Periodic solutions to certain classes of third order delay differential equations

S. A. Olayemi[1] and B. S. Ogundare[2,*]

[1,2] Department of Mathematics, Obafemi Awolowo University, Ile-Ife, Nigeria
* Corresponding Author
E-mail: saheed.olayemi.a@gmail.com[1], ogundareb@yahoo.com[2], bogunda@oauife.edu.ng[2]

Abstract

In this paper, existence of unique periodic solutions for two classes of third order delay differential equations are considered. The two equations are expressed in their integral equivalents with which suitable Greens function is constructed and its associated properties stated. The main tool adopted to establish the existence of unique periodic solutions to the classes of delay differential equations is the Krasnoselskiis fixed point theorem due to its applicability to the combination of a compact and contraction mappings which occur in dealing with perturbed differential operators.

2010 Mathematics Subject Classification. **34K13**. 34K25, 34K99.
Keywords. Third order delay ordinary differential equations, Green's function, Compactness, Contraction, Periodicity..

1 Introduction

In this paper, existence of unique periodic solutions for the classes of third order delay differential equations of the forms

$$x'''(t) + p(t)x''(t) + q(t)x'(t) + w(t)x(t) = c(t)x'(t-\tau(t)) + e(t)x''(t-\tau(t)) + f(t, x(t), x(t-\tau(t))) \tag{1.1}$$

$f : \mathbb{R}^+ \times \mathbb{R}^2 \to \mathbb{R}$

and

$$x'''(t) + p(t)x''(t) + q(t)x'(t) + w(t)x(t) = f(t, x(t), x'(t), x(t-\tau(t)), x'(t-\tau(t))) + c(t)x'(t-\tau(t)) \tag{1.2}$$

$f : \mathbb{R}^+ \times \mathbb{R}^4 \to \mathbb{R}$

where p, q, c, e, τ and w are continuous real-valued periodic functions, with $c, e : \mathbb{R} \to \mathbb{R}$ are continuously differentiable and $\tau : \mathbb{R} \to \mathbb{R}^+$ is twice continuously differentiable, is considered.
Delay differential equations occupy a place of central importance in all areas of science and applied science. The above equations arise from the areas in applied Mathematics and Physics. Various researchers have worked on existence of unique periodic solutions of many delay differential equations ranging from the first order, second order to third order in recent years. Several methods were employed by these researchers which include but not limited to Lyapunov method and Frequency

Tbilisi Mathematical Journal 13(3) (2020), pp. 171–187.
Tbilisi Centre for Mathematical Sciences.
Received by the editors: 07 January 2020.
Accepted for publication: 20 August 2020 .

domain approach ([1], [22], [24], [25], [26], [29]) as well as the fixed point approach.

Raffoul [27], in 2007, considered the nonlinear neutral first order differential equation

$$\frac{d}{dt}\big[x(t) - ax(t-\tau)\big] = r(t)x(t) - f\big(t,\ x(t-\tau)\big) \tag{1.3}$$

with constant delay τ, which arises in the food-limited population models [15], [10], [11], [16] and blood cell models. By transforming the problem to an integral equation, the author was able through the contraction mapping principle and Krasnoselskii's fixed point theorem, to show that the equation has a positive periodic solution.
Wang, Lian and Ge [31] considered the second order delay differential equation,

$$x^{''}(t) + p(t)x^{'}(t) + q(t)x(t) = w(t)x^{'}(t-\tau(t)) + f(t, x(t), x(t-\tau(t))) \tag{1.4}$$

with periodic coefficients. The equation is expressed in its integral equivalence under which the associated Green's function is derived alongside with its important properties. Krasnoselskii's fixed point theorem and the contraction mapping principle were later used to establish some criteria for the existence and uniqueness of periodic solutions to the equation.
Ardjouni and Djoudi in [2] extended the works of Wang *et al.*[31] based on the presence of neutral term which produces the nonlinearity in their own research and consequently, led to different analysis from that of Wang *et al.*. The existence of periodic solutions of the second order nonlinear neutral differential equation,

$$\frac{d^2}{dt^2}x(t) + p(t)\frac{d}{dt}x(t) + q(t)x(t) = \frac{d}{dt}g(t, x(t-\tau(t))) + f(t, x(t), x(t-\tau(t))) \tag{1.5}$$

was proved using the fixed point theorem of Krasnoselskii.
Ardjouni and Djoudi [3] studied the existence of positive periodic solutions of the nonlinear neutral differential equation

$$\frac{d}{dt}[x(t) - g(t, x(t-\tau(t)))] = w(t)x(t) - f(t, x(t-\tau(t))) \tag{1.6}$$

with variable delay where the real-valued functions f, g and w are continuous in their respective arguments. Krasnoselskii's fixed point theorem is applied on the integral equivalence of the equation to prove the existence of positive periodicity of its solution and the result obtained generalizes the work of Raffoul.
Ardjouni *et al.* [4] studied the existence of positive periodic solutions for two types of second order nonlinear neutral differential equations with variable delay.

$$\frac{d^2}{dt^2}[x(t) - g(t, x(t-\tau(t)))] = a(t)x(t) - f(t, x(t-\tau(t))) \tag{1.7}$$

and

$$\frac{d^2}{dt^2}[x(t) - g(t, x(t-\tau(t)))] = -a(t)x(t) + f(t, x(t-\tau(t))) \tag{1.8}$$

where $a, \tau : \mathbb{R} \to (0, \infty)$, $g : \mathbb{R} \times [0, \infty) \to \mathbb{R}$, $f : \mathbb{R} \times [0, \infty) \to [0, \infty)$, are $T-$periodic continuous functions of t and T is a positive constant. With the use of Krasnoselskii's fixed point theorem, the

authors were able to prove their results.
Ardjouni *et al.* [5] worked on the nonlinear neutral functional differential equation with variable delay.

$$x^{''}(t) + p(t)x^{'}(t) + q(t)h(x(t)) = c(t)x^{'}(t - \tau(t)) + f(t, x(t - \tau(t))) \tag{1.9}$$

The equation was transformed to it's equivalent integral equation from which fixed point mapping written as a sum of a large contraction and a compact map, is defined. The authors showed that the maps fit nicely into the framework of Krasnoselskii-Burton's fixed point theorem to ascertain the existence of periodic solutions.

Nouioua, Ardjouni and Djoudi in 2016 employed Krasnoselskii's fixed point theorem and the contraction mapping principle to state and prove the existence and uniqueness of periodic solution to a third order delay differential equation of the form

$$x^{'''}(t) + p(t)x^{''}(t) + q(t)x^{'}(t) + w(t)x(t) = f(t, x(t), x(t - \tau(t))) + c(t)x^{'}(t - \tau(t)) \tag{1.10}$$

The equation is transformed into a non-homogeneous integral equation which is a sum of contraction and compact mappings necessary for application of Krasnoselkii's fixed point theorem.
Motivated by the works in [23], the authors wish to consider the third order delay differential equations (1.1) and (1.2) which are still subjected to further study. It is of utmost importance to consider these general forms of the third order delay differential equations and establish the existence of unique periodic solution of such problems, which are yet to be researched on by the previous authors.
The rest of this article is presented as: sections 2 and 3 present the main tools and preliminary results respectively while our main results for this research work feature in section 4.

