Classes of non-commutative semigroups with finite Fibonacci invariants

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Abstract

For a 2-generated semigroup S we define the invariant $\lambda(S)$ as the minimum of the *Fibonacci lengths* over all generating pairs of the semigroup S, where the Fibonacci length with respect to a generating pair (x, y) is the fundamental period (if exist) of the truncated periodic sequence $z_0 = x, z_1 = y, z_k = z_{k-2}z_{k-1}, (k \geq 2)$ of the elements of S. We name this invariant as the *Fibonacci invariant* of S. Our used notation is the same as of the celebrated work of D.L. Johnson in 2005 on infinite groups. In this paper we examine two classes of semigroups for existence of this invariant. The considered semigroups are the finite semigroup $S = \langle a, b \mid a^{p^\alpha} = a, b^{q^\beta} = b, ab = a\rangle$ of order $p^\alpha q^\beta - 1$ and the infinite semigroup $T = \langle a, b \mid a^{p^\alpha} = a, ab = a\rangle$, for all integers $\alpha, \beta \geq 2$ and all distinct primes p and q. We prove the existence of the Fibonacci invariants of these semigroups, for all parameters. As a numerical result we show that $\lambda(T) \leq \lambda(S) = p^{\alpha+1}$, if p is even.

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1 Introduction

Any ordered pair (x, y) of the elements of an associative algebraic structure $(A, .)$ determines the sequence $z_0 = x, z_1 = y, z_k = z_{k-2}z_{k-1}, (k \geq 2)$ of the elements of A which may be periodic. During the years many efforts have been done in determining the period of this sequence. We list a brief of some known results. The cyclic case for the groups covered by [17] and the sequences of elements of abelian groups studied for their periodicity in [15, 18]. Since then, the *Fibonacci length* of a 2-generated group $G = \langle a, b\rangle$ with respect to the generating set $A = \{a, b\}$, written as $LEN_A(G)$, is defined in [3] as the period of the Fibonacci sequence

$$z_0 = x, z_1 = y, z_k = z_{k-2}z_{k-1}, (k \geq 2)$$

of the elements of a group G. Many interesting contributions illustrated the *Fibonacci length* by examining remarkable 2-generated groups since 1990, one may consult [2, 4, 5, 9, 10, 11]. Also, for the generalization of the subject we may refer to the valuable articles [6, 7, 8], where the groups are mostly k-generated, $(k \geq 3)$. The competent investigation of the subject in [13] motivated us to study the group invariant $\lambda(G)$, for the 2-generated non-commutative finite semigroups. For a 2-generated group $G = \langle a, b\rangle$, $\lambda(G)$ is the minimum of the set $\{LEN_A(G) \mid A = \{x, y\}, \quad G = \langle A\rangle\}$. Reminding the results of [14], this invariant is the least appropriate integer n such that the group G is the homomorphic image of the Fibonacci group $F(2, n)$. As for a 2-generated non-commutative finite semigroup, the Fibonacci sequences are indeed truncated periodic (the sequences which are

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periodic by deleting a finite number of elements from the beginning of the sequence), then for a generating pair $A = \{x, y\}$ of a semigroup S, $LEN_A(S)$ will be used for period of the truncated Fibonacci sequence with the initial values $z_0 = x$ and $z_1 = y$.

The aim of this paper is to study the behavior of Fibonacci sequences of the finitely presented semigroups $S = Sg(\pi_1)$ and $T = Sg(\pi_2)$, where

$$\pi_1 = \langle a, b \mid a^{p^\alpha} = a, b^{q^\beta} = b, ab = a \rangle,$$

$$\pi_2 = \langle a, b \mid a^{p^\alpha} = a, ab = a \rangle,$$

for all integers $\alpha, \beta \geq 2$ and different primes p and q. The semigroup S studied for its finiteness and regularity in [1] and is of order $p^\alpha q^\beta - 1$ however, T is infinite. The results on $\lambda(S)$ will be given in Section 3 and the Section 4 is devoted to study of $\lambda(T)$ and its comparison with $\lambda(S)$.

Our notation is standard, we follow [12] for the preliminaries in semigroup theory, and one may consult [16] for the preliminaries on the semigroups presentation. Throughout the paper we use the notation φ to denote the well known Eulerian function and f_n will be used to denote the n^{th} Fibonacci number defined by the recurrence relation $f_n = f_{n-1} + f_{n-2}$ with the initial values $f_1 = f_2 = 1$.

