New existence criterion of infinitely many solutions for partial discrete Dirichlet problems

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Abstract

In this paper, we present new existence criterion of solutions for partial discrete Dirichlet problems. More precisely, under appropriate assumptions on the nonlinearities that possesses the appropriate behaviors related to the eigenvalues at infinity, the existence result of infinitely many solutions is discussed.

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1 Introduction

Discrete boundary value problems have been intensively studied in the last decade. The modeling of certain nonlinear problems from biological neural networks, economics, optimal control, dynamical systems engineering, and other areas of study have led to the rapid development of the theory of difference equations. As is well known, the existence and non-existence of solutions for nonlinear algebraic systems are very important. In this article, we consider the discrete system:

$$[u(i+1,j)-2u(i,j)+u(i-1,j)]+[u(i,j+1)-2u(i,j)+u(i,j-1)]+f((i,j),u(i,j))=0, \quad (1.1)$$

with boundary conditions

$$u(i,0)=u(i,N+1)=0, \quad \forall i\in[1,M],$$

$$u(0,j)=u(M+1,j)=0, \quad \forall j\in[1,N],$$

where $[1,M]=\{1,2,...,M\}$, $[1,N]=\{1,2,...,N\}$, $f:[1,M]\times[1,N]\times\mathbb{R}\to\mathbb{R}$ denotes a continuous function.

Discrete problems involving functions with two or several discrete variables are very relevant and deeply have been investigated. Such an active interest is due to the advance of modern digital computing devices. In order to establish existence and multiplicity of solutions for discrete problems under some restrictions on f (sublinear or superlinear at infinity), several authors exploited various abstract methods as the topological degree theory, the fixed point theorem, upper and lower solutions method, the Rabinowitz's bifurcation theorem, Morse theory and the critical point theorem; see, for instance, [1, 3, 12] for related topics. We observe that in [9], Ji and Yang studied the structure of the spectrum of the problem (E_λ^f)

$$[u(i+1,j)-2u(i,j)+u(i-1,j)]+[u(i,j+1)-2u(i,j)+u(i,j-1)]$$
$$+\lambda f((i,j),u(i,j))=0, \quad \forall (i,j)\in[1,M]\times[1,N]$$

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with boundary conditions

$$u(i,0) = u(i,N+1) = 0, \quad \forall i \in [1,M],$$

$$u(0,j) = u(M+1,j) = 0, \quad \forall j \in [1,N],$$

where λ is a positive real parameter. they investigated the existence of a positive eigenvector corresponding to the smallest eigenvalue. Successively, in [7], Galewski and Orpel, through variational methods and by using some monotonicity results, analyse problem (E_λ^f) and find the existence of one solution. In recent literatures, in [8] by means of variational methods, the existence of infinitely many solutions for the problem (E_λ^f) has been established. More precisely, unbounded intervals of parameters such that the problem (E_λ^f) admits either an unbounded sequence of solutions, provided that the nonlinearity has a suitable behaviour at infinity, or a pairwise distinct sequence of solutions that strongly converge to zero if a similar behaviour occurs at zero has been determined. In particular, in [4] the problem (E_λ^f) under appropriate assumptions on the nonlinearities, by means of two critical points theorems, exact collections of parameters such that the treated problems admit at least three solutions, were determined. For a complete and exhaustive overview on variational methods we refer the reader to the monograph [6].

In this paper, we shall discuss the existence of infinitely many solutions for problem (1.1) by assuming that the nonlinear term possesses the appropriate behaviors related to the eigenvalues at infinity. As far as the authors know, it is essential to impose the oscillating behaviors on the nonlinear term for the sake of the sequence solutions under most circumstances, as examples in [2, 10]. Applying a new technique to deal with problem (1.1), it is not indispensable to impose the oscillating behaviors on the nonlinear term for the first time in this literature. Example 3.7 illustrates this result vividly.

For an overview on this subject, we refer the reader to the papers [10, 11].

