On bronze Riemannian structures

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Abstract

The paper is devote to the study of bronze Riemannian structures. An integrability condition and curvature for bronze Riemannian structure are investigated via the Tachibana operator which is applied to pure tensor fields. After twin bronze Riemannian metric is studied. Examples of bronze structures are given.

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1 Introduction

Manifold endowed with specific differential geometric structures has an important role in differential geometry. Yano introduced the concept of an $f-$structure which is a tensor field of type $(1,1)$ of constant rank on differentiable manifold M and ensures the equality $f^3+f=0$ [16]. The almost complex and almost contact structure are derived from the $f-$structure. Extending the concept of an $f-$structure, Goldberg and Yano introduced the notion of polynomial structure of degree d on manifold M. The polynomial structure satisfies the equality

$$Q(f)=f^d+a_df^{d-1}+...+a_2f+a_1I=0 \tag{1.1}$$

where I is the identity field and $a_1,\ a_2,...,a_d$ are real numbers [5].

Spinadel introduced the metallic means family [13, 14]. All of the metallic means family members are pozitive quadratic irrational $\sigma_{p,q}=\frac{p+\sqrt{p^2+4q}}{2}$ which are the solutions of quadratic equation $x^2-px-q=0$. Inspired by the metallic means family, Hretcanu and Crasmareanu introduced the metallic structure on manifold M which is determined by a tensor field Ψ of type $(1,1)$ on manifold M satisfying $\Psi^2=p\Psi+qI\ ,\ p,q\in\mathbb{R}$ [6]. A metallic structure is example of polynomial structures. Some remarks on the metallic structures are obtained in [2, 4, 7]. If $p=3$ and $q=1$, then a tensor field Ψ of type $(1,1)$ is called bronze structure on M which satisfies the equation

$$\Psi^2=3\Psi+I \tag{1.2}$$

where I is the identity tensor field of type $(1,1)$ [9]. Using the notion of a bronze ratio $\Psi=\frac{3+\sqrt{13}}{2}\approx 3.302...$ which is a pozitive root of the equation $x^2-3x-1=0$, Pandey and Sameer have studied the notion of a bronze structure on manifold M. Bronze ratio has an important location in studying topics such as dynamical system and quasicrystals.

Let (M,g) be a Riemannian manifold endowed with the Bronze structure Ψ such that

$$g(\Psi A,B)=g(A,\Psi B) \tag{1.3}$$

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for all vector fields A, B. Equivalently to (1.3)

$$g(\Psi A, \Psi B) = g\left(\Psi^2 A, B\right) = g((3\Psi + I)A, B) = 3g(\Psi A, B) + g(A, B). \tag{1.4}$$

The Riemannian metric in equation (1.3) is called $\Psi-$compatible and (M, Ψ, g) is called a Bronze Riemannian manifold.

Let Ψ be a tensor field of type $(1,1)$ on manifold M. A tensor field k of type $(0,r)$ is called a pure tensor field according to Ψ if

$$\begin{aligned} k(\Psi B_1, B_2, ..., B_r) &= k(B_1, \Psi B_2, ..., B_r) \\ &\vdots \\ &= k(B_1, B_2, ..., \Psi B_r) \end{aligned} \tag{1.5}$$

for any vector fields $B_1, B_2, ..., B_r$. The Tachibana operator Φ_Ψ is applied to tensor field of type $(0,r)$ as follow

$$\begin{aligned} (\Phi_\Psi k)(A, B_1, ..., B_r) &= (\Psi A)k(B_1, ..., B_r) - Ak(\Psi B_1, ..., B_r) \\ &\quad + \sum_{\lambda=1}^{r} k(B_1, ..., (L_{B_\lambda}\Psi)A, ..., B_r), \end{aligned}$$

where L_B denotes the Lie derivative according to B [1, 12, 15].

Bronze structure has an important status on Riemannian manifold because this structure has relation with pure Riemannian metrics according to the corresponding structure. Since Riemannian bronze and almost product structures are related to each other [9], the method of $\Phi-$operator used in the theory of almost product structures can be processed to bronze structures.