2 Main tools

Theorem 2.1. **Mean Value Theorem** [30]
If a function f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then

$$f'(\eta) = \frac{f(b) - f(a)}{b - a}$$

for some η in (a, b).

Theorem 2.2. **Ascoli-Arzela Theorem** [7]
If $\{f_n(t)\}$ is a uniformly bounded and equicontinuous sequence of real functions on an interval $[a, b]$, then there is a subsequence that converges uniformly on $[a, b]$ to a continuous function.

Theorem 2.3. **Contraction Mapping Principle** [8]
Let $(\mathcal{X}, d)$ be a complete metric space and $T : \mathcal{X} \to \mathcal{X}$. Then there is a unique $x \in \mathcal{X}$ with $Tx = x$. Furthermore, if $y \in \mathcal{X}$ and if $\{y_n\}$ is defined inductively by $y_1 = Ty_0$ and $y_{n+1} = Ty_n$ then $y_n \to x$, the unique fixed point. In particular, the equation $Tx = x$ has one and only one solution.

Theorem 2.4. **Krasnoselskii's Fixed Point Theorem** [28]
Let $\mathbb{M}$ be a closed convex nonempty subset of a Banach space $(\mathbb{B}, \|.\|)$. Suppose $\mathcal{H}_1, \mathcal{H}_2 : \mathbb{M} \to \mathbb{B}$ such that

(i) $\mathcal{H}_1 x + \mathcal{H}_2 y \in \mathbb{M}$ for every $x, y \in \mathbb{M}$,
(ii) $\mathcal{H}_1$ is continuous and $\mathcal{H}_1\mathbb{M}$ is contained in a compact subset of $\mathbb{B}$, and
(iii) $\mathcal{H}_2$ is a contraction mapping.
Then there exists $z \in \mathbb{M}$ with $z = \mathcal{H}_1 z + \mathcal{H}_2 z$.
In an attempt to establish the desired results, the following basic assumptions are required [23].
(A1) There exist differentiable positive $T-$periodic functions $a_1(t)$ and $a_2(t)$ and a positive real constant ρ such that

$$\left\{\begin{array}{lcl} a_1(t) + \rho & = & p(t), \\ a_1^{'}(t) + a_2(t) + \rho a_1(t) & = & q(t), \\ a_2^{'}(t) + \rho a_2(t) & = & w(t). \end{array}\right. \tag{2.1}$$

(A2) $p, q, w, c, e \in C(\mathbb{R}, \mathbb{R}^+)$ are $T-$periodic functions with $\tau(t) \geq \tau^* > 0$, $\tau'(t) \neq 1$ for all $t \in [0, T]$,

$$\int_0^T p(s)ds > \rho, \qquad \int_0^T q(s)ds > 0 \tag{2.2}$$

(A3) The function $f(t, x, y)$ is continuous, $T-$periodic in t and Lipschitz continuous in x and y. That is, $f(t + T, x, y) = f(t, x, y)$ and there are positive constants k_1 and k_2 such that

$$|f(t, x, y) - f(t, z, s)| \leq k_1|x - z| + k_2|y - s|. \tag{2.3}$$

3 Preliminary results

In order to express delay differential equations (1.1) and (1.2) in their integral forms, we first set their right hand sides to be $g_1(t)$ and $g_2(t)$ respectively. Treating the first equation, we have

$$x^{'''}(t) + p(t)x^{''}(t) + q(t)x^{'}(t) + w(t)x(t) = g_1(t) \tag{3.1}$$

where $g_1(t)$ is a continuous $T-$periodic function.

Lemma 3.1.[23] Suppose equation (2.1) hold, then equation (3.1) becomes

$$\left.\begin{array}{lcl} x^{''}(t) + a_1(t)x^{'}(t) + a_2(t)x(t) & = & y(t) \\ \frac{d}{dt}y(t) + \rho y(t) & = & g_1(t) \end{array}\right\} \tag{3.2}$$

Lemma 3.2.[3] Let $y, g_1 \in P_T :=$ { Set of all continuous scalar periodic functions } with period $T > 0$. Then y is a solution of the first order linear differential equation

$$\frac{d}{dt}y(t) + \rho y(t) = g_1(t)$$

if and only if

$$y(t) = \int_t^{t+T} G_1(t, s)g_1(s)ds \tag{3.3}$$

where

$$G_1(t, s) = \frac{\exp\{\rho(s - t)\}}{\exp(\rho T) - 1}$$

is the required Green's function.

Corollary 3.1.[23] The Green's Function $G_1(t,s)$ satisfies the following
$G_1(t+T,s+T) = G_1(t,s),\ G_1(t,t+T) = G_1(t,t)\exp(\rho T)$,
$G_1(t+T,s) = G_1(t,s)\exp(-\rho T),\ G_1(t,s+T) = G_1(t,s)\exp(\rho T)$,
$\frac{\partial}{\partial t}G_1(t,s) = -\rho G_1(t,s),\ \frac{\partial}{\partial s}G_1(t,s) = \rho G_1(t,s),\ m_1 \le G_1(t,s) \le M_1$,
$m_1 = \frac{1}{\exp(\rho T)-1},\quad M_1 = \frac{\exp(\rho T)}{\exp(\rho T)-1}$

In order to express the second order differential equation in equation (3.2) in its integral equivalence, we will state without proof, the result by Liu and Ge to perform the transformation.

Lemma 3.3. [20] Suppose that **(A1)** and **(A2)** hold, and

$$\frac{R_1\left[exp\left(\int_0^T a_1(v)dv\right)-1\right]}{Q_1 T} \ge 1 \tag{3.4}$$

where

$$R_1 = max_{t\in[0,T]}\left|\int_t^{t+T}\frac{\exp(\int_0^T a_1(v)dv)}{\exp(\int_0^T a_1(v)dv)-1}a_2(s)ds\right|,\quad Q_1 = \left(1+\exp\left(\int_0^T a_1(v)dv\right)\right)^2 R_1^2$$

Then there are continuous $T-$periodic functions a and b such that
$b(t) > 0,\quad \int_0^T a(v)dv > 0,\quad a(t)+b(t) = a_1(t)$ and $b^{'}(t)+a(t)b(t) = a_2(t)$ for $t \in \mathbb{R}$.

