Fractional integral inequalities for generalized convexity

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Abstract

In this paper, we define a new class of functions called generalized φ-convex function. Several variants of Hermite-Hadamard type fractional integral inequalities are presented. This ideas and techniques used in this paper may open new avenues of research and motivate the reader to explore the application of generalized φ-convex functions in various branches of pure and applied sciences.

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1 Introduction

A function $\mathcal{F}_1 : \mathbb{R} \to \mathbb{R}$ is said to be convex, if

$$\mathcal{F}_1(\imath\varnothing_1 + (1-\imath)\varnothing_2) \leq \imath\mathcal{F}_1(\varnothing_1) + (1-\imath)\mathcal{F}_1(\varnothing_2)$$

for all $\varnothing_1,\varnothing_2 \in \mathbb{R}$ and $\imath \in [0,1]$.

The scheme of convexity plays a major role to learn necessary and appropriate conditions to discover the existence of solutions of optimization problems. It's many applications in several disciplines of mathematical sciences like game theory, information analysis and modelling, computing, statistics, optimization and economic [20]. Due to its significance, the notion of convexity has been extended and generalized in many directions. The reader fascinated by such investigations are mentioned in [1, 2, 4, 5, 6, 9, 10, 11, 12, 13, 14, 15, 16, 17] and references mentioned in this. The next inequality is familiar as Hermite-Hadamard inequality as follows:

Let $\mathcal{F}_1 : I \subseteq R \to R$ be a convex function with $\varnothing_1 < \varnothing_2$ and $\varnothing_1, \varnothing_2 \in I$. Then

$$\mathcal{F}_1\left(\frac{\varnothing_1+\varnothing_2}{2}\right) \leq \frac{1}{\varnothing_2-\varnothing_1}\int_{\varnothing_1}^{\varnothing_2}\mathcal{F}_1(x)dx \leq \frac{\mathcal{F}_1(\varnothing_1)+\mathcal{F}_1(\varnothing_2)}{2}. \tag{1.1}$$

For some results associated with this classical results, see [4, 5, 8, 18, 20] and the references in this.

Let us recall some helpful definitions.

Let $\mathcal{F}_1 \in L[\varnothing_1,\varnothing_2]$. The Riemann-Liouville integrals $J^{\boldsymbol{\delta}}_{\varnothing_1+}\mathcal{F}_1$ and $J^{\boldsymbol{\delta}}_{\varnothing_2-}\mathcal{F}_1$ of order $\boldsymbol{\delta} > 0$ with $\varnothing_1 \geq 0$ are defined by

$$J^{\boldsymbol{\delta}}_{\varnothing_1+}\mathcal{F}_1(x) = \frac{1}{\Gamma(\boldsymbol{\delta})}\int_{\varnothing_1}^{x}(x-\imath)^{\boldsymbol{\delta}-1}\mathcal{F}_1(\imath)d\imath, \quad x > \varnothing_1$$

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and

$$J^{\boldsymbol{\delta}}_{\varnothing_2-}\mathcal{F}_1(x)=\frac{1}{\Gamma(\boldsymbol{\delta})}\int_x^{\varnothing_2}(\imath-x)^{\boldsymbol{\delta}-1}\mathcal{F}_1(\imath)d\imath,\quad \varnothing_2>x.$$

Here $J^0_{\varnothing_1+}\mathcal{F}_1(x)=J^0_{\varnothing_2-}\mathcal{F}_1(x)=\mathcal{F}_1(x)$. In the case of $\boldsymbol{\delta}=1$, the fractional integral reduces to the classical integral.

The special function is

$$\mathbf{E}_\alpha(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(1+\alpha k)},\ \alpha\in C,\ R(\alpha)>0,\ z\in C,$$

where Γ is Gamma function.

The general form of the above function is given by

$$\mathbf{E}_{\alpha,\ \beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\beta+\alpha k)},\ \alpha,\ \beta\in C,\ R(\alpha)>0,\ R(\beta)>0,\quad z\in C,$$

with C being set of complex numbers are called Mittag-Leffler functions see [7]. These functions play an active role in fractional calculus.