2 Preliminary results

The relator $ab = a$ of the semigroup S gives us the following partition:

$$S = \{b^i\} \cup \{a^j\} \cup \{b^i a^j\}, \quad 1 \leq i \leq q^\beta - 1, \quad 1 \leq j \leq p^\alpha - 1.$$

Then,

Lemma 2.1. For every values of the integers $\alpha, \beta \geq 2$ and the different primes p and q, S possesses $2\varphi(p^\alpha - 1)\varphi(q^\beta - 1)$ number of generating pairs.

Proof. For every values of i and j if $i < q^\beta$ and $j < p^\alpha$ then the partition of S together with the relators $a^{p^\alpha} = a$ and $b^{q^\beta} = b$ shows that the pair (a^j, b^i) is a generating set for S if and only if $g.c.d.(i, q^\beta - 1) = 1$ and $g.c.d.(j, p^\alpha - 1) = 1$. This yields $\varphi(p^\alpha - 1)\varphi(q^\beta - 1)$ number of generating pairs of the form (a^j, b^i) for S. Obviously, the same argument gives us the number of generating pairs of the form (b^i, a^j). So, there are all together $2\varphi(p^\alpha - 1)\varphi(q^\beta - 1)$ number of generating pairs for S. Q.E.D.

Before identifying the elements of the Fibonacci sequence

$$z_0 = x, z_1 = y, z_k = z_{k-2} z_{k-1}, (k \geq 2)$$

note that, for the pairs $(x, y) = (b^i, a^j)$ and $(x, y) = (a^j, b^i)$, the truncated periods, if exist then are equal, i.e.; $LEN_{\{b^i, a^j\}} = LEN_{\{a^j, b^i\}}$, for every i and j when $g.c.d.(i, q^\beta - 1) = 1$ and $g.c.d.(j, p^\alpha - 1) = 1$. Hence, to study the periodicity of the Fibonacci sequences of S it is sufficient to consider the generating pairs (b, a^j) and (a^j, b).

Lemma 2.2. For every values of the integers $\alpha, \beta \geq 2$ and the different primes p and q, every element z_k, $(k \geq 2)$ of the Fibonacci sequence $z_k = z_{k-2}z_{k-1}$, $(k \geq 2)$, with the initial values $z_0 = b, z_1 = a^j$, may be presented as:

$$z_k = \begin{cases} ba^{jf_k}, & k \text{ is even}, \\ a^{jf_k}, & k \text{ is odd}, \end{cases}$$

for every j satisfies $g.c.d.(j, p^\alpha - 1) = 1$. Moreover, the sequence

$$y_0 = a^j, y_1 = b,\ y_k = y_{k-2}y_{k-1}, \quad (k \geq 2)$$

satisfies $y_k = z_{k-1}$, for every $k \geq 1$.

Proof. For a given integer j where $g.c.d.(j, p^\alpha - 1) = 1$, the relator $ab = a$ shows that a^j is a central element of S. We use an induction method on k. Obviously, $z_2 = z_0z_1 = ba^j = ba^{jf_2}$ and $z_3 = z_1z_2 = a^j.ba^{jf_2} = a^{jf_1}.a^{jf_2} = a^{j(f_2+f_1)} = a^{jf_3}$. Then, by induction hypothesis for z_k and z_{k+1} and considering the relator $ab = a$ we get:

$$z_{k+2} = z_k.z_{k+1} = \begin{cases} ba^{jf_k}.a^{jf_{k+1}} = ba^{jf_k}.a^{jf_{k+1}} = ba^{jf_{k+2}}, & k \text{ is even}, \\ a^{jf_k}.ba^{jf_{k+1}} = a^{jf_k}.a^{jf_{k+1}} = a^{jf_{k+2}}, & k \text{ is odd}. \end{cases}$$

To complete the proof we may check the relator $y_k = z_{k-1}$ by using an induction on k. Q.E.D.

Lemma 2.3. For every values of the integers $\alpha, \beta \geq 2$ and the different primes p and q, if the Fibonacci sequences

$$z_0 = b,\ z_1 = a^j, \quad z_k = z_{k-2}z_{k-1},$$
$$y_0 = a^j,\ y_1 = b, \quad y_k = y_{k-2}y_{k-1},$$

are truncated periodic of periods ℓ and ℓ', respectively, then $\ell = \ell'$.