Lemma 3.4. [31] Suppose the conditions of Lemma 3.3 hold and $y \in P_T$. Then the second order equation

$$x^{''}(t) + a_1(t)x^{'}(t) + a_2(t)x(t) = y(t)$$

has a $T-$ periodic solution

$$x(t) = \int_t^{t+T} G_2(t,s)y(s)ds \tag{3.5}$$

where

$$G_2(t,s) = \frac{\int_t^s \exp\left(\int_t^u b(v)dv + \int_u^s a(v)dv\right)du + \int_s^{t+T}\exp\left(\int_t^u b(v)dv + \int_u^{s+T} a(v)dv\right)du}{\left[\exp\left(\int_0^T a(u)du\right)-1\right]\left[\exp\left(\int_0^T b(u)du\right)-1\right]}$$

is the required Green's function.
Corollary 3.2.[23] The Green's Function $G_2(t,s)$ satisfies the following
$G_2(t+T,s+T) = G_2(t,s),\ G_2(t,t+T) = G_2(t,t)$,
$G_2(t+T,s) = \exp(-\int_0^T b(u)du)\left[G_2(t,s) + \int_t^{t+T} E(t,v)F(v,s)dv\right]$,
$\frac{\partial}{\partial t}G_2(t,s) = -b(t)G_2(t,s) + F(t,s),\ \frac{\partial}{\partial s}G_2(t,s) = a(t)G_2(t,s) - E(t,s)$

where

$$E(t,s)=\frac{\exp\left(\int_t^s b(u)du\right)}{\exp\left(\int_0^T b(u)du\right)-1},\quad F(t,s)=\frac{\exp\left(\int_t^s a(u)du\right)}{\exp\left(\int_0^T a(u)du\right)-1}$$

Lemma 3.5. [20] Let $A=\int_0^T a_1(u)du$ and $B=T^2\exp\left(\frac{1}{T}\int_0^T In(a_2(u))du\right)$. If

$$A^2\geq 4B \tag{3.6}$$

Then

$$min\left\{\int_0^T a(u)du,\int_0^T b(u)du\right\}\geq\frac{1}{2}\left(A-\sqrt{A^2-4B}\right)=l$$

and

$$max\left\{\int_0^T a(u)du,\int_0^T b(u)du\right\}\leq\frac{1}{2}\left(A+\sqrt{A^2-4B}\right)=L$$

Corollary 3.3. [23] The functions $G_2(t,s),\ E(t,s)$ *and* $F(t,s)$ satisfy

$$m_2\leq G_2(t,s)\leq M_2,\quad E(t,s)\leq\frac{e^L}{e^l-1}\ \ and\ F(t,s)\leq e^L$$

where

$$m_2=\frac{T}{(e^L-1)^2}\ \ and\ M_2=\frac{T\exp\left(\int_0^T a_1(u)du\right)}{(e^l-1)^2}.$$

By combining equations (3.3) *and* (3.5), we obtain

$$x(t)=\int_t^{t+T}G(t,s)g_1(s)ds,\qquad G(t,s)=\int_t^{t+T}G_2(t,\eta)G_1(\eta,s)d\eta \tag{3.7}$$

Corollary 3.4.[23] The Green's Function $G(t,s)$ satisfies the following
$G(t+T,s+T)=G(t,s),\ G(t,t+T)=G(t,t)\exp(\rho T)$,
$\frac{\partial}{\partial t}G(t,s)=(\exp(-\rho T)-1)G_1(t,t)G_2(t,s)-b(t)G(t,s)+\int_t^{t+T}F(t,\eta)G_1(\eta,s)d\eta$,
$\frac{\partial}{\partial s}G(t,s)=\rho G(t,s)$ *and* $m\leq G(t,s)\leq M$, where

$$m=\frac{T^2}{(e^l-1)^2(e^{\rho T}-1)}\ \ and\ M=\frac{T^2\exp\left(\rho T+\int_0^T a(u)du\right)}{(e^l-1)^2(e^{\rho T}-1)}$$

By substituting the expression for $g_1(t)=c(t)x^{'}(t-\tau(t))+e(t)x^{''}(t-\tau(t))+f(t,x(t),x(t-\tau(t)))$ into equation (1.1), its integral equivalence is

$$\begin{aligned}x(t)=&(\exp(\rho T)-1)G(t,t)\left[\left(Z(t)-Q(t)\right)x(t-\tau(t))+P(t)x^{'}(t-\tau(t))\right]\\&+\int_t^{t+T}G(t,s)\left[\left(\rho Q(s)+Q^{'}(s)-R(s)\right)x(s-\tau(s))+f\left(s,x(s),x(s-\tau(s))\right)\right]ds\end{aligned} \tag{3.8}$$

In a similar manner, the integral equivalence of delay differential equation (1.2) with substitution $g_2(t) = f\big(t, x(t), x^{'}(t), x(t-\tau(t)), x^{'}(t-\tau(t))\big) + c(t)x^{'}(t-\tau(t))$, replacing $g_1(t)$ as shown above, is given below as

$$\begin{aligned} x(t) &= Z(t)\big(\exp(\rho T) - 1\big)G(t,t)x(t-\tau(t)) \\ &\quad + \int_t^{t+T} G(t,s)\bigg[f\bigg(s, x(s), x^{'}(s), x(s-\tau(s)), x^{'}(s-\tau(s))\bigg) - R(s)x\big(s-\tau(s)\big)\bigg]ds \end{aligned} \tag{3.9}$$

where

$$P(t) = \frac{e(t)}{1-\tau^{'}(t)}, \quad Q(t) = \frac{(e^{'}(t) + \rho e(t))(1-\tau^{'}(t)) + e(t)\tau^{''}(t)}{(1-\tau^{'}(t))^3}, \quad Z(t) = \frac{c(t)}{1-\tau^{'}(t)}$$

and $R(t) = \dfrac{\big(c^{'}(t) + \rho c(t)\big)(1-\tau^{'}(t)) + c(t)\tau^{''}(t)}{(1-\tau^{'}(t))^2}$

4 Main results

4.1 The solution of the third order delay differential equation (1.1)

Consider a mapping $\mathcal{H} : P_T \to P_T$ defined on the integral equivalence of equation (1.1) as follows

$$\begin{aligned} (\mathcal{H}\varphi)(t) &= (\exp(\rho T) - 1)G(t,t)\bigg[\big(Z(t) - Q(t)\big)\varphi(t-\tau(t)) + P(t)\varphi^{'}(t-\tau(t))\bigg] \\ &\quad + \int_t^{t+T} G(t,s)\bigg[(\rho Q(s) + Q^{'}(s) - R(s))\varphi(s-\tau(s)) + f(s, \varphi(s), \varphi(s-\tau(s)))\bigg]ds \end{aligned} \tag{4.1}$$

Constructing two mappings from the above,

$$(\mathcal{H}\varphi)(t) = (\mathcal{H}_1\varphi)(t) + (\mathcal{H}_2\varphi)(t) \qquad where \;\; \mathcal{H}_1, \mathcal{H}_2 : P_T \to P_T$$

which are given by

$$\left.\begin{aligned} (\mathcal{H}_1\varphi)(t) &= \int_t^{t+T} G(t,s)\bigg[(\rho Q(s) + Q^{'}(s) - R(s))\varphi(s-\tau(s)) + f(s, \varphi(s), \varphi(s-\tau(s)))\bigg]ds \\ (\mathcal{H}_2\varphi)(t) &= (\exp(\rho T) - 1)G(t,t)\bigg[(Z(t) - Q(t))\varphi(t-\tau(t)) + P(t)\varphi^{'}(t-\tau(t))\bigg] \end{aligned}\right\} \tag{4.2}$$

