Semitotal domination of Harary graphs

Zeliha Kartal[1,2] and Aysun Aytaç[2]

[1]Computer Programming, Izmir Kavram Vocational School, Izmir, Turkey
[2]Department of Mathematics, Faculty of Science, Ege University, Izmir, Turkey
E-mail: zeliha.kartal@kavram.edu.tr[1], aysun.aytac@ege.edu.tr[2]

Abstract

Let G be a simple finite undirected and an isolate-free graph. A set S of vertices in G is a semitotal dominating set of G if it is a dominating set of G and every vertex in S is within distance 2 of another vertex of S. The semitotal domination number, $\gamma_{t2}(G)$, is the minimum cardinality of such a set. In this paper, we study the semitotal domination number for Harary graphs, which was first introduced by Frank Harary. Since Harary graphs have the maximum possible connectivity with the minimum number of edges, many researchers are interested in studying its stability properties.

2010 Mathematics Subject Classification. **05C12**. 68M10, 68R10.
Keywords. graph theory, domination, Harary graph.

1 Introduction

The domination is one of the concepts in graph theory which has attracted many researchers to work on it because it has the potential to solve many real life problems involving design and analysis of communication network as well as defense surveillance. Many variants of domination models are available in the existing literature. Among them, a few consider the distance that a vertex is from the set, which is distance domination, exponential domination, porous exponential domination, disjunctive domination, semitotal domination and etc.

We begin with a finite, connected and undirected graph $G = (V, E)$ without loops and multiple edges. The set $S \subseteq V$ of vertices in a graph G is called a dominating set if every vertex $v \in V$ is either an element of S or is adjacent to an element of S. The domination number of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G. Cockayne et al. introduced the total domination number that is well-known graph parameter in [2]. A total dominating set, abbreviated a TD-set, of a graph G with no isolated vertex is a set of vertices of G such that every vertex in $V(G)$ is adjacent to at least one vertex in S. The total domination number of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a TD-set of G. A weakly connected dominating set, abbreviated a WCD-set, is a dominating set S with the property that the graph $< S >_w$ has vertex set S and its neighbours, and edge set those edges incident with S. The weakly connected domination number $\gamma_w(G)$ is the minimum cardinality of a WCD-set. Domination number, total domination number and weakly connected domination number are graph parameters that have been studied extensively [4, 5, 10].

Recently, motivated by domination, total domination and weakly connected domination, Goddard, Henning and Mcpillan introduced semitotal domination in [3], which can be viewed as a relaxation of total domination. A set S of vertices in a graph G with no isolated vertices is a semitotal dominating set, abbreviated semi-TD-set, of G if it is a dominating set of G and every vertex in S is within distance 2 of another vertex of S. The semitotal domination number, denoted

Tbilisi Mathematical Journal 13(3) (2020), pp. 11–17.
Tbilisi Centre for Mathematical Sciences.
Received by the editors: 24 January 2019.
Accepted for publication: 20 May 2020.

by $\gamma_{t2}(G)$, is the minimum cardinality of a semi-TD-set of G. S semi-TD-set of G of cardinality $\gamma_{t2}(G)$ is called γ_{t2}-set. Studies on semitotal domination in graphs can be found in [6, 7, 8, 9, 13] and elsewhere.

The semitotal domination number is bounded below by the domination number and above by the total domination number. Hence, we have $\gamma(G) \leq \gamma_{t2}(G) \leq \gamma_t(G)$ [3], which shows that the semitotal domination numbers of interest to study since it is "squeezed" between arguably the two most important domination parameters; namely, the domination number and the total domination number.

We can consider a communication network as a Harary graph $H_{k,n}$ introduced by F. Harary in 1962 [11]. The Harary graph which generates the minimal -connected graphs on n vertices and that means we can remove up to k vertices to partition the network. If we are considering networks that are designed to distribute data to every vertex, Harary graphs allows us to be as strong as we want with a minimal number of edges. The structure of $H_{k,n}$ depends on the parities of k and n. Figure 1 shows graphs $H_{4,8}$, $H_{4,9}$, $H_{5,8}$, $H_{5,9}$, respectively. Given $1 < k < n$, can be constructed by placing n vertices around the circle equally spaced. There are three cases as below.