Assigning some specific values to the parameter α, one obtains some fascinating cases of Mittag-Lefler function $\mathbf{E}_\alpha(z)$:

1. $\mathbf{E}_0(z)=\frac{1}{1-z},\ |z|<1$
2. $\mathbf{E}_1(z)=e^z$
3. $\mathbf{E}_1(i\varphi)=e^{i\varphi}$
4. $\mathbf{E}_2(z)=\cosh(\sqrt{z}),\ z\in C$
5. $\mathbf{E}_2(-z^2)=\cos z,\ z\in C$
6. $\mathbf{E}_3(z)=\frac{1}{2}\left[e^{z^{\frac{1}{3}}}+2e^{-\frac{1}{2}z^{\frac{1}{3}}}\cos\left(\frac{\sqrt{3}}{2}z^{\frac{1}{3}}\right)\right]$
7. $\mathbf{E}_4(z)=\frac{1}{2}\left[\cos(z^{\frac{1}{4}})+\cosh(z^{\frac{1}{4}})\right].$

Let W be a nonempty closed set in $\mathbb{R}^n$, and W° the interior of W. We denote by $\langle .,.\rangle$ and $\|.\|$ the inner product and norm on $\mathbb{R}^n$, respectively. Let $\mathcal{F}_1,\ \varphi:W\to\mathbb{R}$ be continuous mappings.

Definition 1.1. [13] If any nonempty set W satisfies the following condition:

$$\varnothing_1+\imath e^{i\varphi}(\varnothing_2-\varnothing_1)\in W,\ \text{for all}\ \ \imath\in[0,1]\ \text{and}\ \varnothing_1,\ \varnothing_2\in W, \tag{1.2}$$

then the set W is called φ-convex.

Note that If we suppose $\varphi=0$ in (1.2), then φ-convex set reduces to convex set. The converse does not hold in general.

Definition 1.2. [13] If a function $\mathcal{F}_1$ on the φ-convex set W satisfies the following condition:

$$\mathcal{F}_1(\varnothing_1 + \imath e^{i\varphi}(\varnothing_2 - \varnothing_1)) \leq (1-\imath)\mathcal{F}_1(\varnothing_1) + \imath\mathcal{F}_1(\varnothing_2), \qquad \forall\ \varnothing_1, \varnothing_2 \in W \text{ and } \imath \in [0,1],$$

then $\mathcal{F}_1$ is called φ-convex function.

Remember that the function $\mathcal{F}_1$ is said to be φ-concave iff $-\mathcal{F}_1$ is φ-convex. Note that, every convex function is φ-convex but the converse does not hold in general.

Definition 1.3. [13] If any function $\mathcal{F}_1$ on the φ-convex set W satisfies the following condition:

$$\mathcal{F}_1(\varnothing_1 + \imath e^{i\varphi}(\varnothing_2 - \varnothing_1)) \leq \max\left\{\mathcal{F}_1(\varnothing_1), \mathcal{F}_1(\varnothing_2)\right\}, \qquad \forall\ \varnothing_1, \varnothing_2 \in W \text{ and } \quad \imath \in [0,1],$$

then $\mathcal{F}_1$ is called quasi φ-convex function.

Definition 1.4. [13] If any function $\mathcal{F}_1$ on the φ-convex set W satisfies the following condition:

$$\mathcal{F}_1(\varnothing_1 + \imath e^{i\varphi}(\varnothing_2 - \varnothing_1)) \leq (\mathcal{F}_1(\varnothing_1))^{1-\imath}(\mathcal{F}_1(\varnothing_2))^{\imath}, \quad \forall\ \varnothing_1, \varnothing_2 \in W \text{ and } \imath \in [0,1], \tag{1.3}$$

then $\mathcal{F}_1$ is called logarithmically(log) φ-convex function.

For different definition of φ-convex, φ_r-convex and φ_h-convex functions refer to the corresponding references, [22]-[26].

Inspired by the research carried out to obtain different forms of convexity, the purpose of this paper is to introduce a new class of convex sets and convex functions; called generalized φ-convex set and generalized φ-convex function, respectively. We derive several new Hermite-Hadamard fractional integral inequalities for generalized φ-convex functions.

2 Main results

In this section, we introduce the notion of a generalized φ-convex set as follows:

Definition 2.1. If any nonempty set W satisfies the following condition:

$$\varnothing_1 + \imath\mathbf{E}_\alpha(\varnothing_2 - \varnothing_1) \in W, \qquad \forall\ \varnothing_1, \varnothing_2 \in W \text{ and } \imath \in [0,1], \tag{2.1}$$

where $\alpha \in C,\ R(\alpha) > 0$, then W is called generalized $\varphi-$convex set.

If we replace the function $\mathbf{E}_\alpha$ with $\mathbf{E}_1$, a special case given in (3) of Mittag-Leffler function, then generalized φ-convex set reduces to the φ-convex set.