To simplify notations, we introduce the constants:

$$\alpha = \max_{t\in[0,T]}|Z(t)|, \quad \beta = \max_{t\in[0,T]}\{b(t)\}, \quad \gamma = \max_{t\in[0,T]}|R(t)|, \quad \alpha_1 = \max_{t\in[0,T]}|P(t)|, \quad \alpha_2 = \max_{t\in[0,T]}|Q(t)|,$$

$$\alpha_3 = \max_{t\in[0,T]}|Q'(t)|, \quad \delta = \frac{\exp(L)}{\exp(l) - 1}$$

Compactness of $\mathcal{H}_1$

Lemma 4.1. : Suppose **(A1)** − **(A3)**, inequalities (3.4) and (3.6) hold. Then $\mathcal{H}_1 : P_T \to P_T$ is compact.

Proof. We firstly prove that the map $\mathcal{H}_1$ is periodic as follows,

$$\begin{aligned}(\mathcal{H}_1\varphi)(t+T) &= \int_{t+T}^{t+2T} G(t+T,s)\Big[(\rho Q(s)+Q^{'}(s)-R(s))\varphi(s-\tau(s)) \\ &\qquad + f(s,\varphi(s),\varphi(s-\tau(s)))\Big]ds \\ &= \int_{t}^{t+T} G(t,s)\Big[(\rho Q(s)+Q^{'}(s)-R(s))\varphi(s-\tau(s))+ \\ &\qquad + f(s,\varphi(s),\varphi(s-\tau(s)))\Big]ds \\ &= (\mathcal{H}_1\varphi)(t)\end{aligned}$$

To establish the continuity of $\mathcal{H}_1$, let $\varphi, \psi \in P_T$. Choosing $\theta = min\{1, \frac{\varepsilon}{N}\}, \varepsilon > 0$ with $N = MT(k_1 + k_2 + \rho\alpha_2 + \alpha_3 + \gamma)$ and $k_1, k_2 > 0$

$$\begin{aligned}\big\|\mathcal{H}_1\varphi - \mathcal{H}_1\psi\big\| &= \Big\| \int_t^{t+T} G(t,s)\Big[(\rho Q(s)+Q^{'}(s)-R(s))(\varphi(s-\tau(s)-\psi(s-\tau(s))) \\ &\qquad + (f(s,\varphi(s),\varphi(s-\tau(s))) - f(s,\psi(s),\psi(s-\tau(s))))\Big]ds\Big\| \\ &\le M\int_t^{t+T}\Big[(\rho\alpha_2+\alpha_3+\gamma)\big\|\varphi-\psi\big\| + k_1\big\|\varphi-\psi\big\| + k_2\big\|\varphi-\psi\big\|\Big]ds \\ &= MT[k_1+k_2+\rho\alpha_2+\alpha_3+\gamma]\big\|\varphi-\psi\big\| \\ &< MT(k_1+k_2+\rho\alpha_2+\alpha_3+\gamma)\theta \\ &< N\theta \\ &= \varepsilon\end{aligned}$$

Therefore, $\|\mathcal{H}_1\varphi - \mathcal{H}_1\psi\| < \varepsilon$ with $\|\varphi - \psi\| < \theta$. This implies that $\mathcal{H}_1$ is continuous.

To show that $\mathcal{H}_1\varphi$ is contained in a compact set, we define $\mathbb{W} = \{\varphi \in P_T : \|\varphi\| \le \ell\}$ where ℓ is a fixed positive constant. Let $\varphi_n \in \mathbb{W}$, $n \in \mathbb{Z}^+$ and note that by **(A3)**,

$$\begin{aligned}|f(t,x,y)| &= |f(t,x,y) - f(t,0,0) + f(t,0,0)| \\ &\le |f(t,x,y) - f(t,0,0)| + |f(t,0,0)| \\ &\le k_1\|x\| + k_2\|y\| + \mu \quad where\ \mu = \max_{[0,T]}|f(t,0,0)|\end{aligned}$$

$$\begin{aligned}\|\mathcal{H}_1\varphi_n(t)\| &= \left\| \int_t^{t+T} G(t,s)\Big[(\rho Q(s) + Q^{'}(s) - R(s))\varphi_n(s-\tau(s)) \right. \\ &\qquad \left. + f(s,\varphi_n(s),\varphi_n(s-\tau(s)))\Big]ds \right\| \\ &\leq MT[(k_1+k_2+\rho\alpha_2+\alpha_3+\gamma)\ell+\mu] \\ &< D,\end{aligned}$$

for some positive D.

$$\begin{aligned}\frac{d}{dt}(\mathcal{H}_1\varphi_n(t)) &= \int_t^{t+T} \frac{\partial G}{\partial t}\Big[(\rho Q(s) + Q^{'}(s) - R(s))\varphi_n(s-\tau(s)) + f(s,\varphi_n(s),\varphi_n(s-\tau(s)))\Big]ds \\ &+ \Big[(\rho Q(t+T) + Q^{'}(t+T) - R(t+T))\varphi_n(t+T-\tau(t+T)) \\ &+ f(t+T,\varphi_n(t+T),\varphi_n(t+T-\tau(t+T)))\Big]G(t,t+T) \\ &- \Big[(\rho Q(t) + Q^{'}(t) - R(t))\varphi_n(t-\tau(t)) + f(t,\varphi_n(t),\varphi_n(t-\tau(t)))\Big]G(t,t) \\ &= \int_t^{t+T}\Big[(\exp(-\rho T)-1)G_1(t,t)G_2(t,s) - b(t)G(t,s) + \int_t^{t+T} F(t,\eta)G_1(\eta,s)d\eta\Big] \\ &\times \Big[(\rho Q(s) + Q^{'}(s) - R(s))\varphi_n(s-\tau(s)) + f(s,\varphi_n(s),\varphi_n(s-\tau(s)))\Big]ds \\ &+ \Big[(\rho Q(t) + Q^{'}(t) - R(t))\varphi_n(t-\tau(t)) + f(t,\varphi_n(t),\varphi_n(t-\tau(t)))\Big] \\ &\times \Big[exp(\rho T) - 1\Big]G(t,t)\end{aligned}$$

