Meromorphic solutions of some non-linear $q-$shift difference equations

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Abstract

With the help of Nevanlinna theory, we investigate the existence of meromorphic solution of certain class of q-difference differential equations and consequently we are able to show that no transcendental meromorphic function with some conditions will satisfy such equations.

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1 Introduction and results

A meromorphic function which always mean meromorphic in the complex plane $\mathbb{C}$. We adopt the basic notations in the Nevanlinna's Value distribution theory of meromorphic functions ([17], [21]).

Complex differential equations and complex difference equations play an important role in the Complex analysis. Recently the study of meromorphic solutions of complex differential or difference equations by Nevanlinna theory approach has become a topic of great interest.

Throughout the paper we denote by $b_j(z)$ meromorphic functions which satisfy $T(r, b_j(z)) = S(r, f)$ for $j = 0, 1, ..., n$.
If $b_n(z) \not\equiv 0$, we call

$$P[f] = b_n f^n + b_{n-1} f^{n-1} + ... + b_1 f + b_0$$

a polynomial in f with degree n.
If $n_0, n_1, ..., n_k$ are non negative integers, we say

$$M[f] = f^{n_0}(f')^{n_1}...(f^{(k)})^{n_k}$$

a differential monomial in f of degree $\gamma_M = n_0 + n_1 + ... + n_k$ and of weight $\Gamma_M = n_0 + 2n_1 + ... + (k+1)n_k$.
If $M_1, M_2..., M_s$ are differential monomials in f, then we call

$$Q[f] = \sum_{j=1}^{s} d_j(z) M_j[f]$$

a differential polynomial in f. The degree γ_Q and the weight Γ_Q of $Q[f]$ are defined by $\gamma_Q = \max\limits_{1\le j\le s} \gamma_{M_j}$ and $\Gamma_Q = \max\limits_{1\le j\le s} \Gamma_{M_j}$ respectively.

In 2014, Pujari proved the following two theorems.

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Theorem 1.A ([19]). No transcendental meromorphic function f of finite order ρ with $N(r,f) = O(r^{\rho-1+\varepsilon}) + S(r,f)$ will satisfy the non-linear difference equation of the form

$$a_1(z)P(f)f(z+c) + a_2(z)f(z+c) + a_3(z) \equiv 0$$

where $c \in \mathbb{C}$, $a_1(z)(\not\equiv 0)$, $a_2(z)$ and $a_3(z)(\not\equiv 0)$ are small functions in the sense of $T(r,a_i) = O(r^{\rho-1+\varepsilon}) + S(r,f)$, $i = 1,2,3$ and

$$P(f) = b_n f^n + b_{n-1}f^{n-1} + ... + b_1 f + b_0$$

where n is a positive integer, $b_n(\not\equiv 0)$, $b_{n-1}, ..., b_0$ are small functions in the sense of $T(r,b_j) = O(r^{\rho-1+\varepsilon}) + S(r,f)$, $j = 1,2,...,n$.

Theorem 1.B ([19]). No transcendental meromorphic function f of finite order ρ with $N(r,f) + N(r,\frac{1}{f}) = O(r^{\rho-1+\varepsilon}) + S(r,f)$ will satisfy the difference equation of the form

$$a_1(z)P(f)\Pi(f) + a_2(z)\Pi(f) + a_3(z) \equiv 0$$

where $n \geq 1$, $a_1(z)(\not\equiv 0)$, $a_2(z)$ and $a_3(z)(\not\equiv 0)$ are small functions in the sense of $T(r,a_i) = O(r^{\rho-1+\varepsilon}) + S(r,f)$, $i = 1,2,3$ and

$$\Pi(f) = \sum_{\lambda} a_\lambda(z) f(z)^{l_0} f(z+c_1)^{l_1} ... f(z+c_\lambda)^{l_\lambda}$$

is a difference polynomial of degree n, where $n = \max \sum_{j=1}^{\lambda} l_j$ and $c_1, c_2, ..., c_\lambda$ are distinct values in $\mathbb{C}$ and $T(r,a_\lambda) = S(r,f)$.

In this paper, we consider another type of non-linear $q-$difference differential equation and proved the following theorems.