2 Preliminaries

Let $(X, \|.\|)$ be a finite dimensional Banach space. As ambient space X, we consider the MN-dimensional Banach space $\mathbb{R}^{MN}$ endowed by the norm $\|u\| := (\sum_{k=1}^{MN} u_k^2)^{\frac{1}{2}}$. Moreover, $\mathfrak{M}_{\ell\times\ell}$, for every $\ell \in \mathbb{N}$, stands for the linear space of all the matrices of order ℓ with real entries. Let $h : [1,M] \times [1,N] \to [1,MN]$ be the bijection defined by $h(i,j) := i + M(j-1)$, for every $(i,j) \in [1,M] \times [1,N]$. It can be displayed, as pointed out in [8]. Let us denote $w_k := u(h^{-1}(k))$ and $g_k(w_k) := f(h^{-1}(k), w_k)$, for every $k \in [1,MN]$. So, problem (1.1) can be written as the nonlinear algebraic system

$$Aw = g(w),$$

where A is given by

$$A := \begin{pmatrix} L & -I_M & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ -I_M & L & -I_M & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & -I_M & L & -I_M & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & -I_M & L & \cdots & 0 & 0 & 0 & 0 \\ & & & & \ddots & & & & \\ 0 & 0 & 0 & 0 & \cdots & L & -I_M & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & -I_M & L & -I_M & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -I_M & L & -I_M \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -I_M & L \end{pmatrix} \in \mathfrak{M}_{MN\times MN}(\mathbb{R}),$$

L is defined by

$$L := \begin{pmatrix} 4 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ -1 & 4 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & -1 & 4 & -1 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 4 & \cdots & 0 & 0 & 0 & 0 \\ & & & & \ddots & & & & \\ 0 & 0 & 0 & 0 & \cdots & 4 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & -1 & 4 & -1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 4 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -1 & 4 \end{pmatrix} \in \mathfrak{M}_{M\times M}(\mathbb{R})$$

$I_M \in \mathfrak{M}_{M\times M}(\mathbb{R})$ is the identity matrix and $g(w) := (g_1(w_1), ..., g_{MN}(w_{MN}))^t$ for every $w \in X$. Moreover, a direct computation shows that

$$\sum_{k=1}^{MN} a_{ij} = 2(M+1)N - 2(N-1)M = 2(M+N)$$

Example 2.1. For $M = N = 3$ problem (1.1) assumes the form

$$\begin{cases} 4u(1,1) - u(2,1) - u(1,2) = f((1,1), u(1,1)) \\ -u(3,1) + 4u(2,1) - u(1,1) - u(2,2) = f((2,1), u(2,1)) \\ -u(2,1) - u(3,2) + 4u(3,1) = f((3,1), u(3,1)) \\ 4u(1,2) - u(2,2) - u(1,1) = f((1,2), u(1,2)) \\ -u(3,2) + 4u(2,2) - u(1,2) - u(2,1) = f((2,2), u(2,2)) \\ -u(2,2) - u(3,1) + 4u(3,2) = f((3,2), u(3,2)) \\ 4u(1,3) - u(2,3) - u(1,2) = f((1,3), u(1,3)) \\ -u(3,3) + 4u(2,3) - u(1,3) - u(2,2) = f((2,3), u(2,3)) \\ -u(2,3) - u(3,2) + 4u(3,3) = f((3,3), u(3,3)) \end{cases}$$

and, through map h, it can be also written as follows:

$$\begin{cases} 4w_1 - w_2 - w_4 = g_1(w_1) \\ -w_1 + 4w_2 - w_3 - w_5 = g_2(w_2) \\ -w_2 + 4w_3 - w_6 = g_3(w_3) \\ -w_1 + 4w_4 - w_5 = g_4(w_4) \\ -w_2 - w_4 + 4w_5 - w_6 = g_5(w_5) \\ -w_3 - w_5 + 4w_6 = g_6(w_6) \\ -w_4 + 4w_7 - w_8 = g_7(w_7) \\ -w_5 - w_7 + 4w_8 - w_9 = g_8(w_8) \\ -w_6 - w_8 + 4w_9 = g_9(w_9) \end{cases}$$

where $w_k := u(h^{-1}(k))$ and $g_k(w_k) := f(h^{-1}(k), w_k)$, for every $k \in [1, 9]$. Clearly, in this case

$$A = \begin{pmatrix} 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 4 & 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 4 & -1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 4 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 4 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 4 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 \end{pmatrix}.$$