$$\begin{aligned}\left|\frac{d}{dt}(\mathcal{H}_1\varphi_n(t))\right| &\leq \Big[(\exp(-\rho T)-1)M_1M_2 + M\beta + M_1\delta T\Big]\Big[(k_1+k_2+\rho\alpha_2+\alpha_3+\gamma)\ell+\mu\Big]T \\ &+ \Big[(k_1+k_2+\rho\alpha_2+\alpha_3+\gamma)\ell+\mu\Big]\Big[exp(\rho T)-1\Big]M \\ &= \Big[(k_1+k_2+\rho\alpha_2+\alpha_3+\gamma)\ell+\mu\Big]\Big[\Big((\exp(-\rho T)-1)M_1M_2 + M\beta + M_1\delta T\Big)T \\ &+ \Big(exp(\rho T)-1\Big)M\Big] \\ &\leq K,\end{aligned}$$

for some positive K. Hence, the sequence $\{\mathcal{H}_1\varphi_n\}$ is uniformly bounded and equicontinuous. By the statement of Ascoli-Arzela theorem, there is a subsequence $\{\mathcal{H}_1\varphi_{n_k}\}$ of $\{\mathcal{H}_1\varphi_n\}$ which converges

uniformly to a continuous $T-$periodic function. Thus $\mathcal{H}_1$ is continuous and $\mathcal{H}_1(\mathbb{W})$ is contained in a compact subset of P_T.

Contractive property of $\mathcal{H}_2$

Lemma 4.2. Let $(\mathcal{H}_2\varphi)(t) = (\exp(\rho T)-1)G(t,t)\big[(Z(t)-Q(t))\varphi(t-\tau(t)) + P(t)\varphi^{'}(t-\tau(t))\big]$ with $(\exp(\rho T)-1)(\alpha+\alpha_2)M < 1$, then $\mathcal{H}_2 : P_T \to P_T$ is a contraction.

Proof. Let $\varphi, \psi \in P_T$.

$$\begin{aligned}\|\mathcal{H}_2\varphi - \mathcal{H}_2\psi\| &= \|(\exp(\rho T)-1)G(t,t)[(Z(t)-Q(t))(\varphi-\psi) + P(t)(\varphi^{'}-\psi^{'})]\| \\ &\leq (\exp(\rho T)-1)M[(\alpha+\alpha_2)\|\varphi-\psi\| + \alpha_1\|\varphi^{'}-\psi^{'}\|] \\ &= (\exp(\rho T)-1)M(\alpha+\alpha_2)\|\varphi-\psi\| \ by\ mean-value\ theorem.\end{aligned}$$

Therefore, $\|\mathcal{H}_2\varphi - \mathcal{H}_2\psi\| \leq (\exp(\rho T)-1)M(\alpha+\alpha_2)\|\varphi-\psi\|$ which implies that $\mathcal{H}_2 : P_T \to P_T$ is a contraction mapping with $(\exp(\rho T)-1)(\alpha+\alpha_2)M < 1$.

Theorem 4.1. Suppose that conditions **(A1) – (A3)**, inequalities (3.4) and (3.6) hold and $(\exp(\rho T)-1)(\alpha+\alpha_2)M < 1$. Also, suppose there is a positive constant J satisfying the inequality $\big[(\exp(\rho T)-1)(\alpha+\alpha_2)MJ + (k_1+k_2+M(\rho\alpha_2+\alpha_3+\gamma))J + \mu\big]T \leq J$. Then the third order delay differential equation (1.1) has a solution $x \in P_T$ such that $\|x\| \leq J$.

Proof. Define $\mathbb{M} = \{\varphi \in P_T : \|\varphi\| \leq J\}$. By Lemma(4.1), the operator $\mathcal{H}_1 : \mathbb{M} \to P_T$ is compact and continuous. Also, from Lemma(4.2), the operator $\mathcal{H}_2 : \mathbb{M} \to P_T$ is a contraction. The above two assertions confirm that conditions (ii) *and* (iii) of Krasnoselskii's theorem are satisfied. It only remains condition (i) to be fulfilled.

Now let $\varphi, \psi \in \mathbb{M}$. Then

$$\begin{aligned}\Big\|\mathcal{H}_1\varphi + \mathcal{H}_2\psi\Big\| &= \Big\| \int_t^{t+T} G(t,s)\Big[(\rho Q(s) + Q^{'}(s) - R(s))\varphi(s-\tau(s)) \\ &\qquad + f(s,\varphi(s),\varphi(s-\tau(s)))\Big]ds \\ &\quad + (\exp(\rho T)-1)G(t,t)\Big[(Z(t)-Q(t))\psi(t-\tau(t)) + P(t)\psi^{'}(t-\tau(t))\Big]\Big\| \\ &\leq \Big[(k_1+k_2+M(\rho\alpha_2+\alpha_3+\gamma))\|\varphi\| + \mu\Big]T \\ &\quad + (\exp(\rho T)-1)M\Big[(\alpha+\alpha_2)\|\psi\| + \alpha_1\|\psi^{'}\|\Big] \\ &\leq [(k_1+k_2+M(\rho\alpha_2+\alpha_3+\gamma))J + \mu]T + (\exp(\rho T)-1)M(\alpha+\alpha_2)J\end{aligned}$$

So $\|(\mathcal{H}_1\varphi)(t) + (\mathcal{H}_2\psi)(t)\| \leq J$ and so $\mathcal{H}_1\varphi + \mathcal{H}_2\psi \in \mathbb{M}$. All the conditions of Krasnoselskii's fixed point theorem are satisfied and consequently, the operator $\mathcal{H}$ defined in equation (4.1) has a fixed point in $\mathbb{M}$. This fixed point is the solution of the delay differential equation (1.1) and the proof is complete.

Theorem 4.2. Suppose that conditions **(A1) – (A3)**, inequalities (3.4) and (3.6) hold and $\Big[(\exp(\rho T)-1)(\alpha+\alpha_2) + T(k_1+k_2+\rho\alpha_2+\alpha_3+\gamma)\Big]M < 1$. Then the third order delay differential

equation (1.1) has a unique $T-$periodic solution.

Proof. By the mapping defined in equation (4.1),

$$(\mathcal{H}\varphi)(t) = (\exp(\rho T)-1)G(t,t)\big[(Z(t)-Q(t))\varphi(t-\tau(t)) + P(t)\varphi^{'}(t-\tau(t))\big] + \int_{t}^{t+T} G(t,s)\big[(\rho Q(s)+Q^{'}(s)-R(s))\varphi(s-\tau(s)) + f(s,\varphi(s),\varphi(s-\tau(s)))\big]ds$$

We need to ascertain the existence of unique periodic solution of the third order delay differential equation (1.1). With $\varphi,\psi \in \mathbb{M}$, we have

$$\begin{aligned}
\|\mathcal{H}\varphi - \mathcal{H}\psi\| &= \Big\|(\exp(\rho T)-1)G(t,t)\Big((Z(t)-Q(t))(\varphi(t-\tau(t)-\psi(t-\tau(t))\Big)\\
&+ P(t)\Big(\varphi^{'}(t-\tau(t)-\psi^{'}(t-\tau(t)\Big)\\
&+ \int_{t}^{t+T} G(t,s)\Big[(\rho Q(s)+Q^{'}(s)-R(s))(\varphi-\psi)(t-\tau(t))\\
&+ \Big(f(s,\varphi(s),\varphi(s-\tau(s))-f(s,\psi(s),\psi(s-\tau(s))\Big)\Big]ds\Big\|\\
&\leq \|(\exp(\rho T)-1)M(\alpha+\alpha_2)\|\varphi-\psi\|\\
&+ MT(k_1+k_2+\rho\alpha_2+\alpha_3+\gamma)\|\varphi-\psi\|\\
&= [(\exp(\rho T)-1)M(\alpha+\alpha_2)+MT(k_1+k_2+\rho\alpha_2+\alpha_3+\gamma)]\|\varphi-\psi\|
\end{aligned}$$

By contraction mapping principle, $\mathcal{H}$ has a fixed point in P_T and by consequence, the map defined by equation (4.1) has a fixed point which is the solution of the third order delay differential equation (1.1).