- Case 1: If k is even, form $H_{k,n}$ by making each vertex adjacent to the nearest $\frac{k}{2}$ vertices in each direction around the circle.
- Case 2: If k is odd and n is even, form $H_{k,n}$ by making each vertex adjacent to the nearest $\frac{k-1}{2}$ vertices in each direction and to the diametrically opposite vertex.
- Case 3: If k and n are both odd, we construct $H_{k,n}$ from $H_{k-1,n}$ and then adding edges joining vertex 1 to $1+\frac{n-1}{2}$ and $1+\frac{n+1}{2}$ and vertices i to vertex $i+\frac{n+1}{2}$ for each $2 \leq i < \frac{n+1}{2}$.

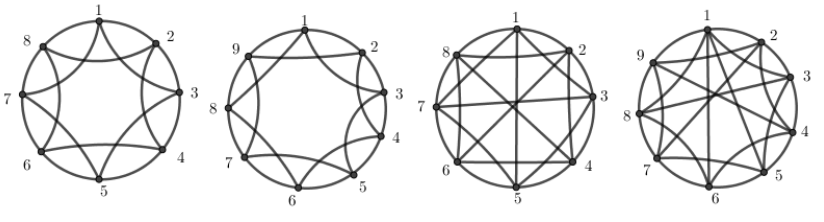


FIGURE 1. The graphs $H_{4,8}, H_{4,9}, H_{5,8}, H_{5,9}$

Further, for $i \in \{1, 2, ..., n\}$, if we visit vertex $(i + \lfloor \frac{k}{2} \rfloor)$modn or $(i - \lfloor \frac{k}{2} \rfloor)$modn, then it is called a city-tour, otherwise it is called village-tour [1]. Domination, total domination and forcing domination in Harary graphs are studied in [12]. This paper determines semitotal domination number of Harary graph $H_{k,n}$ in all cases of k and n.

For notation and graph theory terminology that are not defined herein, we refer the reader to [14]. Let $G = (V, E)$ be a graph with vertex set $V = V(G)$ of order $n = |V|$ and edge set $E = E(G)$ of size $m = |E|$, and let v be a vertex in V. We denote the degree of v in G by $deg(v)$. The maximum (minimum) degree among the vertices of G is denoted by $\Delta(G)(\delta(G), respectively)$. For any $v \in V(G)$, the open neighborhood of $v \in G$ is $N_G(v) = \{u|uv \in E(G)\}$. The closed neighborhood of $v \in G$ is defined as $N_G[v] = \{v\} \cup N_G(v)$. The distance between two vertices u, v

in G is the number of edges is a shortest path connecting them, denoted by $d_G(u, v)$. Further, throughout the paper "one unit" means if we move one unit from vertex i, we can reach vertex $i + 1$.

2 Basic results

Lemma 2.1. [3] For a graph G with every component of order $n \geq 4$, we have $\gamma_{t2}(G) \leq \frac{n}{2}$.

Observation 2.2. [3] If G is a graph with no isolated vertices,then

$$\gamma(G) \leq \gamma_{t2}(G) \leq \begin{cases} \gamma_t(G) \\ max(\gamma_w(G), 2). \end{cases}$$

Lemma 2.3. [3] For $n \geq 3$, the semitotal domination number of a path and cycle is $\gamma_{t2}(P_n) = \gamma_{t2}(C_n) = \lceil \frac{2n}{5} \rceil$.

Lemma 2.4. [3] If G has n vertices and maximum degree Δ, then

$$\gamma_{t2}(G) \geq \frac{2n}{2\Delta + 1}.$$

Lemma 2.5. [3] If G is connected and nontrivial, then
$(a)\gamma_{t2}(G) \leq 2\gamma(G) - 1$;
$(b)\gamma_t(G) \leq 2\gamma_{t2}(G) - 1$;
and these bounds are sharp.