Theorem 1.1. No transcendental meromorphic function $f(z)$ of zero order with $N(r,f(z)) = S(r,f(z))$ will satisfy the non-linear $q-$difference differential equation of the form

$$a_1(z)F(f)f(qz+c) + a_2(z)f(qz+c) + a_3(z) \equiv 0, \tag{1.1}$$

where $q, c \in \mathbb{C} \setminus \{0\}$, $a_1(z)(\not\equiv 0)$, $a_2(z)$ and $a_3(z)(\not\equiv 0)$ are small functions in the sense of $T(r,a_i) = S(r,f)$, $i = 1,2,3$ and the polynomial

$$F(f) = b_n f^n + b_{n-1}f^{n-1} + Q[f] \quad for \quad n \geq 1,$$

where $Q[f]$ is a differential polynomial in f of degree at most $n-2$.

Theorem 1.2. No transcendental meromorphic function $f(z)$ of finite order ρ with $N(r,f(z)) = O(r^{\rho-1+\varepsilon}) + S(r,f)$ will satisfy the non-linear difference differential equation of the form

$$a_1(z)F(f)f(z+c) + a_2(z)f(z+c) + a_3(z) \equiv 0, \tag{1.2}$$

where $c \in \mathbb{C} \setminus \{0\}$, $a_1(z)(\not\equiv 0)$, $a_2(z)$ and $a_3(z)(\not\equiv 0)$ are small functions in the sense of $T(r,a_i) = O(r^{\rho-1+\varepsilon}) + S(r,f)$ for $i = 1,2,3$ and $F(f)$ is defined as in the Theorem 1.1.

Theorem 1.3. No transcendental meromorphic function $f(z)$ of zero order with $N(r, f(z))) + N(r, \frac{1}{f(z)}) = S(r, f(z))$ will satisfy the $q-$difference differential equation of the form

$$a_1(z)F(f)U(f) + a_2(z)U(f) + a_3(z) \equiv 0, \tag{1.3}$$

where $a_1(z)(\not\equiv 0)$, $a_2(z)$ and $a_3(z)(\not\equiv 0)$ are small functions in the sense of $T(r, a_i) = S(r, f)$ for $i = 1, 2, 3$ and

$$U(f) = \sum_{\mu} e_\mu(z) f(z)^{v_{\mu,0}} f(q_1 z + c_1)^{v_{\mu,1}} ... f(q_t z + c_t)^{v_{\mu,t}}$$

a q-difference polynomial of degree $p_1 = \max_{\mu} \sum_{h=0}^{t} v_{\mu,h}$ for non negative integers $v_{\mu,0}, v_{\mu,1}, \cdots v_{\mu,t}$ and $q_1, q_2, ..., q_t$, $c_1, c_2, ..., c_t$ are distinct non zero values in $\mathbb{C}$, $T(r, e_\mu) = S(r, f)$ and $F(f)$ is as defined in the Theorem 1.1.

Theorem 1.4. No transcendental meromorphic function $f(z)$ of finite order ρ with $N(r, f(z)) + N(r, \frac{1}{f(z)}) = O(r^{\rho-1+\varepsilon}) + S(r, f(z))$ will satisfy the difference differential equation of the form

$$a_1(z)F(f)U_1(f) + a_2(z)U_1(f) + a_3(z) \equiv 0, \tag{1.4}$$

where $a_1(z)(\not\equiv 0)$, $a_2(z)$ and $a_3(z)(\not\equiv 0)$ are small functions in the sense of $T(r, a_i) = O(r^{\rho-1+\varepsilon}) + S(r, f)$ for $i = 1, 2, 3$ and

$$U_1(f) = \sum_{\beta} g_\beta(z) f(z)^{w_{\beta,0}} f(z + c_1)^{w_{\beta,1}} ... f(z + c_u)^{w_{\beta,u}}$$

a difference polynomial of degree $p_2 = \max_{\beta} \sum_{j=0}^{u} w_{\beta,j}$ for non negative integers $w_{\beta,0}, w_{\beta,1}, \cdots w_{\beta,u}$ and $c_1, c_2, ..., c_u$ are distinct non zero values in $\mathbb{C}$, $T(r, g_\beta) = S(r, f)$ and $F(f)$ is as defined in the Theorem 1.1.

For particular cases of $Q[f]$, we have the following Corollaries.

Corollary 1.1. If $Q[f] = b_{n-2}f^{n-2} + b_{n-3}f^{n-3} + ... + b_1 f + b_0$ in the Theorem 1.2, then the Theorem 1.2 reduces to the Theorem 1.A.

Corollary 1.2. If $Q[f] = b_{n-2}f^{n-2} + b_{n-3}f^{n-3} + ... + b_1 f + b_0$ in the Theorem 1.4, then the Theorem 1.4 reduces to the Theorem 1.B.