2 Locally decomposable bronze Riemannian structures

Let (M, g) be a Riemannian manifold endowed with almost product structure P,i.e. it called an almost product Riemannian manifold and represented with the triple (M, g, P). The almost product Riemannian manifold is endowed with an almost product structure P of type $(1,1)$ such that

$$P^2 = I, \quad g(PA, B) = g(A, PB) \tag{2.1}$$

for all vector fields A, B and g is a Riemannian metric. The almost product structure P on manifold M is derived from the polynomial structure on manifold M. A necessary and sufficient condition for P to be integrable is that $\nabla P = 0$, where ∇ is the Riemannian connection of g. An almost product Riemannian manifold with an integrable product structure P is called locally product Riemannian manifold. If a pure tensor field k ensures the equivalence $\Phi_\varepsilon k = 0$ that ε is a tensor field of type $(1,1)$ and Φ is Tachibana operator, then it is called $\Phi-$tensor. If ε is a product structure, a $\Phi-$tensor is a decomposable tensor [15]. The condition $\Phi_P g = 0$ means that the pure metric g is decomposable tensor, where P is a product structure and

$$(\Phi_P g)(A, B_1, B_2) = (PA)(g(B_1, B_2)) - A(g(PB_1, B_2)) + g((L_{Y_1}P)A, B_2) + g(B_1, (L_{B_2}P)A). \tag{2.2}$$

For an almost product Riemannian manifold equipped with the pure metric g, the condition $\nabla P = 0$ is equavalent to the condition $\Phi_P g = 0$, where ∇ is the Riemannian connection of g [12, 10]. Then,

a locally product Riemannian manifold with decomposable tensor g is called locally decomposable Riemannian manifold [17].

We can obtain condition of integrability for a bronze Riemannian structure via the Tachibana operator.

Theorem 2.1. Let M be a bronze Riemannian manifold endowed with a bronze structure Ψ and a Riemanian metric g. Then:

a) Ψ is integrable if $\Phi_\Psi g = 0$,

b) The condition $\Phi_\Psi g = 0$ is equivalent to $\nabla\Psi = 0$, where ∇ is the Riemannian connection of g.

Proof. By using purity of g and the condition $\nabla g = 0$, we have

$$g\left(B_1, \left(\nabla_A \Psi\right) B_2\right) = g\left(\left(\nabla_A \Psi\right) B_1, B_2\right) \tag{2.3}$$

for all vector fields A, B_1, B_2.

By using the equation (2.3) and $[A, B] = \nabla_A B - \nabla_B A$, we can write the equation (2.2) in a different form as follows:

$$\left(\Phi_\Psi g\right)(A, B_1, B_2) = -g\left(\left(\nabla_A \Psi\right) B_1, B_2\right) + g\left(\left(\nabla_{B_1} \Psi\right) A, B_2\right) + g\left(B_1, \left(\nabla_{B_2} \Psi\right) A\right). \tag{2.4}$$

By replacing A and B_2, we have

$$\left(\Phi_\Psi g\right)(B_2, B_1, A) = -g\left(\left(\nabla_{B_2} \Psi\right) B_1, A\right) + g\left(\left(\nabla_{B_1} \Psi\right) B_2, A\right) + g\left(B_1, \left(\nabla_A \Psi\right) B_2\right). \tag{2.5}$$

If we add the equations (2.4) and (2.5), we obtain

$$\left(\Phi_\Psi g\right)(A, B_1, B_2) + \left(\Phi_\Psi g\right)(B_2, B_1, A) = 2g\left(A, \left(\nabla_{B_1} \Psi\right) B_2\right). \tag{2.6}$$

From the equation (2.6), we reach the items a and b of Theorem 2.1. Q.E.D.

Now, we are going to give the relationships between the almost product structures and bronze structures.