Further, matrix A is positive-definite, see [9]; by denoting as $\lambda_1, ..., \lambda_{MN}$ its eigenvalues satisfying the ordering $0 < \lambda_1 < \lambda_2 \leq ... \leq \lambda_{MN}$ it follows that, for every $w \in X$,

$$\lambda_1 \|w\|^2 \leq w^t A w \leq \lambda_{MN} \|w\|^2. \tag{2.1}$$

At this point it is easy to observe that the solutions of (1.1) are the critical points of the C^1-functional

$$I(w) = \frac{w^t A w}{2} - \sum_{k=1}^{MN} \int_0^{w_k} g_k(t)dt, \quad \forall w \in X.$$

Indeed, a column vector $z = (z_1, ..., z_{MN})^t \in X$ is a critical point of the functional I if the gradient of I at z is zero, i.e.,

$$\frac{\partial I(w)}{\partial w_1} \mid_{w=z} = 0, \frac{\partial I(w)}{\partial w_2} \mid_{w=z} = 0, ..., \frac{\partial I(w)}{\partial w_{MN}} \mid_{w=z} = 0.$$

Moreover, it is well known that

$$\frac{\partial w^t A w}{\partial w_k} = 2(Aw)_k,$$

where $(Aw)_k := \sum_{j=1}^{MN} a_{kj} w_j$. Thus

$$\frac{\partial I(w)}{\partial w_k} = (Aw)_k - g_k(w_k).$$

Now we recall the following strong maximum principle for the problem (1.1) as given in [8, Proposition 2.2].

Proposition 2.2. Assume that there exists $\hat{u} : [0, M+1] \times [0, N+1] \to \mathbb{R}$ such that

$$[\hat{u}(i+1,j) - 2\hat{u}(i,j) + \hat{u}(i-1,j)] + [\hat{u}(i,j+1) - 2\hat{u}(i,j) + \hat{u}(i,j-1)] \leq 0$$

for every $(i,j) \in [0,M] \times [0,N]$, and

$$\hat{u}(i,0) = \hat{u}(i,N+1) = 0, \quad \forall i \in [1,M],$$
$$\hat{u}(0,j) = \hat{u}(M+1,j) = 0, \quad \forall j \in [1,N].$$

Then, either $\hat{u}$ is the identically zero function or $\hat{u}(i,j) > 0$ for every $(i,j) \in [0,M] \times [0,N]$.

Note that when $f : [0,M] \times [0,N] \times \mathbb{R} \to \mathbb{R}$ is a non-negative function, the above proposition ensures that every solution of problem (1.1) is either zero or positive.

By denoting $\bar{w} := \frac{\sum_{k=1}^{MN} w_k}{MN} \in \mathbb{R}$ and $\tilde{X} := \{w \in X : \sum_{k=1}^{MN} w_k = 0\}$, We can gap X into $\mathbb{R} \oplus \tilde{X}$ and so, for every $w \in X$ can be decomposed as

$$w = \bar{w} + \tilde{w}, \quad \text{with} \quad \tilde{w} \in \tilde{X}, \quad \text{and} \quad \bar{w} \in \mathbb{R}.$$

$I(w)$ can be expressed as the following equivalent form

$$I(w) = \frac{w^t A w}{2} - \sum_{k=1}^{MN} F_k(w_k),$$

where $F_k(t) := \int_0^t g_k(s)ds$, for every $t \in \mathbb{R}$ and $k \in [1, MN]$.

To introduce the following Lemma 2.3 and get the main result, we need the following condition.

(H_1) $k \to g_k(t)$ is measurable for $t \in \mathbb{R}$, $|g_k(t)| \leq \alpha(k)|t|$ with $\max\{\alpha(k)\} < \lambda_1$ for $k \in [1, MN]$ and $t \in \mathbb{R}$.