4.2 The solution of the third order delay differential equation(1.2)

By integral equivalence of equation (1.2), let the mapping $\mathcal{H}: P_T \to P_T$ be defined as

$$\begin{aligned}
\mathcal{H}\varphi(t) &= Z(t)\big(\exp(\rho T)-1\big)G(t,t)\varphi(t-\tau(t))\\
&+ \int_{t}^{t+T} G(t,s)[f(s,\varphi(s),\varphi^{'}(s),\varphi(s-\tau(s)),\varphi^{'}(s-\tau(s))) - R(s)\varphi(s-\tau(s))]ds
\end{aligned} \tag{4.3}$$

Similarly, we define two mappings $\mathcal{H}_1, \mathcal{H}_2 : P_T \to P_T$ by

$$\left.\begin{aligned}
(\mathcal{H}_1\varphi)(t) &= \textstyle\int_{t}^{t+T} G(t,s)[f(s,\varphi(s),\varphi^{'}(s),\varphi(s-\tau(s)),\varphi^{'}(s-\tau(s))) - R(s)\varphi(s-\tau(s))]ds\\
(\mathcal{H}_2\varphi)(t) &= Z(t)\big(\exp(\rho T)-1\big)G(t,t)\varphi(t-\tau(t))
\end{aligned}\right\} \tag{4.4}$$

Compactness of $\mathcal{H}_1$

Lemma 4.3. Suppose **(A1)** – **(A3)**, inequalities (3.4) and (3.6) hold. Then $\mathcal{H}_1 : P_T \to P_T$ is compact.

Proof. We check the periodicity of $\mathcal{H}_1$ as follows,

$$\begin{aligned}(\mathcal{H}_1\varphi)(t+T) &= \int_{t+T}^{t+2T} G(t+T,s)[f(s,\varphi(s),\varphi^{'}(s),\varphi(s-\tau(s)),\varphi^{'}(s-\tau(s))) \\ &\qquad - R(s)\varphi(s-\tau(s))]ds \\ &= \int_{t}^{t+T} G(t,s)[f(s,\varphi(s),\varphi^{'}(s),\varphi(s-\tau(s)),\varphi^{'}(s-\tau(s))) - R(s)\varphi(s-\tau(s))]ds \\ &= (\mathcal{H}_1\varphi)(t)\end{aligned}$$

To check the continuity of $\mathcal{H}_1$, let $\varphi, \psi \in P_T$. Given $\varepsilon > 0$, take $\theta = \frac{\varepsilon}{N}$ with $N = [(k_1 + k_2 + \gamma)T + k_3 + k_4]$ where k_1, k_2, k_3 *and* k_4 are Lipschitz constants. Suppose $\|\varphi - \psi\| < \theta$, then we obtain,

$$\begin{aligned}\|\mathcal{H}_1\varphi - \mathcal{H}_1\psi\| &= \left\| \int_t^{t+T} G(t,s)\left[\left(f(s,\varphi(s),\varphi^{'}(s),\varphi(s-\tau(s)),\varphi^{'}(s-\tau(s)))\right.\right.\right. \\ &- \left.\left.\left. f(s,\psi(s),\psi^{'}(s),\psi(s-\tau(s)),\psi^{'}(s-\tau(s)))\right) - R(s)\left((\varphi-\psi)(s-\tau(s))\right)\right]\right\| \\ &\leq M\int_t^{t+T}\left[\left\|f(s,\varphi(s),\varphi^{'}(s),\varphi(s-\tau(s)),\varphi^{'}(s-\tau(s)))\right.\right. \\ &- \left.\left. f(s,\psi(s),\psi^{'}(s),\psi(s-\tau(s)),\psi^{'}(s-\tau(s)))\right\| + \gamma\|\varphi-\psi\|\right]ds \\ &\leq M\int_t^{t+T}[(k_1+k_2+\gamma)\|\varphi-\psi\| + (k_3+k_4)\|\varphi^{'}-\psi^{'}\|]ds \\ &= MT(k_1+k_2+\gamma)\|\varphi-\psi\|,\end{aligned}$$

by mean-value Theorem.
Let $N = MT(k_1 + k_2 + \gamma)$, then

$$\begin{aligned}\|\mathcal{H}_1\varphi - \mathcal{H}_1\psi\| &\leq N\|\varphi-\psi\| \\ &< N\theta \\ &= \varepsilon\end{aligned}$$

Therefore, $\|\mathcal{H}_1\varphi - \mathcal{H}_1\psi\| < \varepsilon$ This implies that $\mathcal{H}_1$ is continuous.
To show that the image of $\mathcal{H}_1$ is contained in a compact set, we consider $\mathbb{W} = \{\varphi \in P_T : \|\varphi\| \leq \ell\}$ where ℓ is a fixed positive constant and let $\varphi_n \in \mathbb{W}$, $n \in \mathbb{Z}^+$ as defined earlier, then we need to

show that $\|\mathcal{H}_1\varphi_n\| \leq D$ for some positive D.

$$\begin{aligned}\|\mathcal{H}_1\varphi_n\| &= \left\| \int_t^{t+T} G(t,s)\Big[f(s,\varphi_n(s),\varphi_n^{'}(s),\varphi_n(s-\tau(s)),\varphi_n^{'}(s-\tau(s))) \right. \\ &\qquad \left. - R(s)\varphi_n(s-\tau(s))\Big]ds \right\| \\ &\leq M\int_t^{t+T}[(k_1+k_2+\gamma)\|\varphi_n\| + (k_3+k_4)\|\varphi_n^{'}\| + \mu]ds \\ &= M[(k_1+k_2+\gamma)\|\varphi_n\| + \mu]T \\ &\leq M[(k_1+k_2+\gamma)\ell + \mu]T \\ &= D\end{aligned}$$

Therefore, $\|\mathcal{H}_1\varphi_n\| \leq D$ for some positive D.