Theorem 2.6. [13] If G is a graph with $\delta \geq 2$ and D is a minimal dominating set of G, then $V(G) \setminus D$ is a semi-TD-set of G.

Theorem 2.7. [13] If G is a graph with $\delta \geq 2$ and S is a minimal semi-TD-set of G, then $V(G) \setminus S$ is a dominating set of G.

Observation 2.8. [8] The paths P_2, P_7 and P_n, where $n \equiv 0(mod5)$, all have unique minimum semi-TD-sets.

Observation 2.9. [8] If G is a connected graph that is not a star, then there is a $\gamma_{t2}(G)$-set that contains no leaf of G.

Conjecture 2.10. [3] For all graph of order n and minimum degree at least 3, $\gamma_{t_2} \leq \frac{2n}{5}$.

3 Semitotal domination number of Harary graph

In this section, we have calculated the semitotal domination number of Harary graphs.

Theorem 3.1. Let $H_{k,n}$ be a Harary graph on n vertices, where $k = 2m$ is even. If $n = (4m+1)l+r$ and $n \geq 4m + 1$, where $0 \leq r \leq 4m$, then

$$\gamma_{t2}(H_{2m,n}) = \begin{cases} 2l, & \text{if } r = 0, \\ 2l + 1, & \text{if } 1 \leq r \leq 2m, \\ 2l + 2, & \text{if } 2m + 1 \leq r \leq 4m. \end{cases}$$

Proof. Let S be semi-TD-set of $H_{2m,n}$. As $H_{2m,n}$ is vertex transitive, we can start from any vertex v in the graph to form S. For this reason, let $v = 1 \in S$. By definiton of semitotal dominating set, there must be one vertex in S which is within distance 2 to semitotally dominate the vertex $v = 1$. Define $S^{'} = \{w|d(v,w) \leq 2\}$. Therefore we must choose $w \in S^{'} = \{v-2m, v-2m+1, ..., v-1, v+1, ..., v+2m\}$. Since S has minimum cardinality, vertex $w = v+2m$ or vertex $w = v-2m$ that is distance 2 from v must be added to S. Since $H_{2m,n}$ is vertex transitive, these two conditions are equal to each other. If $w = v+2m$, then $N(v) \cup N(w) = \{v-m, ..., v, ..., v+m, ..., v+2m, ..., v+3m\}$. This means that we can choose two vertices to semitotatlly dominate to each $4m+1$ vertices. Thus, we can consider $H_{2m,n}$ as $n = (4m+1)l+r$. S can be partitioned into two sets of S_0 and S_1, where $S_0 = \{v_i = 1 + (4m+1)i \ |0 \leq i \leq l-1\}$, $S_1 = \{w_i = v_i + 2m | v_i \in S_0\}$ and $S = S_0 \cup S_1$. If $r = 0$, then S is a γ_{t2}-set for $H_{2m,n}$ and $\gamma_{t2} = 2l$. If $1 \leq r \leq 2m$, then for remaining r vertices, the vertex $(n-m-r+1) + \lfloor \frac{r}{2} \rfloor$ must be added to S. Further, the vertex $(n-m-r+1) + \lfloor \frac{r}{2} \rfloor$ is within distance 2 to vertex 1, since $r \leq 2m$. Thus $S \cup \{(n-m-r+1) + \lfloor \frac{r}{2} \rfloor\}$ is a γ_{t2}-set for $H_{2m,n}$ and $\gamma_{t2} = 2l+1$. If $2m+1 \leq r \leq 4m$, then for remaining consecutive r vertices, two vertices which in these vertices must be added to S. Thus $S \cup \{\alpha, \beta\}$ is γ_{t2}-set for $H_{2m,n}$, where $1 \leq \beta - \alpha \leq 2m$ and α and β semitotally dominate the consecutive r vertices. Hence, we get $\gamma_{t2} = 2l+2$. Q.E.D.