Proposition 2.2. If Ψ is a bronze structure on manifold M, then

$$P = \frac{1}{\sqrt{13}}\left(2\Psi - 3I\right) \tag{2.7}$$

is an almost product structure on M. Conversely, every almost product structure P on manifold M induces two bronze structures on manifold M, given as follows:

$$\Psi_1 = \frac{1}{2}\left(3I + \sqrt{13}P\right), \quad \Psi_2 = \frac{1}{2}\left(3I - \sqrt{13}P\right). \tag{2.8}$$

Proof. Let P be an almost product structure on a Riemannian manifold (M, g), i.e. $P^2 = I$. Then each of the structures $\Psi_1 = \frac{1}{2}\left(3I + \sqrt{13}P\right)$ and $\Psi_2 = \frac{1}{2}\left(3I - \sqrt{13}P\right)$ obtained from the almost product P is a Bronze structure. In fact,

$$\begin{aligned}\Psi_1^2 &= \tfrac{9I+6\sqrt{13}P+13I}{4}\\ &= \tfrac{3\sqrt{13}P+11I}{2}\\ &= \tfrac{1}{2}\left(11I+3\sqrt{13}\tfrac{2\Psi_1-3I}{\sqrt{13}}\right)\\ &= \tfrac{1}{2}\left(11I+6\Psi_1-9I\right)\\ &= 3\Psi_1+I.\end{aligned}$$

Similarly, the equation $\Psi_2^2-3\Psi_2-I=0$ is obtained for bronze structure Ψ_2.

Conversely, let Ψ be a Bronze structure on a Riemannian manifold (M,g). Then the structure $P=\frac{1}{\sqrt{13}}\left(2\Psi-3I\right)$ induced by the Bronze structure Ψ is an almost product structure. In fact,

$$P^2=\frac{4\Psi^2-12\Psi+9I}{13}=\frac{4\left(\Psi^2-3\Psi\right)+9I}{13}=\frac{13I}{13}=I.$$

Q.E.D.

A Riemannian metric g is pure according to a Bronze structure if and only if the Riemannian metric g is pure accoding to corresponding almost product structure P. The relation between the Tachibana operators $\Phi_P g$ and $\Phi_\Psi g$ is given as

$$\Phi_P g=\frac{2}{\sqrt{13}}\Phi_\Psi g. \tag{2.9}$$

We know that if the Riemannian metric g satisfies the condition $\Phi_P g=0$, the Riemannian metric g is decomposable. Considering the Theorem 2.1, we can deduce that the Bronze Riemannian structure Ψ is integrable if the Riemannian metric g is decomposable. If (M,Ψ,g) is a locally Bronze Riemannian manifold with decomposable Riemannian metric, then (M,Ψ,g) is called a locally decomposable Bronze Riemannian manifold. So, we get following proposition

Proposition 2.3. Let M be a Bronze Riemannian manifold endowed with a bronze structure Ψ and a Riemannian metric g. The manifold M is a locally decomposable Bronze Riemannian manifold if and only if $\Phi_P g=0$, where P is the corresponding almost product structure associated with Ψ.

The twin bronze Riemannian metric is defined by

$$G\left(B_1,B_2\right)=g\left(\Psi B_1,B_2\right) \tag{2.10}$$

for all vector fields B_1,B_2 on M. It is easily seen that the twin bronze Riemannian metric G is pure according to the bronze structure Ψ. If we apply the $\Phi_\Psi-$operator to the twin bronze Riemannian metric G, we obtain

$$\left(\Phi_\Psi G\right)\left(A,B_1,B_2\right)=\left(\Phi_\Psi g\right)\left(A,\Psi B_1,B_2\right)+g\left(N_\Psi\left(A,B_1\right),B_2\right) \tag{2.11}$$

where N_Ψ is Nijenhuis tensor consructed from Ψ.So, we have

Proposition 2.4. Let (M,Ψ,g) be a locally decomposable bronze Riemannian manifold, where Ψ is a bronze structure and g is a bronze Riemannian metric. Then, the twin bronze Riemannian metric G is $\Phi-$tensor field.

Theorem 2.5. Let (M, Ψ, g) be a bronze Riemannian manifold, where Ψ is a bronze structure and g is a bronze Riemannian metric. The Riemannian curvature tensor field R is a $\Phi-$ tensor field.