$$\begin{aligned}\frac{d}{dt}(\mathcal{H}_1\varphi_n) &= \int_t^{t+T}\frac{\partial G}{\partial t}\Big[f(s,\varphi_n(s),\varphi_n^{'}(s),\varphi_n(s-\tau(s)),\varphi_n^{'}(s-\tau(s))) - R(s)\varphi_n(s-\tau(s))\Big]ds \\ &+ \Big[G(t,t+T) - G(t,t)\Big]\Big[f(t,\varphi_n(t),\varphi_n^{'}(t),\varphi_n(t-\tau(t)),\varphi_n^{'}(t-\tau(t))) \\ &- R(s)\varphi_n(t-\tau(t))\Big] \\ &= \int_t^{t+T}\Big[(\exp(-\rho T)-1)G_1(t,t)G_2(t,s) - b(t)G(t,s) + \int_t^{t+T}F(t,\eta)G_1(\eta,s)d\eta\Big] \\ &\times \Big[f(s,\varphi_n(s),\varphi_n^{'}(s),\varphi_n(s-\tau(s)),\varphi_n^{'}(s-\tau(s))) - R(s)\varphi_n(s-\tau(s))\Big]ds \\ &+ \Big[f(t,\varphi_n(t),\varphi_n^{'}(t),\varphi_n(t-\tau(t)),\varphi_n^{'}(t-\tau(t))) - R(t)\varphi_n(t-\tau(t))\Big]\Big[exp(\rho T)-1\Big]G(t,t)\end{aligned}$$

$$\begin{aligned}\|\frac{d}{dt}(\mathcal{H}_1\varphi_n)\| &\leq [(\exp(-\rho T)-1)M_1M_2 + \beta M + M_1\delta T](k_1+k_2+\gamma+\mu)\ell T \\ &\leq K,\end{aligned}$$

for some positive K.

Hence, the sequence $\{\mathcal{H}_1\varphi_n\}$ is uniformly bounded and equicontinuous. The Ascoli-Arzela theorem implies that a subsequence $\{\mathcal{H}_1\varphi_{n_k}\}$ of $\{\mathcal{H}_1\varphi_n\}$ converges uniformly to a continuous $T-$periodic function. Thus, $\mathcal{H}_1$ is continuous and $\mathcal{H}_1\mathbb{W}$ is contained in a compact subset of P_T.

Contractive property of $\mathcal{H}_2$

Lemma 4.4. Let $(\mathcal{H}_2\varphi)(t) = Z(t)\big(\exp(\rho T)-1\big)G(t,t)\varphi(t-\tau(t))$ with $(\exp(\rho T)-1)\alpha M < 1$, then $\mathcal{H}_2 : P_T \to P_T$ is a contraction.

Proof. We need to show that $\mathcal{H}_2$ is a contraction. Let $\varphi, \psi \in P_T$.

$$\begin{aligned}\|\mathcal{H}_2\varphi - \mathcal{H}_2\psi\| &= \|Z(t)\big(\exp(\rho T)-1\big)G(t,t)[\varphi(t-\tau(t)) - \psi(t-\tau(t))]\| \\ &\leq \big(\exp(\rho T)-1\big)\alpha M\|\varphi-\psi\|\end{aligned}$$

Hence, $\mathcal{H}_2 : P_T \to P_T$ is a contraction. That is, $\|\mathcal{H}_2\varphi - \mathcal{H}_2\psi\| \leq \big(\exp(\rho T) - 1\big)\alpha M\|\varphi - \psi\|$ with $\big(\exp(\rho T) - 1\big)\alpha M < 1$.

Theorem 4.3. Suppose that conditions **(A1) – (A3)**, inequalities (3.4) and (3.6) hold and $\alpha\big(\exp(\rho T) - 1\big)M < 1$. Also, suppose there is a positive constant J satisfying the inequality $M[(k_1 + k_2 + \gamma)J + \mu]T + \alpha\big(\exp(\rho T) - 1\big)MJ \leq J$. Then the third order delay differential equation (1.2) has a solution $x \in P_T$ such that $\|x\| \leq J$.

Proof. Now conditions (ii) and (iii) of Krasnoselskii's fixed point theorem are satisfied, it remains the validity of its condition (i) which is provided below.

Define $\mathbb{M} = \{\varphi, \psi \in P_T : \|\varphi\| \leq J, \|\psi\| \leq J\}$.

$$\begin{aligned}
\big\|\mathcal{H}_1\varphi + \mathcal{H}_2\psi\big\| &\leq \left\| \int_t^{t+T} G(t,s)\Big[f(s,\varphi(s),\varphi^{'}(s),\varphi(s-\tau(s)),\varphi^{'}(s-\tau(s))) \right. \\
&\qquad \left. - R(s)\varphi(s-\tau(s))\Big]ds \right\| + \|Z(t)\big(\exp(\rho T) - 1\big)G(t,t)\psi(t-\tau(t))\| \\
&\leq M[\|f(s,\varphi(s),\varphi^{'}(s),\varphi(s-\tau(s)),\varphi^{'}(s-\tau(s)))\| + \gamma\|\varphi\|]T \\
&\qquad + \alpha\big(\exp(\rho T) - 1\big)M\|\psi\| \\
&\leq M[(k_1 + k_2 + \gamma)\|\varphi\| + \mu]T + \alpha\big(\exp(\rho T) - 1\big)M\|\psi\| \\
&\leq M[(k_1 + k_2 + \gamma)J + \mu]T + \alpha\big(\exp(\rho T) - 1\big)MJ \\
&\leq J
\end{aligned}$$

Thus, $\|\mathcal{H}_1\varphi + \mathcal{H}_2\psi\| \leq J$, for a positive constant J.

Theorem 4.4**.** Suppose that conditions **(A1) – (A3)**, inequalities (3.4) and (3.6) hold and $M[(k_1 + k_2 + \gamma)T + \alpha\big(\exp(\rho T) - 1\big)] < 1$. Then the third order delay differential equation (1.2) has a unique $T-$periodic solution.

Proof. By the mapping defined in equation (4.3), the presence of fixed point is ascertained as follows

$$\begin{aligned}
\|\mathcal{H}\varphi - \mathcal{H}\psi\| &\leq \left\| \int_t^{t+T} G(t,s)\Big[f(s,\varphi(s),\varphi^{'}(s),\varphi(s-\tau(s)),\varphi^{'}(s-\tau(s))) \right. \\
&- \left. f(s,\psi(s),\psi^{'}(s),\psi(s-\tau(s)),\psi^{'}(s-\tau(s)))\Big] - R(s)[\varphi(s-\tau(s)) - \psi(s-\tau(s))]ds \right\| \\
&+ \left\| Z(t)\big(\exp(\rho T) - 1\big)G(t,t)[\varphi(t-\tau(t)) - \psi(t-\tau(t))] \right\| \\
&\leq M\int_t^{t+T} [(k_1 + k_2 + \gamma)\|\varphi - \psi\| + (k_3 + k_4)\|\varphi^{'} - \psi^{'}\|]ds \\
&+ \alpha\big(\exp(\rho T) - 1\big)M\|\varphi - \psi\| \\
&= M[(k_1 + k_2 + \gamma)T + \alpha\big(\exp(\rho T) - 1\big)]\|\varphi - \psi\|
\end{aligned}$$

Therefore, $\|\mathcal{H}\varphi - \mathcal{H}\psi\| \leq M[(k_1 + k_2 + \gamma)T + \alpha\big(\exp(\rho T) - 1\big)]\|\varphi - \psi\|$, which is showing that $\mathcal{H}$, with $M[(k_1 + k_2 + \gamma)T + \alpha\big(\exp(\rho T) - 1\big)] < 1$, by contraction mapping principle, has a fixed point in P_T and this fixed point is a unique solution to the third order delay differential equation (1.2).