Definition 2.2. If a function $\mathcal{F}_1$ on the generalized φ-convex set W satisfies the following condition:

$$\mathcal{F}_1(\varnothing_1 + \imath\mathbf{E}_\alpha(\varnothing_2 - \varnothing_1)) \leq (1-\imath)\mathcal{F}_1(\varnothing_1) + \imath\mathcal{F}_1(\varnothing_2), \qquad \forall\ \varnothing_1, \varnothing_2 \in W \text{ and } \imath \in [0,1],$$

then $\mathcal{F}_1$ is called generalized φ-convex function on W.

Obviously generalized φ-convex function is φ-convex function by replacing $\mathbf{E}_\alpha$ with $\mathbf{E}_1$, a special case given in (3) of Mittag-Leffler function.

Definition 2.3. If a function $\mathcal{F}_1$ on the φ-convex set W satisfies the following condition:

$$\mathcal{F}_1(\varnothing_1 + \imath \mathbf{E}_\alpha(\varnothing_2 - \varnothing_1)) \leq \max\left\{\mathcal{F}_1(\varnothing_1), \mathcal{F}_1(\varnothing_2)\right\}, \quad \forall\ \varnothing_1, \varnothing_2 \in W \text{ and } \imath \in [0,1],$$

then $\mathcal{F}_1$ is called generalized quasi φ-convex function on set W.

Definition 2.4. If a function $\mathcal{F}_1$ on the φ-convex set W satisfies the following condition:

$$\mathcal{F}_1(\varnothing_1 + \imath \mathbf{E}_\alpha(\varnothing_2 - \varnothing_1)) \leq (\mathcal{F}_1(\varnothing_1))^{1-\imath}(\mathcal{F}_1(\varnothing_2))^{\imath},\ \forall\ \varnothing_1, \varnothing_2 \in W \text{ and } \imath \in [0,1],$$

then $\mathcal{F}_1$ is called generalized logarithmically φ-convex function on set W.

Let $W = [\varnothing_1, \varnothing_1 + \mathbf{E}_\alpha(\varnothing_2 - \varnothing_1)] \subseteq \mathbb{R}$. Then $\mathcal{F}_1$ is generalized φ-convex function on W if and only if

$$\begin{vmatrix} 1 & 1 & 1 \\ \varnothing_1 & x & \varnothing_1 + \mathbf{E}_\alpha(\varnothing_2 - \varnothing_1) \\ \mathcal{F}_1(\varnothing_1) & \mathcal{F}_1(x) & \mathcal{F}_1(\varnothing_1 + \mathbf{E}_\alpha(\varnothing_2 - \varnothing_1)) \end{vmatrix} \geq 0,$$

where $x = \varnothing_1 + \imath \mathbf{E}_\alpha(\varnothing_2 - \varnothing_1)$.

Remark 2.5. The generalized φ-convex sets and generalized φ-convex functions are nonconvex.

Now, we are in position to obtain several variants of generalized φ-Hermite-Hadamard fractional integral inequalities.

Theorem 2.6. Let $\mathcal{F}_1 : W = [\varnothing_1, \varnothing_1 + \mathbf{E}_\alpha(\varnothing_2 - \varnothing_1)] \to (0, \infty)$ be generalized φ-convex function on the interval W° (the interior of W) and $\varnothing_1, \varnothing_2 \in W^\circ$ with $\mathbf{E}_\alpha(\varnothing_2 - \varnothing_1) > 0$. Then for $\boldsymbol{\delta} > 0$, the following inequalities holds:

$$\begin{aligned} & \mathcal{F}_1\left(\frac{2\varnothing_1 + \mathbf{E}_\alpha(\varnothing_2 - \varnothing_1)}{2}\right) \\ \leq\ & \frac{\Gamma(\boldsymbol{\delta}+1)}{2[\mathbf{E}_\alpha(\varnothing_2 - \varnothing_1)]^{\boldsymbol{\delta}}}\left[J^{\boldsymbol{\delta}}_{\varnothing_1^+}\mathcal{F}_1((\varnothing_1 + \mathbf{E}_\alpha(\varnothing_2 - \varnothing_1)) + J^{\boldsymbol{\delta}}_{(\varnothing_1 + \mathbf{E}_\alpha(\varnothing_2 - \varnothing_1))^-}\mathcal{F}_1(\varnothing_1)\right] \\ \leq\ & \frac{\mathcal{F}_1(\varnothing_1) + \mathcal{F}_1(\varnothing_2)}{2}. \end{aligned}$$