Lemma 2.3. Assume that (H_1) is fulfilled, then any sequence $\{w_n\} \subset X$ is bounded when both $I(w_n)$ and $\bar{w}_n$ are bounded.

Proof. Taking both (2.1) and (H_1) into account, one has

$$\begin{aligned}
I(w) &= \frac{w^t A w}{2} - \sum_{k=1}^{MN} F_k(w_k) \geq \frac{\lambda_1}{2}\|w\|^2 - \sum_{k=1}^{MN} F_k(w_k) \\
&= \frac{\lambda_1}{2}\sum_{k=1}^{MN}(\bar{w}^2 + \tilde{w}^2 + 2\bar{w}\tilde{w}) - \sum_{k=1}^{MN}\int_0^1 \frac{d}{ds}F_k(\bar{w}_k + s\tilde{w}_k)ds - \sum_{k=1}^{MN} F_k(\bar{w}) \\
&= \frac{\lambda_1}{2}\|\tilde{w}\|^2 + \frac{MN\lambda_1}{2}\bar{w}^2 + \lambda_1\bar{w}\sum_{k=1}^{MN}\tilde{w} - \sum_{k=1}^{MN}\int_0^1 g_k(\bar{w} + s\tilde{w}_k)\tilde{w}_k ds - \sum_{k=1}^{MN} F_k(\bar{w}) \\
&\geq \frac{\lambda_1}{2}\|\tilde{w}\|^2 + \frac{MN\lambda_1}{2}\bar{w}^2 - \sum_{k=1}^{MN}\int_0^1 \alpha(k)|\bar{w} + s\tilde{w}_k|\tilde{w}_k - \sum_{k=1}^{MN} F_k(\bar{w}) \\
&\geq \frac{\lambda_1}{2}\|\tilde{w}\|^2 + \frac{MN\lambda_1}{2}\bar{w}^2 - |\bar{w}|\max\{\alpha(k)\}\sum_{k=1}^{MN}|\tilde{w}_k| - \frac{1}{2}\max\{\alpha(k)\}\sum_{k=1}^{MN}|\tilde{w}_k|^2 \\
&\quad - \sum_{k=1}^{MN} F_k(\bar{w})
\end{aligned}$$

$$\geq \frac{1}{2}(\lambda_1 - \max\{\alpha(k)\})\|\tilde{w}\|^2 - |\bar{w}|\max\{\alpha(k)\}\sqrt{MN}\|\tilde{w}\| - \sum_{k=1}^{MN} F_k(\bar{w}) + \frac{MN\lambda_1}{2}\bar{w}^2. \tag{2.2}$$

Thus, we conclude that any sequence $\{w_n\} \subset X$ is bounded when both $I(w_n)$ and $\bar{w}_n$ are bounded. Q.E.D.

3 Main results

Theorem 3.1. Suppose that both (H_1) and the following conditions are valid

(H_2) $\limsup_{s\to\infty} \inf_{a\in\mathbb{R},\ \ |a|=s} \big((M+N)a^2 - \sum_{k=1}^{MN} F_k(a)\big) = +\infty$;

(H_3) $\liminf_{r\to\infty} \sup_{b\in\mathbb{R},\ \ |b|=r} \big(\frac{MN(b\max\{\alpha(k)\})^2}{2(\lambda_1-\max\{\alpha(k)\})} + \sum_{k=1}^{MN} F_k(b) - \frac{MN\lambda_1}{2}b^2\big) < +\infty$.

Then problem (1.1) admits infinitely many solutions.

We will prove Theorem 3.1 with a sequence of claims.

Claim 3.2. There exists a sequence $\{s_n\}$ with $\lim_{n\to\infty} s_n = +\infty$ such that

$$\lim_{n\to\infty} \big(\inf_{a\in\mathbb{R},\ \ |a|=s_n} I(a)\big) = +\infty.$$

Proof. Choose $w = (w_1, ..., w_{MN})$ with $w_k = a$ for every $k \in [1, MN]$. We have $w^t A w = 2(M+N)a^2$, one has

$$I(a) = (M+N)a^2 - \sum_{k=1}^{MN} F_k(a), \quad \forall\ a \in \mathbb{R}.$$

Following from (H_2), the claim is proved. Q.E.D.