2 Some preliminary results

To prove the main results of this paper, we require the following Lemmas.

Lemma 2.1 ([17] P 69-70)**.** Suppose that f is meromorphic and not constant in the plane, that

$$F = a_n f^n + a_{n-1} f^{n-1} + Q[f]$$

where $Q[f]$ is a differential polynomial of degree at most $n-2$ in f. If $N(r,f)+N(r,\frac{1}{F})=S(r,f)$, then

$$F=a_n\left(f+\frac{a_{n-1}}{na_n}\right)^n.$$

Lemma 2.2 ([18])**.** Let q and η be two non-zero finite complex constants and let f be a non-constant meromorphic function of zero order, then

$$N(r,f(qz+\eta))\leq N(r,f(z))+S(r,f).$$

Lemma 2.3 ([20])**.** Let $f(z)$ be a non-constant meromorphic function of zero order and let $q,c\in\mathbb{C}\setminus\{0\}$. Then

$$m\left(r,\frac{f(qz+c)}{f(z)}\right)=o\{T(r,f)\}$$

on a set of logarithmic density 1.

Lemma 2.4 ([18])**.** Let q and η be two non-zero finite complex constants and let f be a non-constant meromorphic function of zero order, then

$$T(r,f(qz+\eta))\leq T(r,f(z))+S(r,f).$$

Lemma 2.5 ([2])**.** Let f be a meromorphic function with exponent of convergence of poles $\lambda(\frac{1}{f})=\lambda<+\infty,\quad \eta\neq 0$ be fixed, then for each $\varepsilon>0$,

$$N(r,f(z+\eta))=N(r,f)+O(r^{\lambda-1+\varepsilon})+O(logr).$$

Lemma 2.6 ([15, 16])**.** Let $f(z)$ be a meromorphic function of finite order σ and let c be a fixed non-zero complex constant. Then for each $\varepsilon>0$, we have

$$m\left(r,\frac{f(z+c)}{f(z)}\right)+m\left(r,\frac{f(z)}{f(z+c)}\right)=O(r^{\sigma-1+\varepsilon}).$$

Lemma 2.7 ([2])**.** Let $f(z)$ be a meromorphic function of finite order σ and let η be a fixed non-zero complex constant, then for each $\varepsilon>0$, we have

$$T(r,f(z+\eta))=T(r,f)+O(r^{\sigma-1+\varepsilon})+O(logr).$$

3 Proof of the main results

3.1 Proof of the Theorem 1.1

Assume, to the contrary, that there exists a transcendental meromorphic function $f(z)$ of zero order with

$$N(r,f)=S(r,f), \tag{3.1}$$

and satisfying the q-difference differential equation

$$a_1(z)F(f)f(qz+c)+a_2(z)f(qz+c)+a_3(z)\equiv 0. \tag{3.2}$$

This implies

$$(a_1b_nf^n + a_1b_{n-1}f^{n-1})f(qz+c) + Q_1[f]f(qz+c) + a_3 \equiv 0, \tag{3.3}$$

where $Q_1[f] = a_1Q[f] + a_2$. By the virtue of (3.1), Lemma 2.2 gives

$$N(r, f(qz+c)) = S(r, f). \tag{3.4}$$

Note that $a_1(z)(\not\equiv 0)$. Equation (3.3) can be written as

$$b_nf^n + b_{n-1}f^{n-1} + \frac{Q_1[f]}{a_1} \equiv \frac{-a_3}{a_1f(qz+c)}.$$

Denote this by $H(z)$, that is

$$H(z) \equiv b_nf^n + b_{n-1}f^{n-1} + \frac{Q_1[f]}{a_1} \equiv \frac{-a_3}{a_1f(qz+c)}. \tag{3.5}$$

Combing this with (3.4), we get

$$\begin{aligned} N\left(r, \frac{1}{H}\right) &= N\left(r, \frac{a_1f(qz+c)}{a_3}\right) \\ &= S(r, f). \end{aligned}$$

This with (3.1) yields

$$N\left(r, \frac{1}{H}\right) + N(r, f) = S(r, f).$$

Using this and (3.5) in the Lemma 2.1, we get

$$H(z) \equiv b_n\left(f(z) + \frac{b_{n-1}}{nb_n}\right)^n. \tag{3.6}$$

It follows from (3.5) and (3.6) that

$$b_n\left(f(z) + \frac{b_{n-1}}{nb_n}\right)^n \equiv \frac{-a_3}{a_1f(qz+c)},$$

which implies

$$f(qz+c)\left(f(z) + \frac{b_{n-1}}{nb_n}\right)^n \equiv \frac{-a_3}{a_1b_n}.$$