Proof. By using Lemma 2.2 on the presenting of the sequences we get:

$$y_{\ell'+2} = y_2 = z_1 = z_{\ell+1},$$
$$y_{\ell'+3} = y_3 = z_2 = z_{\ell+2}.$$

Hence, the equations hold if and only if $\ell = \ell'$, as required. Q.E.D.

We need a numerical property of the Fibonacci numbers in our computation.

Lemma 2.4. For every positive integers k, k' and t where $k' \geq 3$, the equations $f_{k+1} \equiv 1(mod\ t)$ and $f_{k+2} \equiv 1(mod\ t)$ yield $f_{k+k'} \equiv f_{k'}(mod\ t)$.

Proof. Let k be a fixed positive integer. We use an induction method on k'. For $k' = 3$ the equations $f_{k+1} \equiv 1(mod\ t)$ and $f_{k+2} \equiv 1(mod\ t)$ give us the equation $f_{k+3} = f_{k+1} + f_{k+2} \equiv 2(mod\ t)$. Then $f_{k+3} \equiv f_3(mod\ t)$. In a similar way we get $f_{k+4} \equiv f_4(mod\ t)$. By the induction hypothesis suppose that

$$\begin{cases} f_{k+k'} & \equiv f_{k'}(mod\ t), \\ f_{k+(k'-1)} & \equiv f_{k'-1}(mod\ t). \end{cases}$$

To prove the assertion for $k' + 1$ we use the well-known identity $f_{i+j} = f_if_{j+1} + f_{i-1}f_j$ of the Fibonacci sequence and get:
$f_{k+(k'+1)} = f_{k+k'}f_2 + f_{k+(k'-1)}f_1 = f_{k+k'} + f_{k+(k'-1)} \equiv f_{k'} + f_{k'-1} = f_{k'+1}\ (mod\ t)$.
This completes the proof. Q.E.D.

3 The invariant $\lambda(S)$

Using the results of the last section we attempt to calculate the Fibonacci invariant. Our first result is:

Proposition 3.1. For every integer j where $g.c.d.(j, p^\alpha - 1) = 1$, the Fibonacci sequence $z_0 = b, z_1 = a^j,\ z_k = z_{k-2}z_{k-1},\ \ (k \geq 2)$ is truncated periodic with the period $\ell = LEN_{\{b,a^j\}}(S)$ if ℓ satisfies both of the numerical equations:

$$f_{\ell+1} \equiv\ 1(\mathrm{mod}(\ p^\alpha - 1)),$$
$$f_{\ell+2} \equiv\ 1(\mathrm{mod}(\ p^\alpha - 1)).$$

Moreover, $\lambda(S) = \ell$.

Proof. We show that the sequence is truncated periodic with only one omitted term from the beginning. Since S is finite then the sequence is periodic and using the results of Lemma 2.2 $\ell = LEN_{\{b,a^j\}}$ is the least positive integer satisfies the relators:

$$z_{\ell+1} = z_1 = a^j,$$
$$z_{\ell+2} = z_2 = ba^j.$$

Equivalently, ℓ satisfies the relators $a^{jf_{\ell+1}} = a^j$ and $ba^{jf_{\ell+2}} = ba^j$. Consequently, ℓ satisfies the numerical equations $jf_{\ell+1} \equiv j(\mathrm{mod}\ (\ p^\alpha - 1))$ and $jf_{\ell+2} \equiv j(\mathrm{mod}\ (\ p^\alpha - 1))$, because of the relator $a^{p^\alpha} = a$. Since $g.c.d.(j, p^\alpha - 1) = 1$ then ℓ satisfies the desired equations. To complete the proof we see that ℓ is independent of choosing the generating pairs (b, a^j). Hence, $\lambda(S) = \ell$. Q.E.D.

We are now able to estimate the Fibonacci invariant $\lambda(S)$ by considering two cases for the prime p as $p = 2$ and $p > 2$. For the case $p = 2$ we give an explicit value for $\lambda(S)$.

Proposition 3.2. For every integers $\alpha \geq 2$, $\beta \geq 1$ and every odd prime q, $\lambda(S) = p^{\alpha+1}$ if $p = 2$.