Proof. The Riemannian curvature tensor field R of the bronze Riemannian metric g is pure according to the bronze structure Ψ, i.e.

$$R(\Psi B_1, B_2, B_3, B_4) = R(B_1, \Psi B_2, B_3, B_4) = R(B_1, B_2, \Psi B_3, B_4) = R(B_1, B_2, B_3, \Psi B_4) \quad (2.12)$$

for all vector fields B_1, B_2, B_3, B_4 on M. Applying the Tachibana operator to the Riemannian curvature R of type $(0, 4)$ as in equation (2.2), we write

$$(\Phi_\Psi R)(A, B_1, B_2, B_3, B_4) = (\nabla_{\Psi A} R)(B_1, B_2, B_3, B_4) - (\nabla_A R)(\Psi B_1, B_2, B_3, B_4). \quad (2.13)$$

Considering the Riemannian curvature tensor R to be pure and applying the Bianchi's 2nd identity to (2.13), we obtain

$$(\Phi_\Psi R)(A, B_1, B_2, B_3, B_4) = 0. \quad (2.14)$$

Since the interim calculations are similar to proof in [3], we omit standart calculations. Q.E.D.

By (2.7) and (2.13), we can find, in a similar way like the equation (2.9),

$$\Phi_P R = \frac{2}{\sqrt{13}} \Phi_\Psi R, \quad (2.15)$$

where Ψ is the bronze structure and P is its corresponding almost product structure. Based on Theorem 2.5 and the equation (2.15), we obtain following proposition

Proposition 2.6. Let (M, Ψ, g) be a bronze Riemannian manifold, where Ψ is a bronze structure and g is a bronze Riemannian metric. The Riemannian curvature tensor field is a decomposable tensor field.

3 Examples

Example 3.1. Let (M, g) be a Riemannian manifold with dimension n and $T(M)$ be its tangent bundle with the bundle projection $\pi : T(M) \to M$ that the bundle projection defines the natural bundle structure of tangent bundle $T(M)$ over manifold M. Then $T(M)$ is a $2n$ dimensional smooth manifold. A system of local coordinates (U, x^i) in manifold M, $U \subset M$, induces to a system of local coordinates $\left(\pi^{-1}(U), x^i, x^{\bar{i}} = y^i\right)$ $i = 1, ..., n$, $\bar{i} = n+1, ..., 2n$ in tangent bundle $T(M)$, where (y^i) are the Cartesian coordinates in each tangent space $T_P(M)$ at $P \in M$ according to natural base.

Let $A = A^i \frac{\partial}{\partial x^i}$ be the local expression in U of a vector field A. Then the vertical lift ${}^V A$ and horizontal lift ${}^H A$ of A are given according to induced coordinates in $T(M)$

$$\begin{aligned} {}^V A &= \begin{pmatrix} {}^V A^i \\ {}^V A^{\bar{i}} \end{pmatrix} = \begin{pmatrix} 0 \\ A^i \end{pmatrix} \\ {}^H A &= \begin{pmatrix} {}^H A^i \\ {}^H A^{\bar{i}} \end{pmatrix} = \begin{pmatrix} A^i \\ -A^s y^t \Gamma^i_{ts} \end{pmatrix} \end{aligned} \quad (3.1)$$

where Γ_{ts}^{i} are the coefficients of the Riemannian connection ∇ of g.

The Sasaki metric ${}^{S}g$ on the tangent bundle $T(M)$ is defined by

$$
\begin{aligned}
{}^{S}g\left({}^{V}A,\ {}^{V}B\right) &= {}^{V}\left(g\left(A,B\right)\right) && (3.2)\\
{}^{S}g\left({}^{V}A,\ {}^{H}B\right) &= {}^{S}g\left({}^{H}A,\ {}^{V}B\right) = 0 && (3.3)\\
{}^{S}g\left({}^{H}A,\ {}^{H}B\right) &= {}^{V}\left(g\left(A,B\right)\right) && (3.4)
\end{aligned}
$$

for any vector fields A, B on manifold M [18].

We define a bronze structure $\widehat{J}$ on $T(M)$

$$
\begin{cases}
\widehat{J}\left({}^{H}A\right) = \frac{1}{2}\left(3\ {}^{H}A + \sqrt{13}\ {}^{V}A\right)\\
\widehat{J}\left({}^{V}A\right) = \frac{1}{2}\left(3\ {}^{V}A + \sqrt{13}\ {}^{H}A\right)
\end{cases}
\tag{3.5}
$$

which implies $\widehat{J}^{2} - 3\widehat{J} - I = 0$.