Claim 3.3. There exists a sequence $\{r_n\}$ with $\lim_{n\to\infty} r_n = +\infty$. For some constant L, such that $\lim_{n\to\infty} \big(\inf_{b\in\mathbb{R},\ |b|=r_n,\ \tilde{w}\in\tilde{X}} I(b+\tilde{w})\big) =: L > -\infty$.

Proof. By thanks for the inequality (2.2), one has

$$I(w) \geq -\frac{MN(|\bar{w}|\max\{\alpha(k)\})^2}{2(\lambda_1 - \max\{\alpha(k)\})} - \sum_{k=1}^{MN} F_k(\bar{w}) + \frac{MN\lambda_1}{2}\bar{w}^2, \quad \forall\ w \in X. \tag{3.1}$$

Letting $r > 0$, $b \in \mathbb{R}$ as $|b| = r$ and replacing b with $\bar{w}$ in (3.1), one has

$$\inf_{b\in\mathbb{R},\ |b|=r,\ \tilde{w}\in\tilde{X}} I(b+\tilde{w}) \geq \inf_{b\in\mathbb{R},\ |b|=r} \big(- \frac{MN(b\max\{\alpha(k)\})^2}{2(\lambda_1 - \max\{\alpha(k)\})} - \sum_{k=1}^{MN} F_k(b) + \frac{MN\lambda_1}{2}b^2\big).$$

Following from (H_3), the claim is proved. Q.E.D.

Claim 3.4. There exists a sequence $\{w_n^*\}$ such that $\{w_n^*\}$ is the local minimum point sequence of I.

Proof. Set $B_n = \{w \in X : |\bar{w}| \leq s_n\}$ for all $n \in \mathbb{N}$.
By (3.1), one has $-\frac{MN(|\bar{w}|\max\{\alpha(k)\})^2}{2(\lambda_1-\max\{\alpha(k)\})} - \sum_{k=1}^{MN} F_k(\bar{w}) + \frac{MN\lambda_1}{2}\bar{w}^2$ that is bounded on B_n. It follows that $I(w)$ is bounded from below on B_n by (3.1). Let

$$\mu_n := \inf_{w\in B_n} I(w), \quad \forall\ n \in \mathbb{N} \tag{3.2}$$

and $\{w_m\} \subset B_n$ be a minimizing sequence such that $I(w_m) \to \mu_n$ for $m \to \infty$, one has $\{w_m\}$ is bounded on X by Lemma 2.3. Since X is a finite dimensional Banach space, we get that $\{w_m\}$ possesses a convergent subsequence with $w_m \to w_n^* \in X$. More than that, $B_n \subset X$ is a closed convex subset. Then, one has $w_n^* \in B_n$. Furthermore, by (2.2), one has I is weakly lower semi-continuous. It follows that

$$\mu_n = \inf_{w\in B_n} I(w) \leq I(w_n^*) \leq \lim I(w_m) = \mu_n,$$

which gives rise to

$$\inf_{w\in B_n} I(w) = I(w_n^*) = \mu_n.$$

Then, the claim is proved. Q.E.D.

Claim 3.5. The local minimum point sequence $\{w_n^*\}$ is the critical point sequence of I and $w_n^* \notin \partial B_n$.

Proof. Let $B_{m_n} = \{w \in X : |\bar{w}| \leq s_{m_n}\}$ and $\partial B_{m_n} = \{b \in \mathbb{R} : |b| = s_{m_n}\} + \tilde{X}$. By Claim 3.3 for some constant $c \in (L, +\infty)$, there exist a subsequence $\{r_{m_n}\}$ of $\{r_n\}$ and some positive integer $n_0 \in \mathbb{N}$ such that $\inf_{\bar{w}=r_{m_n}} I(w) < c$, for all $n > n_0$. Fixing a subsequence $\{s_{m_n}\}$ of $\{s_n\}$ such that $0 < r_{m_n} < s_{m_n}$, one has