Proof. As a result of the Proposition 3.1 we have to prove the equations:

$$f_{2^{\alpha+1}+1} \equiv 1(\mathrm{mod}\ (\ 2^\alpha - 1)),$$
$$f_{2^{\alpha+1}+2} \equiv 1(\mathrm{mod}\ (\ 2^\alpha - 1)),$$

for every values of $\alpha \geq 2$. We use an induction method on α. For $\alpha = 2$ we have:

$$f_9 = 34 \equiv 1(\mathrm{mod}\ \ 3),$$
$$f_{10} = 55 \equiv 1(\mathrm{mod}\ \ 3).$$

Then, by using Lemma 2.4 we get:

$$f_{2^{\alpha+1}+2^{\alpha+1}+1} \equiv f_{2^{\alpha+1}+1}(\mathrm{mod}\ (\ 2^\alpha - 1)).$$

So, $f_{2^{\alpha+2}+1} \equiv 1(\mathrm{mod}\ (\ 2^\alpha - 1))$, and in a similar way we get $f_{2^{\alpha+2}+2} \equiv 1(\mathrm{mod}\ (\ 2^\alpha - 1))$. This completes the proof. Q.E.D.

Proposition 3.3. For every integers $\alpha, \beta \geq 1$ and every different odd primes p and q where $p < q$, $\ell = LEN_{\{b,a^j\}}(S) \equiv 0(\mathrm{mod}\ \ 3)$, for every j co-prime with $p^\alpha - 1$. Moreover, $\lambda(S)$ is a multiple of 6.

Proof. For the given integers $\alpha, \beta \geq 1$ and the odd prime q where $p < q$ the results

$$f_{\ell+1} \equiv 1(\text{mod } (p^\alpha - 1)),$$
$$f_{\ell+2} \equiv 1(\text{mod } (p^\alpha - 1)),$$

of the Proposition 3.1 yield the equation $f_\ell \equiv 0(\text{mod } (p^\alpha - 1))$. Hence, f_ℓ is an even integer because of the fact that $p^\alpha - 1$ is an even integer, for every odd prime p. This means that f_ℓ is even and then $\ell \equiv 0(\text{mod } 3)$.

To complete the proof we consider three possible cases:
case 1: $\alpha = 1$ and $p = 3$.
case 2: $\alpha = 2$ and $p \geq 3$.
case 3: $\alpha \geq 3$ and $p \geq 3$.

In the first case the possible generating pairs for S are indeed (b, a) and (a, b). The Fibonacci sequence with respect to the first generating pair is:

$$z_0 = b,\ z_1 = a,\ z_2 = ba,\ z_3 = a^2,\ z_4 = ba,\ z_5 = a,\ z_6 = ba^2,\ z_7 = a,\ z_8 = ba, \ldots,$$

and for the second generating pair is:

$$y_0 = a,\ y_1 = b,\ y_2 = a,\ y_3 = ba,\ y_4 = a^2,\ y_5 = ba,\ y_6 = a,\ y_7 = ba^2,\ y_8 = a, y_9 = ba, \ldots.$$

Evidently, both of the sequences are truncated periodic (first sequence with one truncated term and the second sequence with two truncated terms from the beginning) and are of the period $\ell = 6$. Hence, $\lambda(S) = 6$, in the first case.

The proof in the second case needs to verify the following properties concerning the prime p and the Fibonacci sequence of numbers.

(i). For every odd prime p, $p^2 - 1$ is divisible by 8.

(ii). For every positive integer k, $f_{6k} \equiv 0(\text{mod } 8)$.

The property (i) may be checked easily by considering two possible cases for p as $p \equiv \pm 1(\text{mod } 4)$. And (ii) may be proved by an induction method on k by using the identity $f_{i+j} = f_i f_{j+1} + f_{i-1} f_j$ of the Fibonacci numbers. Hence, (i) together with the equation $f_\ell \equiv 0(\text{mod } (p^\alpha - 1))$ yields $8 \mid f_\ell$. Since then, (ii) proves that $\lambda(S)$ is divisible by 6, as desired.