We write

$$
K\left(\widehat{A},\widehat{B}\right) = {}^{S}g\left(\widehat{J}\widehat{A},\widehat{B}\right) - {}^{S}g\left(\widehat{A},\widehat{J}\widehat{B}\right) \tag{3.6}
$$

for any vector fields $\widehat{A},\widehat{B}$. Replacing $\widehat{A},\widehat{B}$ with ${}^{V}A,\ {}^{V}B$ or ${}^{H}A,\ {}^{H}B$, respectively, from (3.2)-(3.4) and (3.6) we have $K\left(\widehat{A},\widehat{B}\right) = 0$. It is mean that ${}^{S}g$ is pure according to the bronze structure $\widehat{J}$. So we obtain

Theorem 3.2. Let (M,g) be a Riemannian manifold and $T(M)$ be its tangent bundle endowed with the Sasaki metric ${}^{S}g$ and Bronze structure $\widehat{J}$ defined by (3.5). The triple $\left(T(M),\widehat{J},{}^{S}g\right)$ is a bronze Riemannian manifold.

We determined the Sasaki metric ${}^{S}g$ and the bronze structure in equations (3.2) − (3.4) and (3.5), respectively. Using the several properties of lifts on tangent bundle $T(M)$ that ${}^{V}A\,{}^{V}\left(g\left(B_1,B_2\right)\right) = 0$ and ${}^{H}A\,{}^{V}\left(g\left(B_1,B_2\right)\right) = {}^{V}\left(Ag\left(B_1,B_2\right)\right)$, we calculate

$$
\begin{aligned}
\left(\Phi_{\widehat{J}}\,{}^{S}g\right)\left(\widehat{A},\widehat{B_1},\widehat{B_2}\right) &= \left(\widehat{J}\widehat{A}\right)\left({}^{S}g\left(\widehat{B_1},\widehat{B_2}\right)\right) - \widehat{A}\left({}^{S}g\left(\widehat{J}\widehat{B_1},\widehat{B_2}\right)\right)\\
&\quad + {}^{S}g\left(\left(L_{\widehat{B_1}}\widehat{J}\right)\widehat{A},\widehat{B_2}\right) + {}^{S}g\left(\widehat{B_1},\left(L_{\widehat{B_2}}\widehat{J}\right)\widehat{A}\right)
\end{aligned}
\tag{3.7}
$$

for all vector fields A, B_1, B_2. Then we obtain

$$
\begin{aligned}
\left(\Phi_{\widehat{J}}\,{}^{S}g\right)\left({}^{V}A,\ {}^{V}B_1,\ {}^{V}B_2\right) &= 0\\
\left(\Phi_{\widehat{J}}\,{}^{S}g\right)\left({}^{V}A,\ {}^{V}B_1,\ {}^{H}B_2\right) &= -\tfrac{\sqrt{13}}{2}\,{}^{S}g\left({}^{V}B_1,\ {}^{V}\left(R\left(B_2,A\right)y\right)\right)\\
\left(\Phi_{\widehat{J}}\,{}^{S}g\right)\left({}^{V}A,\ {}^{H}B_1,\ {}^{V}B_2\right) &= -\tfrac{\sqrt{13}}{2}\,{}^{S}g\left({}^{V}\left(R\left(B_1,A\right)y\right),\ {}^{V}B_2\right)\\
\left(\Phi_{\widehat{J}}\,{}^{S}g\right)\left({}^{V}A,\ {}^{H}B_1,\ {}^{H}B_2\right) &= 0\\
\left(\Phi_{\widehat{J}}\,{}^{S}g\right)\left({}^{H}A,\ {}^{V}B_1,\ {}^{V}B_2\right) &= 0\\
\left(\Phi_{\widehat{J}}\,{}^{S}g\right)\left({}^{H}A,\ {}^{V}B_1,\ {}^{H}B_2\right) &= 0\\
\left(\Phi_{\widehat{J}}\,{}^{S}g\right)\left({}^{H}A,\ {}^{H}B_1,\ {}^{V}B_2\right) &= 0\\
\left(\Phi_{\widehat{J}}\,{}^{S}g\right)\left({}^{H}A,\ {}^{H}B_1,\ {}^{H}B_2\right) &= \tfrac{\sqrt{13}}{2}\,{}^{S}g\left({}^{H}\left(R\left(B_1,A\right)y\right),\ {}^{H}B_2\right)\\
&\quad + \tfrac{\sqrt{13}}{2}\,{}^{S}g\left({}^{H}B_1,\ {}^{H}\left(R\left(B_2,A\right)y\right)\right).
\end{aligned}
\tag{3.8}
$$