$$I(w_{m_n}^*) = \inf_{w\in B_{m_n}} I(w) = \inf_{\bar{w}\leq s_{m_n}} I(w) \leq \inf_{\bar{w}\leq r_{m_n}} I(w) < \inf_{\bar{w}= r_{m_n}} I(w) < c, \tag{3.3}$$

for all $n > n_0$. For the above constant c, by Claim 3.2, there exists some positive integer $n_1 \in \mathbb{N}$ such that

$$\inf_{w\in \partial B_{m_n}} I(w) > c, \tag{3.4}$$

for all $n > n_1$. Combining (3.3) with (3.4), one has $w_{m_n}^* \notin \partial B_{m_n}$ for $n > \max\{n_0, n_1\}$. Together with (3.2), one has $I(w_{m_n}^*) = 0$. we conclude that $w_{m_n}^*$ is the critical point of I and $w_{m_n}^* \notin \partial B_{m_n}$. Q.E.D.

Claim 3.6. The local minimum point sequence $\{w_n^*\}$ is the solution sequence of problem (1.1).

Proof. By Claim 3.4, $w_n^* \in X$. By Claim 3.5, $w_n^* \in \mathrm{int}B_n$. Then, B_n contains an open $X-$neighborhood of w_n^*. It follows that w_n^* is the free critical point of I in B_n and the critical point of I in X. Then, one has $\{w_n^*\} \subset X$ is the solution sequence of problem (1.1). Q.E.D.

In fact, by (3.3), one has

$$I(w_n^*) \leq \inf_{a\in\mathbb{R},\ \bar{a}=r_n} I(a),$$

for all $n > \max\{n_0, n_1\}$. So we conclude that $\lim_{n\to\infty} I(w_n^*) \leq c$.

Eventually, we present a concrete application of Theorem 3.1.

Example 3.7. Choosing $M = N = 2$ and $\alpha(k) = 1$ and $g_k(t) = \frac{t^3}{1+t^2}$, one has $\sum_{k=1}^{4} F_k(t) = \sum_{k=1}^{4} \frac{1}{2}(t^2 - \ln|1+t^2|)$. Indeed, the matrix A associated with the problem is

$$A = \begin{pmatrix} 4 & -1 & -1 & 0 \\ -1 & 4 & 0 & -1 \\ -1 & 0 & 4 & -1 \\ 0 & -1 & -1 & 4 \end{pmatrix}.$$

Clearly, its characteristic polynomial can be easily computed obtaining that

$$\begin{aligned} p(\lambda) &:= \det(A - \lambda I_4) = \lambda^4 - 16\lambda^3 + 92\lambda^2 - 224\lambda + 192 \\ &= (\lambda - 2)(\lambda - 4)^2(\lambda - 6). \end{aligned}$$

Hence, the first eigenvalue λ_1 is equal to 2. Moreover, by setting $w_n = \sqrt{n}$ for $n \in \mathbb{N}$, one has

$$\lim_{n\to\infty} \big((M+N)w_n^2 - \sum_{k=1}^{MN} F_k(u_n)\big) \to +\infty$$

and

$$\lim_{n\to\infty} \big(\frac{MN(w_n \max\{\alpha(k)\})^2}{2(\lambda_1 - \max\{\alpha(k)\})} + \sum_{k=1}^{MN} F_k(w_n) - \frac{MN\lambda_1}{2} w_n^2\big) \to -\infty$$

Thus, (H_1), (H_2) and (H_3) are valid. Hence, problem (1.1) admits infinitely many solutions.

4 Conclusion

In this paper, we have discussed the existence of infinitely many solutions for the problem (1.1), by proving a sequence of claims, which has not been investigated in the literature so far. By applying critical point theory, for the problem (E_λ^f) in [4], the existence of at least three positive solutions was discussed, while in [5], by using two consequences of a local minimum theorem, the existence of at least one nontrivial solution for the problem (E_λ^f) was ensured. Here, we have not investigated the uniqueness problem, our assumptions just allow us to discuss the existence of infinitely many solutions for the problem (1.1).

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