Thus

$$T\left(r, f(qz+c)\left(f(z) + \frac{b_{n-1}}{nb_n}\right)^n\right) = S(r, f). \tag{3.7}$$

Using (3.4), we deduce that

$$N\left(r, \frac{f(qz+c)}{f(z)}\right) = N\left(r, \frac{1}{f(z)}\right) + S(r, f). \tag{3.8}$$

With the help of Lemma 2.3 and (3.8), we deduce that

$$\begin{aligned} T\left(r, \frac{f(qz+c)}{f(z)}\right) &= m\left(r, \frac{f(qz+c)}{f(z)}\right) + N\left(r, \frac{f(qz+c)}{f(z)}\right) \\ &= N\left(r, \frac{1}{f(z)}\right) + S(r,f) \\ &\leq T(r,f) + S(r,f). \end{aligned} \tag{3.9}$$

By the first fundamental theorem of Nevanlinna, (3.7) and (3.9), we deduce that

$$\begin{aligned} T\left(r, f(z)\left(f(z)+\frac{b_{n-1}}{nb_n}\right)^n\right) &= T\left(r, \frac{1}{f(z)\left(f(z)+\frac{b_{n-1}}{nb_n}\right)^n}\right) + O(1) \\ &\leq T\left(r, \frac{f(qz+c)}{f(z)}\right) + T\left(r, \frac{1}{f(qz+c)\left(f(z)+\frac{b_{n-1}}{nb_n}\right)^n}\right) \\ &\quad + O(1) \\ &\leq T(r,f) + S(r,f). \end{aligned} \tag{3.10}$$

On the other hand, we have

$$T\left(r, f(z)\left(f(z)+\frac{b_{n-1}}{nb_n}\right)^n\right) = (n+1)T(r,f) + S(r,f). \tag{3.11}$$

Therefore, from (3.10) and (3.11) we get

$$\begin{aligned} (n+1)T(r,f) + S(r,f) &\leq T(r,f) + S(r,f), \\ \textit{that is, } nT(r,f) &\leq S(r,f). \end{aligned}$$

Since $n \geq 1$, we get contradiction to the assumption that f is transcendental. Hence, we conclude that there do not exists transcendental meromorphic function f of zero order with $N(r,f) = S(r,f)$ satisfying the non-linear $q-$difference diferential eqution (1.1).

3.2 Proof of Theorem 1.2

Using the same argument as in the proof of the Theorem 1.1 with the help of the Lemmas 2.5 - 2.7 instead of the Lemmas 2.2 - 2.4, we get that no transcendental meromorphic function $f(z)$ of finite order ρ with $N(r,f) = O(r^{\rho-1+\varepsilon}) + S(r,f)$ will satify the non-linear difference differential equation (1.2).

3.3 Proof of Theorem 1.3

On the contrary, suppose that there exists a transcendental meromorphic function $f(z)$ of zero order with

$$N\left(r, \frac{1}{f}\right) + N(r,f) = S(r,f), \tag{3.12}$$

and satisfying the equation

$$a_1(z)F(f)U(f) + a_2(z)U(f) + a_3(z) \equiv 0. \tag{3.13}$$

This implies

$$(a_1b_nf^n + a_1b_{n-1}f^{n-1})U(f) + Q_1[f]U(f) + a_3 \equiv 0, \tag{3.14}$$

where $Q_1[f] = a_1Q[f] + a_2$ and q-difference differential polynomial $U(f)$ as

$$\begin{aligned} U(f) &= \sum_{\mu} e_\mu(z)f(z)^{v_{\mu,0}}f(q_1z+c_1)^{v_{\mu,1}}...f(q_tz+c_t)^{v_{\mu,t}} \\ &= f(z)^{p_1}\sum_{\mu}\frac{e_\mu(z)f(z)^{v_{\mu,0}}f(q_1z+c_1)^{v_{\mu,1}}...f(q_tz+c_t)^{v_{\mu,t}}}{f(z)^{p_1}} \\ &= f(z)^{p_1}\sum_{\mu} e_\mu(z)\left(\frac{f(q_1z+c_1)}{f(z)}\right)^{v_{\mu,1}}\left(\frac{f(q_2z+c_2)}{f(z)}\right)^{v_{\mu,2}}...\left(\frac{f(q_tz+c_t)}{f(z)}\right)^{v_{\mu,t}}. \end{aligned}$$

