left(\int_0^{\max\left\{\begin{array}{c}\omega(y_{n_k},z),\omega(y_{n_k},fy_{n_k}),\omega(z,fz) \\ \frac{\omega(y_{n_k},fy_{n_k})\omega(z,fz)}{1+\omega(y_{n_k},z)}, \frac{\omega(y_{n_k},fy_{n_k})\omega(z,fz)}{1+H_\omega^+(fy_{n_k},fz)}\end{array}\right\}} \varkappa(s)ds\right).\end{aligned}$$

Taking the limit $k \to \infty$ from (3.8) and using the continuity of F, we have $\tau + F(\omega(z, fz)) \leq F(\omega(z, fz))$, which is a contradiction. Thus, we get $z \in \overline{fz} = fz$. This completes the proof. ∎

Corollary 3.3. Let (ϑ, ω) be a complete weak partial metric space and $f : \vartheta \to K(\vartheta)$ be a rational

type F-contraction of multivalued integral type mapping, i.e. there exists $\tau > 0$ such that

$$H^+_\omega(fm, fn) > 0 \Longrightarrow \tau + F\left(\int_0^{H^+_\omega(fm,fn)} \varkappa(s)ds\right) \leq F\left(\int_0^{\omega(m,n)} \varkappa(s)ds\right)$$

for $m, n \in \vartheta$ where $\varkappa : \mathbb{R}^+ \to \mathbb{R}^+$ is Lebesque integrable mapping which is summable on each compact subset of $\mathbb{R}^+$,nonnegative and

$$\int_0^\varepsilon \varkappa(s)ds > 0$$

for each $\varepsilon > 0$. If f or F is continuous, then f has a fixed point.

Corollary 3.4. Let (ϑ, ω) be a complete weak partial metric space and $f : \vartheta \to K(\vartheta)$ be a rational type contraction of multivalued integral type mapping, i.e. there exists $\tau > 0$ such that

$$\int_0^{H^+_\omega(fm,fn)} \varkappa(s)ds \leq \int_0^{\omega(m,n)} \varkappa(s)ds$$

for $m, n \in \vartheta$ where $\varkappa : \mathbb{R}^+ \to \mathbb{R}^+$ is Lebesque integrable mapping which is summable on each compact subset of $\mathbb{R}^+$,nonnegative and

$$\int_0^\varepsilon \varkappa(s)ds > 0$$

for each $\varepsilon > 0$. If f or F is continuous, then f has a fixed point.

References

[1] I. Beg, H. K. Pathak, *A variant of Nadler's theorem on weak partial metric spaces with application to homotopy result*, Vietnam J. Math., 46 (2018), 693-706.

[2] Y. Feng and S. Liu, *Fixed point theorems for multi-valued increasing operators in partial ordered spaces*, Soochow Journal of Mathematics, vol.30, no.4, pp.461–469,2004.

[3] N. Mizoguchi and W. Takahashi, *Fixed point theorems for multivalued mappings on complete metric spaces*, Journal of Mathematical Analysis and Applications, vol.141, no.1, pp.177–188, 1989.

[4] S. B. Nadler, *Multivalued contraction mappings*, Pac. J. Math., 30(1969), 475-488.

[5] S. Reich, *Some problems and results in fixed point theory*, Contemporary Mathematics, vol.21, pp.179–187, 1983.

[6] T. Suzuki, *Mizoguchi-Takahashi's fixed point theorem is areal generalization of Nadler's*, Journal of Mathematical Analysis and Applications, vol.340, no.1, pp.752–755, 2008.

[7] D. Wardowski, *Fixed points of a new type of contractive mappings in complete metric spaces*, Fixed Point Theory Appl. 2012, 2012:94, 6 pp.