Theorem 3.2. Let $H_{k,n}$ be a Harary graph on n vertices, where $k = 2m$ is even. If $n = (4m+1)l+r$ and $2m < n < 4m+1$, where $0 \leq r \leq 4m$, then $\gamma_{t2}(H_{2m,n}) = 2$.

Proof. By definition of Harary graphs, we know $n > k = 2m$. Further, since $n < 4m+1$, we get $2m+1 \leq r \leq 4m$. It is easy to see that $l = 0$ and $n = r$ for $n = (4m+1)l+r$. Under these conditions this case equals to $2m+1 \leq r \leq 4m$ case within the proof of the Theorem 3.1. As a result of this case, when we put $l = 0$ in its place in formulation $\gamma_{t2} = 2l+2$, we get $\gamma_{t2}(H_{2m,n}) = 2$ and the vertices α and β taken from Theorem 3.1. semitotally dominate all vertices of $H_{2m,n}$. Q.E.D.

Theorem 3.3. Let $H_{k,n}$ be a Harary graph on n vertices, where k is odd and n is even. If $n = kl+r$ with $0 \leq r \leq k-1$, then

$$\gamma_{t2}(H_{k,n}) = \lceil \frac{n}{k} \rceil.$$

Proof. The result is straightforward to verify for $k < n \leq 3k$. Suppose that $n > 3k$. Let S be a semi-TD-set of $H_{k,n}$. As $H_{k,n}$ is vertex transitive, we can start from any vertex v in the graph to form S. For this reason, let $v = 1 \in S$. We first establish upper bound on $\gamma_{t2}(H_{k,n})$. Let $n = kl + r$ with $0 \leq r \leq k-1$ and $S_1 = \bigcup_{i=0}^{\lceil \frac{n}{k} \rceil - 1} v_i$, where $v_i = ki+1$. In this case, after making 2city-tour plus 1 unit from any vertex of S_1, we arrive at its consecutive vertex in S_1. This implies that $n > 2\frac{k-1}{2} + 1 = k$. Let we now show that the set S_1 is a semi-TD-set of $H_{k,n}$. If $v_i \leq \frac{n}{2}$ for any $v_i \in S_1$ then there exists $\exists w_i \in S_1$, where either $w_i \in \{v_i + \frac{n}{2}, v_i + \frac{n}{2} + 1, ..., v_i + \frac{n}{2} + \frac{k-1}{2}\}$ or $w_i \in \{v_i + \frac{n}{2} - \frac{k-1}{2}, ..., v_i + \frac{n}{2} - 1, \}$. Since we know that v_i is adjacent $v_i + \frac{n}{2}$ in $H_{k,n}$, it easy to see that $d(v_i, w_i) \leq 2$. When $v_i > \frac{n}{2}$, similarly, there exists $\exists w_i \in S_1$, where either $w_i \in \{v_i - \frac{n}{2}, v_i - \frac{n}{2} + 1, ..., v_i - \frac{n}{2} + \frac{k-1}{2}\}$ or $w_i \in \{v_i - \frac{n}{2} - \frac{k-1}{2}, ..., v_i - \frac{n}{2} - 1, \}$. Since we know that v_i is adjacent $v_i - \frac{n}{2}$ in $H_{k,n}$, it easy to see that $d(v_i, w_i) \leq 2$. It is easy to see that the semitotal domination definiton is provided for this situation. Thus S_1 is a semi-TD-set of $H_{k,n}$ and then we can consider $H_{k,n}$ as $n = kl+r$, where $0 \leq r \leq k-1$. In all cases of r, the set $S = S_1$ is semi-TD-set

of $H_{k,n}$. Hence,

$$\gamma_{t2}(H_{k,n}) \leq \lceil \frac{n}{k} \rceil. \tag{3.1}$$

We now go on the prove the reverse inequality. Let D be a γ_{t2}-set of $H_{k,n}$. Assume that any v_i vertex in D can not reach the vertex to its consecutive vertex through 2 city-tor plus 1 unit. This acceptance is examined in two cases.