Proof. Since $\mathcal{F}_1$ is a generalized φ-convex function, we have

$$\begin{aligned} & \mathcal{F}_1\left(\frac{2\varnothing_1 + \mathbf{E}_\alpha(\varnothing_2 - \varnothing_1)}{2}\right) \\ =\ & \mathcal{F}_1\left(\frac{\varnothing_1 + \varnothing_1 + \mathbf{E}_\alpha(\varnothing_2 - \varnothing_1)}{2}\right) \\ =\ & \mathcal{F}_1\left(\frac{1}{2}\begin{bmatrix} (1-\imath)\varnothing_1 + \imath(\varnothing_1 + \mathbf{E}_\alpha(\varnothing_2 - \varnothing_1)) \\ +\imath\varnothing_1 + (1-\imath)(\varnothing_1 + \mathbf{E}_\alpha(\varnothing_2 - \varnothing_1)) \end{bmatrix}\right) \\ \leq\ & \frac{1}{2}\begin{bmatrix} \mathcal{F}_1((1-\imath)\varnothing_1 + \imath(\varnothing_1 + \mathbf{E}_\alpha(\varnothing_2 - \varnothing_1))) \\ +\mathcal{F}_1(\imath\varnothing_1 + (1-\imath)(\varnothing_1 + \mathbf{E}_\alpha(\varnothing_2 - \varnothing_1))) \end{bmatrix}. \end{aligned}$$

Multiplying both sides of the above inequality with $\imath^{\boldsymbol{\delta}-1}$ and integrating with respect to $\imath$ on $[0,1]$, we have

$$\begin{aligned}
&\int_0^1 \imath^{\boldsymbol{\delta}-1}\mathcal{F}_1\left(\frac{2\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1)}{2}\right)d\imath\\
\leq\ & \frac{1}{2}\int_0^1 \imath^{\boldsymbol{\delta}-1}\mathcal{F}_1((1-\imath)\varnothing_1+\imath(\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1)))d\imath\\
+\ & \frac{1}{2}\int_0^1 \imath^{\boldsymbol{\delta}-1}\mathcal{F}_1(\imath\varnothing_1+(1-\imath)(\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1)))d\imath\\
=\ & \frac{\Gamma(\boldsymbol{\delta})}{2[\mathbf{E}_\alpha(\varnothing_2-\varnothing_1)]^{\boldsymbol{\delta}}}\left[J^{\boldsymbol{\delta}}_{\varnothing_1^+}\mathcal{F}_1((\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1))+J^{\boldsymbol{\delta}}_{(\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1))^-}\mathcal{F}_1(\varnothing_1)\right].
\end{aligned}$$

Hence

$$\begin{aligned}
&\mathcal{F}_1\left(\frac{2\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1)}{2}\right)\\
\leq\ & \frac{\Gamma(\boldsymbol{\delta}+1)}{2[\mathbf{E}_\alpha(\varnothing_2-\varnothing_1)]^{\boldsymbol{\delta}}}\left[J^{\boldsymbol{\delta}}_{\varnothing_1^+}\mathcal{F}_1((\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1))+J^{\boldsymbol{\delta}}_{(\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1))^-}\mathcal{F}_1(\varnothing_1)\right].
\end{aligned}$$

Now we show that

$$\begin{aligned}
&\frac{\Gamma(\boldsymbol{\delta}+1)}{2[\mathbf{E}_\alpha(\varnothing_2-\varnothing_1)]^{\boldsymbol{\delta}}}\left[J^{\boldsymbol{\delta}}_{\varnothing_1^+}\mathcal{F}_1((\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1))+J^{\boldsymbol{\delta}}_{(\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1))^-}\mathcal{F}_1(\varnothing_1)\right]\\
\leq\ & \frac{\mathcal{F}_1(\varnothing_1)+\mathcal{F}_1(\varnothing_2)}{2}.
\end{aligned}$$