So, from Proposition 2.3 and equation (2.9), we obtain following theorem

Theorem 3.3. Let (M,g) be a Riemannian manifold and let $T(M)$ be its tangent bundle endowed with the Sasaki metirc ${}^{S}g$ and the bronze structure $\widehat{J}$ defined by (3.5). The triple $\left(T(M),\widehat{J},\ {}^{S}g\right)$ is locally decomposable bronze Riemannian manifold if and only if the Riemannian manifold is locally flat.

Example 3.4. Let (M,g) be a Riemannian manifold with dimension n and $T^{*}(M)$ be its cotangent bundle with the bundle projection $\pi: T^{*}(M)\to M$ that the bundle projection defines the natural bundle structure of cotangent bundle $T^{*}(M)$ over manifold M. Then $T^{*}(M)$ is a $2n$ dimensional smooth manifold. A system of local coordinates $\left(U,x^{i}\right)$ in manifold M, $U\subset M$, induces to a system of local coordinates $\left(\pi^{-1}(U),x^{i},x^{\overline{i}}=p_{i}\right)$ $i=1,...,n,\ \overline{i}=n+1,...,2n$ in cotangent bundle $T^{*}(M)$, where (p_{i}) are the Cartesian coordinates in each cotangent space ${T^{*}}_{P}(M)$ at $P\in M$ according to natural base.

Let $\alpha=\alpha_{j}dx^{j}$ be the local expression in U of a $1-$form α. Then the vertical lift ${}^{V}\alpha$ of $1-$form α and horizontal lift ${}^{H}A$ of vector field A are given according to induced coordinates in $T^{*}(M)$

$$\begin{aligned} {}^{V}\alpha &= \begin{pmatrix} {}^{V}\alpha_{j} \\ {}^{V}\alpha_{\overline{j}} \end{pmatrix} = \begin{pmatrix} 0 \\ \alpha_{j} \end{pmatrix} \\ {}^{H}A &= \begin{pmatrix} {}^{H}A^{j} \\ {}^{H}A^{\overline{j}} \end{pmatrix} = \begin{pmatrix} A^{j} \\ p_{s}\Gamma^{s}_{ji}A^{i} \end{pmatrix}. \end{aligned} \tag{3.9}$$

The Sasaki metric ${}^{S}g$ on the cotangent bundle $T^{*}(M)$ is defined by

$${}^{S}g\left({}^{V}\alpha,\ {}^{V}\beta\right) = {}^{V}\left(g^{-1}(\alpha,\beta)\right) = g^{-1}(\alpha,\beta)\circ\pi, \tag{3.10}$$

$${}^{S}g\left({}^{V}\alpha,\ {}^{H}B\right) = 0, \tag{3.11}$$

$${}^{S}g\left({}^{H}A,\ {}^{H}B\right) = {}^{V}\left(g(A,B)\right) = g(A,B)\circ\pi, \tag{3.12}$$

for any vector fields A,B and $1-$forms α,β. The Sasaki metric ${}^{S}g$ is assigned by equations (3.10) – (3.12). The Riemannian connection ${}^{S}\nabla$ of ${}^{S}g$ compensates the following relations:

$$\begin{aligned} &(i)\quad {}^{S}\nabla_{\ {}^{V}\alpha}\ {}^{V}\beta = 0 \\ &(ii)\quad {}^{S}\nabla_{\ {}^{V}\alpha}\ {}^{H}B = \tfrac{1}{2}\ {}^{H}\left(p\left(g^{-1}\circ R(\ ,B)\,\widetilde{\alpha}\right)\right) \\ &(iii)\quad {}^{S}\nabla_{\ {}^{H}A}\ {}^{V}\beta = {}^{V}\left(\nabla_{A}\beta\right)+\tfrac{1}{2}{}^{H}\left(p\left(g^{-1}\circ R(\ ,A)\,\widetilde{\beta}\right)\right) \\ &(iv)\quad {}^{S}\nabla_{\ {}^{H}A}\ {}^{H}B = {}^{H}\left(\nabla_{A}B\right)+\tfrac{1}{2}{}^{V}\left(pR(A,B)\right) \end{aligned} \tag{3.13}$$

for all vector fields A,B and $1-$forms α,β, where $\widetilde{\alpha}=g^{-1}\circ\alpha$ is a vector field, $R(\ ,A)\,\widetilde{\beta}$ is a tensor field of type $(1,1)$ and $g^{-1}\circ R(\ ,A)\,\widetilde{\beta}$ is a tensor field of type $(2,0)$ [11, 8].

We define a bronze structure $\widehat{J}^{*}$ on $T^{*}(M)$ by

$$\begin{cases} \widehat{J}^{*}\left({}^{H}A\right)=\frac{1}{2}\left(3\ {}^{H}A+\sqrt{13}\ {}^{V}\widetilde{A}\right) \\ \widehat{J}^{*}\left({}^{V}\alpha\right)=\frac{1}{2}\left(3\ {}^{V}\alpha+\sqrt{13}\ {}^{H}\widetilde{\alpha}\right) \end{cases} \tag{3.14}$$

for any vector field A and any $1-$form α, where $\widetilde{A}=g\circ A$ is a $1-$form, $\widetilde{\alpha}=g^{-1}\circ\alpha$ is a vector field. And the Sasaki metric is pure according to bronze structure $\widehat{J}^{*}$. Then we obtain following theorem

Theorem 3.5. Let (M,g) be a Riemannian manifold and let $T^*(M)$ be its cotangent bundle endowed with the Sasaki metric ${}^Sg$ and bronze structure $\widehat{J}^*$ defined by (3.14). The triple $\left(T^*(M),\widehat{J}^*,{}^Sg\right)$ is a Bronze Riemannian manifold.

We investigate the covariant derivative of $\widehat{J}^*$. Considering the equations $(i)-(ii)$ of (3.13) and (3.14), we have

$$\begin{array}{rcl}
\left({}^S\nabla_{{}^HA}\widehat{J}^*\right)\left({}^HB\right) & = & \frac{\sqrt{13}}{4}\,{}^H\left(p\left(g^{-1}\circ\left(R\left(\ ,A\right)B-R\left(A,B\right)\right)\right)\right)\\
\left({}^S\nabla_{{}^V\alpha}\widehat{J}^*\right)\left({}^HB\right) & = & -\frac{\sqrt{13}}{4}\,{}^V\left(pR\left(\ ,B\right)\widetilde{\alpha}\right)\\
\left({}^S\nabla_{{}^HA}\widehat{J}^*\right)\left({}^V\beta\right) & = & \frac{\sqrt{13}}{4}\,{}^V\left(pR\left(A,\widetilde{\beta}\right)-pR\left(\ ,A\right)\widetilde{\beta}\right)\\
\left({}^S\nabla_{{}^V\alpha}\widehat{J}^*\right)\left({}^V\beta\right) & = & \frac{\sqrt{13}}{4}\,{}^H\left(p\left(g^{-1}\circ R\left(\ ,\widetilde{\beta}\right)\widetilde{\alpha}\right)\right)
\end{array}\tag{3.15}$$

for all vector fields A,B and $1-$forms α,β. From equations (3.15) we obtain

Theorem 3.6. Let (M,g) be a Riemannian manifold and $T^*(M)$ be its cotangent bundle endowed with the Sasaki metric ${}^Sg$ and bronze structure $\widehat{J}^*$ defined by (3.14). The triple $\left(T^*(M),\widehat{J}^*,{}^Sg\right)$ is locally decomposable Bronze Riemannian manifold if and only if The Riemannian manifold is flat.

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