Suppose firstly that after making 2 city-tour from any vertex, we arrive at its consecutive vertex. Which means that $n = (k-1)l_1 + r_1$ with $0 \leq r_1 \leq k-2$. Let $S_2 = \bigcup_{i=0}^{\lceil \frac{n}{k-1} \rceil - 1} v_i$, where $v_i = (k-1)i+1$. In this case for all cases of r_1, $D = S_2$ is a semi-TD-set of $H_{k,n}$. Hence, $\gamma_{t2}(H_{k,n}) = |D| \leq \lceil \frac{n}{k-1} \rceil$. This contradicts our earlier upper bound (3.1) established on $\gamma_{t2}(H_{k,n})$.

Suppose secondly that after making 2 city-tour plus 2 units from any vertex, we arrive at the consecutive vertex. Which means that $n = (k+1)l_2 + r_2$, with $0 \leq r_2 \leq k$. Let $S_3 = \bigcup_{i=0}^{\lceil \frac{n}{k+1} \rceil - 1} v_i$, where $v_i = (k+1)i + 1$. When this S_3 set is considered, either the undominated vertices remain or the semi-total definiton is not provided for the vertices of the S_3 set. Therefore, we must add some new vertices to S_3, and then this make the cardinality of the set is larger than the cardinality of S. This contradicts our earlier upper bound (3.1) established on $\gamma_{t2}(H_{k,n})$.

Therefore, γ_{t2}-set D has two vertices which are arrived to each other after making 2 city-tour plus 1 unit which means that $\{1, k+1\} \subseteq D$ and then $D \cap \{k+2, k+3, ..., 2k\} = \varnothing$. Let $D' = D - \{1, k+1\}$. It follows that $|D'| = |D| - 2$. Let G' be a graph obtained from $G = H_{k,n}$ by deleting the vertices i for $1 \leq i \leq 2k$, adding the new edge $n(2k+1)$ between the vertices n and $2k+1$ and reorganizing all other edges except for the edges on cycle. It follows that $G' = H_{k,n'}$, where $n' = n - 2k > k$. This implies that $n > 3k$. Since D is a semi-TD-set of G, the set D' is a semi-TD-set of G'. It follows that $\gamma_{t2}(G') \leq |D'| = |D| - 2$. When we apply the inductive hypothesis to G', we have $\gamma_{t2}(G') = \lceil \frac{n'}{k} \rceil = \lceil \frac{n}{k} \rceil - 2$ for all cases of r, in which $n = kl + r$. This implies that

$$\gamma_{t2}(H_{k,n}) = |D| \geq \lceil \frac{n}{k} \rceil. \tag{3.2}$$

The proof is completed from (3.1) and (3.2). Q.E.D.

Theorem 3.4. Let $H_{2m+1,2n+1}$ be a Harary graph on $2n+1$ vertices. If $2n+1 = (4m+2)l + 2r + 1$, where $0 \leq r \leq 2m$, then

$$\gamma_{t2}(H_{2m+1,2n+1}) = \begin{cases} 2l+1, & \text{if } 0 \leq r \leq m, \\ 2l+2, & \text{if } m+1 \leq r \leq 2m. \end{cases}$$