References

[1] Afuwape, A. U. and Adesina, O. A. *Frequency Domain Approach to Stability and Periodic Solutions of certain Fourth Order Nonlinear Differential Equations*,Nonlinear Studies, 12(3) (2005), pp. 259-269.

[2] Ardjouni, A. and Djoudi, A. *Periodic solutions for a second-order nonlinear neutral differential equation with variable delay*, Electron. J. Differential Equations, No. 128(2011) , 7 pp.

[3] Ardjouni, A. and Djoudi, A. *Existence of Positive periodic solutions for a nonlinear neutral differential equations with variable delay*, Applied Mathematics E-Notes, 12 (2012), 94-101.

[4] Ardjouni, A. and Djoudi, A. *Existence of Positive periodic solutions for two types of second order nonlinear neutral differential equations with variable delay*, Proyecciones Journal of Mathematics, Vol. 32, No. 4 (2013), pp. 377-391.

[5] Ardjouni, A. and Djoudi, A. *Existence of periodic solutions for a second-order nonlinear neutral functional differential equation*, Electronic Journal of Mathematical Analysis and Applications, volume 2(1) (2014), 117-126.

[6] Ardjouni, A., Djoudi, A. and Rezaiguia, A. *Existence of positive periodic solutions for two types of third-order nonlinear neutral differential equations with variable delay*, Appl. Math. E-Notes, 14 (2014), 86-96.

[7] Burton, T. A. *Volterra Integral and Differential Equations*, Elsevier B. V., Mathematics in Science and Engineering, Volume 202 (2005).

[8] Burton, T. A. *Stability and Periodic Solutions of Ordinary and Functional Differential equations*, Academic Press, Inc., Mathematics in Science and Engineering, Volume 178 (1985).

[9] Carmen Chicone, *Ordinary Differential Equations with Applications*, Springer-Verlag New York, Inc. (1999).

[10] Chen, F. D. *Positive periodic solutions of neutral Lotka-Volterra system with feedback controls*, Applied Mathematics and Computation, 162(3) (2005), 1279-1302.

[11] Chen, F. D., Lin, F. X. and Chen, X. X. *Sufficient conditions for the existence of positive periodic solutions of a class of neutral delay models with feedback control*, Applied Mathematics and Computation, 158(1) (2004), 45-68.

[12] Dib, Y. M., Maroun M. R. and Raffoul Y. N. *Periodicity and stability in neutral nonlinear differential equations with functional delay*, Electronic Journal of Differential Equations, No. 142 (2005), 11 pp.

[13] Driver, R. D. *Ordinary and Delay Differential Equations*, Springer-Verlag New York, Applied Mathematical Sciences 20 (1977).

[14] Eberhard Zeidler, *Nonlinear Functional Analysis and its Applications, I: Fixed Point Theorems*, Springer-Verlag, New York, Inc. (1986).

[15] Fan, M. and Wang, K. *Global periodic solutions of a generalized n−species Gilpin-Ayala competition model*, Comput. Math. Appl., 40 (2000), 1141-1151.

[16] Gilpin, M. E. and Ayala, F. J. Global models of growth and competition, Proceeding of the National Academy of Sciences USA, 70 (1973), 3590-3593.

[17] Jack Hale, *Theory of Functional Differential Equations*, Springer-Verlag New York, Applied Mathematical Sciences Volume 3 (1977).

[18] Kreyszig Erwin, *Introductory Functional Analysis with Applictaions,* John Wiley and Sons Inc. (1978).

[19] Kuang Yang, *Delay Differential Equations with Application in Population Dynamics*, Academic Press, New York, Mathematics in Science and Engineering, Volume 191 (1993).

[20] Liu, Y. and Ge, W. *Positive periodic solutions of nonlinear Duffing equations with delay and variable coefficients*, Tamsui Oxf. J. Math. Sci., 20 (2004), 235-255.

[21] Li Yong Kun, *Periodic Solution of Periodic Neutral Delay Equation*, Journal of Mathematical Analysis and Applications, 214 (1997), 11-21 Article No. AY975576.

[22] Murat Adivar and Youssef N. Raffoul, *Inequalities and Exponential Decay in time varying Delay Differential Equations*, Mathematical and Computer Modelling, 54 (2011), 794-802.

[23] Nouioua, F., Ardjouni, A., and Djoudi, A. *Periodic Solutions for a Third Order Delay Differential Equation*, Applied Mathematics E-Notes, 16 (2016), 210-221.

[24] Ogundare, B. S. *On Boundedness and Stability of Solutions of certain Third Order Delay Differential Equation*, Journal of the Nigerian Mathematical Society, Vol. 31 (2012), pp. 55-68.

[25] Ogundare, B. S. *On Stability and Boundedness of Solutions of certain Fourth Order Delay Differential Equation*, International Journal of Differential Equations and Applications, Volume 11, No. 4 (2012), 197-213.

[26] Ogundare, B. S. and Afuwape, A. U. *Unified Qualitative Properties of Solution of certain Third Order Non-linear Differential Equations*, International Journal of Pure and Applied Mathematics, Volume 26, No.2 (2006), 173-185.

[27] Raffoul N. Youssef, *Positive Periodic Solutions in Neutral Nonlinear Differential Equations*, Electronic Journal of Qualitative Theory of Differential Equations, No. 16 (2007), 1-10.

[28] Smart, D. R. *Fixed Point Theorems* Cambridge Tracts in Mathematics, Cambridge University Press, London-New York, No. 66 (1974).

[29] Tejumola, H. O. *A Note on the Boundedness of Solutions of Some non-linear Differential Equations of the Third Order*, Ghana Journal of Science, 11, No.2 (1969), 117-118.

[30] Trench, F. William, *Introduction to Real Analysis*, Pearson Education (2003).

[31] Wang, Y., Lian, H. and Ge, W. *Periodic solutions for a second order nonlinear functional differential equation*, Appl. Math. Lett., 20 (2007), 110-115.

[32] Yankson Ernest, *Existence of Periodic Solutions for Totally Nonlinear Neutral Differential Equations with Functional Delay*, Opuscula Mathematica, Vol. 32, No. 3 (2012).