For $j = 1, 2, \cdots t$, by using the Lemma 2.2 and (3.12) we get

$$N\left(r, \frac{f(q_jz+c_j)}{f(z)}\right) \le N(r, f(q_jz+c_j)) + N\left(r, \frac{1}{f(z)}\right) = S(r, f) \tag{3.15}$$

Combining this with (3.12) and $T(r, e_\mu) = S(r, f)$, we obtain

$$N(r, U(f)) + p_1N\left(\frac{1}{f}\right) = N\left(r, \frac{U(f)}{f(z)^{p_1}}\right) = S(r, f). \tag{3.16}$$

Since $a_1(z)(\not\equiv 0)$, equation (3.14) can be written as

$$b_nf^n + b_{n-1}f^{n-1} + \frac{Q_1[f]}{a_1} \equiv \frac{-a_3}{a_1U(f)} \equiv \Psi(z). \tag{3.17}$$

From (3.16) and (3.17), we have

$$N\left(r, \frac{1}{\Psi(z)}\right) \equiv N\left(r, \frac{a_1U(f)}{a_3}\right) = S(r, f). \tag{3.18}$$

By (3.12) and (3.18), we have

$$N(r, f) + N(r, \frac{1}{\Psi}) = S(r, f).$$

Using this with (3.17), the Lemma 2.1 gives

$$\Psi(z) \equiv b_n\left(f(z) + \frac{b_{n-1}}{nb_n}\right)^n. \tag{3.19}$$

With the help of (3.17) and (3.19), we have

$$b_n\left(f(z) + \frac{b_{n-1}}{nb_n}\right)^n \equiv \frac{-a_3}{a_1U(f)},$$

which implies

$$U(f)\left(f(z) + \frac{b_{n-1}}{nb_n}\right)^n \equiv \frac{-a_3}{a_1b_n}.$$

Thus

$$T\left(r, U(f)\left(f(z)+\frac{b_{n-1}}{nb_n}\right)^n\right) = S(r,f). \tag{3.20}$$

Also, we have

$$m\left(r, \frac{U(f)}{f(z)^{p_1}}\right) \leq \sum_{\mu}\left[m(r,e_\mu) + \sum_{j=1}^{t} v_{\mu,j}\; m\left(r, \frac{f(q_j z + c_j)}{f(z)}\right)\right].$$

By using the Lemma 2.3 and $m(r,e_\mu) = S(r,f)$, we have

$$m\left(r, \frac{U(f)}{f(z)^{p_1}}\right) \leq S(r,f).$$

Using this with (3.16), we obtain

$$T\left(r, \frac{U(f)}{f(z)^{p_1}}\right) = S(r,f) \tag{3.21}$$

Now, by the first fundamental theorem of Nevanlinna, (3.20) and (3.21), we deduce that

$$\begin{aligned} T\left(r, f(z)^{p_1}\left(f(z)+\frac{b_{n-1}}{nb_n}\right)^n\right) &= T\left(r, \frac{1}{f(z)^{p_1}\left(f(z)+\frac{b_{n-1}}{nb_n}\right)^n}\right) + O(1) \\ &\leq T\left(r, \frac{U(f)}{f(z)^{p_1}}\right) + T\left(r, \frac{1}{U(f)\left(f(z)+\frac{b_{n-1}}{nb_n}\right)^n}\right) \\ &\quad + O(1) \\ &= S(r,f). \end{aligned} \tag{3.22}$$

On the other hand, we have

$$T\left(r, f(z)^{p_1}\left(f(z)+\frac{b_{n-1}}{nb_n}\right)^n\right) = (n+p_1)T(r,f) + S(r,f). \tag{3.23}$$

Therefore, from (3.22) and (3.23) we get

$$(n+p_1)T(r,f) + S(r,f) \quad = \quad S(r,f).$$

This contradicts to the assumption that $f(z)$ is transcendental. Hence we conclude that there do not exists transcendental meromorphic function $f(z)$ of zero order with $N(r,f)+N(r,\frac{1}{f}) = S(r,f)$ satisfying the $q-$difference differential equation (1.3).

3.4 Proof of Theorem 1.4

Using the same argument as in the proof of the Theorem 1.3 and applying the Lemmas 2.5 - 2.7 instead of the Lemmas 2.2 - 2.4, we obtain that no transcendental meromorphic function $f(z)$ of finite order ρ with $N(r,f) + N(r,\frac{1}{f}) = O(r^{\rho-1+\varepsilon}) + S(r,f)$ will satisfy the difference differential equation (1.4).

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