Proof. In $H_{2m+1,2n+1}$, vertex 1 connected to vertices $1+n$ and $1+(n+1)$ and vertex i connected to vertex $i+(n+1)$ for $2 \leq i \leq n$. Further, $H_{2m+1,2n+1}$ is not vertex transitive. Vertex 1 is directly connected vertex $1+(n+1)$ which divide into graph almost half. Thus, we assume that diametrically opposite vertex of 1 is vertex $1+(n+1)$. The degree of vertex 1 is $2m+2$ and the degree of the other vertices of $H_{2m+1,2n+1}$ is $2m+1$. Thus, vertex 1 is considered as originator vertex. Let S be semi-TD-set of $H_{2m+1,2n+1}$ and $1 \in S$. Similar to proof of Theorem 3.1., since S

has minimum cardinality, in order to semitotally dominate the vertex 1, vertex $w = 1+(n+1)+m$ that is at distance 2 from vertex 1 must be added to S. Hence, $4m+2$ vertices are semitotally dominated and then we can consider $H_{2m+1,2n+1}$ as $2n+1 = (4m+2)l+2r+1$ and $S = S_0 \cup S_1$, where $S_0 = \{v_i = 1+(2m+1)i | 0 \leq i \leq l\}$ and $S_1 = \{w_i = v_i + (n+1) + m | v_i \in S_0\}$. If $0 \leq r \leq m$, then since $l = \lfloor \frac{2n+1}{4m+2} \rfloor$ and $r \leq m$, the vertex $w_l = 1+(2m+1)l+n+1+m > 2n+1$, $w_l \notin S$. Now obviously the vertex $v_l = 1+(2m+1)l$ is semitotally dominated by 1 since $r \leq m$ and $l = \lfloor \frac{n}{2m+1} \rfloor$. Further, the set of $S - \{w_l\}$ semitotally dominate all vertices of $H_{2m+1,2n+1}$. Thus, $\gamma_{t2} = 2(l+1) - 1 = 2l+1$. If $m+1 \leq r \leq 2m$, then all vertices of $H_{2m+1,2n+1}$ are semitotally dominated by S. Thus, $\gamma_{t2} = 2l+2$. Q.E.D.

Acknowledgement

This paper is supported by Ege University PhD Scientific Research Project (BAP) under project number FDK-2018-20266.

References

[1] P. Bhabak, H.A. Harutyunyan and S. Tanna, *Broadcasting in Harary-Like Graphs.*, In computational Science and Engineering (CSE), 2014 IEEE 17th International Conference (2014) 1269–1276.

[2] E.J. Cockayne, R.M. Dawes and S.T. Hedetniemi, *Total domination in graphs*, Networks **10(3)** (1980) 211–219.

[3] W. Goddard, M.A. Henning, C.A. McPillan, *Semitotal domination in graphs*, Utilitas Math. **94** (2014) 67–81.

[4] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, P.J., *Fundamentals of Domination in Graphs*, Marcel Dekker Inc., New York **208** (1998).

[5] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, P.J., *Domination in Graphs: Advanced Topics*, Marcel Dekker Inc., New York (1998).

[6] M.A. Henning, *Edge Weighting Functions on Semitotal Dominating Sets*, Graphs Combin. **33** (2017) 403–417.

[7] M.A. Henning and A.J. Marcon, *On matching and semitotal domination in graphs*, Discr. Math. **324** (2014) 13–18.

[8] M.A. Henning and A.J. Marcon, *Vertices contained in all or in no minimum semitotal dominating set of a tree*, Discuss. Math. Graph Theory **36** (2014) 71–93.

[9] M.A. Henning and A.J. Marcon, *Semitotal domination in claw-free cubic graphs*, Ann. Comb. **20(4)** (2016) 799–813.

[10] M.A. Henning and A. Yeo, *Total domination in graphs*, Springer Monographs in Mathematics,(2013) ISBN:978-1-4614-6524-9(print) 978-1-4614-6525-6(online).

[11] F. Harary, *The maximum connectivity of a graph*, Proceedings of the National Academy of Sciences **48(7)** (1962) 1142–1146.

[12] A. Khodkar, D. Mojdeh, A.P. Kazemi, *Domination in Harary graphs*, Bull. ICA**49** (2007) 61–78.

[13] A.J. Marcon, *Semitotal domination in graphs*, PhD dissertation, University Johannesburg.

[14] G. Chartrand and L. Lesniak, *Graphs and Digraphs*, Chapman and Hall, California (1996).