Finally, in the case 3 for every j where $g.c.d.(j, p^\alpha - 1) = 1$, the equation $\ell = LEN_{\{b,a^j\}}(S) \equiv 0(\text{mod } 3)$ shows that there are two possible cases for ℓ as $\ell = 6k + 3$ or $\ell = 6k$. To complete the proof we refute the first case. Indeed, if $\ell = 6k + 3$ then the equations $f_{\ell+1} \equiv 1(\text{mod } (p^\alpha - 1))$ and $f_{\ell+2} \equiv 1(\text{mod } (p^\alpha - 1))$ yield that $f_\ell \equiv 0(\text{mod } (p^\alpha - 1))$. Substituting for $\ell = 6k + 3$ we get the equations

$$\begin{aligned} f_{6k+3} &\equiv f_{6(k-1)+3}(\text{mod } (p^\alpha - 1)) \\ &\equiv f_{6(k-2)+3}(\text{mod } (p^\alpha - 1)) \\ &\vdots \\ &\equiv f_{6+3}(\text{mod } (p^\alpha - 1)) \\ &\equiv f_3(\text{mod } (p^\alpha - 1)) \\ &\equiv 2(\text{mod } (p^\alpha - 1)), \end{aligned}$$

consecutively. So, the equation $2 \equiv 0(\text{mod } (p^\alpha - 1))$ is a contradiction. Consequently, $\lambda(S)$ is a multiple of 6.

Q.E.D.

4 The invariant $\lambda(T)$

As well as in the last sections the semigroup T may be partitioned as

$$S = \{b^i\} \cup \{a^j\} \cup \{b^i a^j\}, \quad i \geq 1, \quad 1 \leq j \leq p^\alpha - 1.$$

For this semigroup there are exactly $2\varphi(p^\alpha - 1)$ number of generating pairs (b, a^j) and (a^j, b) such that $g.c.d.(j, p^\alpha - 1) = 1$. For every j their Fibonacci sequences are:

$$\begin{aligned} z_0 = b, z_1 = a^j, z_k = z_{k-2}z_{k-1} \quad k \geq 2, \\ y_0 = a^j, y_1 = b, y_k = y_{k-2}y_{k-1} \quad k \geq 2. \end{aligned}$$

The Lemmas 2.2, 2.3 and the Proposition 3.1 are confirmed for T except when they concerning the parameter β. Our main result on the invariant $\lambda(T)$ is:

Proposition 4.1. For every integer $\alpha \geq 1$ and the prime $p = 2$, $\lambda(T) \leq p^{\alpha+1}$.

Proof. Since $p = 2$ then we get $LEN_{\{b,a^j\}}(T) = 2^{\alpha+1}$, as well as in the Proposition 3.2. So, the result follows at once. Q.E.D.

5 Conclusion

The semigroup $S = \langle a, b \mid a^{p^\alpha} = a, b^{q^\beta} = b, ab = a \rangle$ is of order $n = p^\alpha q^\beta - 1$ and the Proposition 3.1 gives an explicit formula for $\lambda(S)$ when $p = 2$ which shows that $\lambda(S)$ is independent of the parameters q and β. In turn this proves that infinite number of non-isomorphic semigroups may have the same Fibonacci invariant. Although we didn't manage to formulate $\lambda(S)$ for all odd primes p, however, the Proposition 3.1 helped us to get explicit values for this invariant for certain values of the prime p as in the following table. In each case $q \neq p$ is an arbitrary odd prime and $\beta \geq 2$ is an arbitrary integer.

$p = 3$

α	1	2	3	4	5	6
$\lambda(S)$	6	12	84	120	330	336

$p = 5$

α	1	2	3	4	5	6
$\lambda(S)$	6	24	30	168	210	240

$p = 7$

α	1	2	3	4	5	6
$\lambda(S)$	24	24	72	1200	4200	792

This investigation answers in part, the natural question: "Is there any finite or infinite 2-generated non-commutative semigroup S corresponds to an even integer ℓ such that $\ell = \lambda(S)$? " Indeed, we answered the question when $\ell \equiv 0 (\text{mod } 6)$ and when $\ell \equiv \pm 2 (\text{mod } 6)$, $(\ell \geq 4)$. For $\ell = 2$ we may consider the semigroup $\langle a, b \mid a^2 = a, ab = a \rangle$ where we see that the Fibonacci sequence

$$z_0 = b, z_1 = a, z_2 = ba, z_3 = a^2 = a, z_4 = ba^2 = ba, z_5 = aba = a^2 = a, \ldots$$

is truncated periodic of the length $\ell = 2$. So, the case when $\ell \equiv 3 (\text{mod } 6)$ remains as a question.

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