Consider

$$\begin{aligned}
&\frac{1}{2}\int_0^1 \imath^{\boldsymbol{\delta}-1}\mathcal{F}_1((1-\imath)\varnothing_1+\imath(\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1)))d\imath\\
+\ & \frac{1}{2}\int_0^1 \imath^{\boldsymbol{\delta}-1}\mathcal{F}_1(\imath\varnothing_1+(1-\imath)(\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1)))d\imath\\
\leq\ & \frac{1}{2}\int_0^1 \imath^{\boldsymbol{\delta}-1}((1-\imath)\mathcal{F}_1(\varnothing_1)+\imath\mathcal{F}_1(\varnothing_2))d\imath\\
&+\frac{1}{2}\int_0^1 \imath^{\boldsymbol{\delta}-1}(\imath\mathcal{F}_1(\varnothing_1)+(1-\imath)\mathcal{F}_1(\varnothing_2))d\imath\\
=\ & \frac{\mathcal{F}_1(\varnothing_1)+\mathcal{F}_1(\varnothing_2)}{2\boldsymbol{\delta}}.
\end{aligned}$$

This completes the proof. ∎

Corollary 2.7. Let $\mathcal{F}_1 : W = \left[\varnothing_1, \varnothing_1 + e^{i\varphi}(\varnothing_2-\varnothing_1)\right] \to (0,\infty)$ be φ-convex function on the interval W° (the interior of W) and $\varnothing_1$,$\varnothing_2 \in W^\circ$ with $e^{i\varphi}(\varnothing_2-\varnothing_1) > 0$. Then for $\boldsymbol{\delta} > 0$, the

following inequalities holds:

$$\begin{aligned}
&\mathcal{F}_1\left(\frac{2\varnothing_1+e^{i\varphi}(\varnothing_2-\varnothing_1)}{2}\right)\\
\leq\ &\frac{\Gamma(\boldsymbol{\delta}+1)}{2[e^{i\varphi}(\varnothing_2-\varnothing_1)]^{\boldsymbol{\delta}}}\Big[J^{\boldsymbol{\delta}}_{\varnothing_1^+}\mathcal{F}_1((\varnothing_1+e^{i\varphi}(\varnothing_2-\varnothing_1))+J^{\boldsymbol{\delta}}_{(\varnothing_1+e^{i\varphi}(\varnothing_2-\varnothing_1))^-}\mathcal{F}_1(\varnothing_1)\Big]\\
\leq\ &\frac{\mathcal{F}_1(\varnothing_1)+\mathcal{F}_1(\varnothing_2)}{2}. \qquad (2.2)
\end{aligned}$$

Proof. We can easily prove our this result by using the idea of Theorem 2.6 and particular case (3) of Mittag-Lefler function. ∎

Theorem 2.8. Let $\mathcal{F}_1,\mathcal{F}_2 : W=[\varnothing_1,\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1)]\to(0,\infty)$ be generalized φ-convex functions on the interval W° (the interior of W) and $\varnothing_1$,$\varnothing_2\in W^\circ$ with $\mathbf{E}_\alpha(\varnothing_2-\varnothing_1)>0$. Then for $\boldsymbol{\delta}>0$, the following inequalities holds:

$$\begin{aligned}
&\mathcal{F}_1\left(\frac{2\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1)}{2}\right)\mathcal{F}_2\left(\frac{2\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1)}{2}\right)\\
\leq\ &\frac{\Gamma(\boldsymbol{\delta}+1)}{4[\mathbf{E}_\alpha(\varnothing_2-\varnothing_1)]^{\boldsymbol{\delta}}}\Big[J^{\boldsymbol{\delta}}_{\varnothing_1^+}\mathcal{F}_1((\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1))\mathcal{F}_2((\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1))\\
&+J^{\boldsymbol{\delta}}_{(\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1))^-}\mathcal{F}_1(\varnothing_1)\mathcal{F}_2(\varnothing_1)\Big]\\
&+\frac{1}{4}\Big[\frac{2\boldsymbol{\delta}}{(\boldsymbol{\delta}+1)(\boldsymbol{\delta}+2)}(M(\mathcal{F}_1,\mathcal{F}_2;\varnothing_1,\varnothing_2)+N(\mathcal{F}_1,\mathcal{F}_2;\varnothing_1,\varnothing_2))\\
&+N(\mathcal{F}_1,\mathcal{F}_2;\varnothing_1,\varnothing_2)\Big]\\
\leq\ &\frac{\boldsymbol{\delta}}{(\boldsymbol{\delta}+1)(\boldsymbol{\delta}+2)}((\boldsymbol{\delta}+1)M(\mathcal{F}_1,\mathcal{F}_2;\varnothing_1,\varnothing_2)+N(\mathcal{F}_1,\mathcal{F}_2;\varnothing_1,\varnothing_2))\\
&+\frac{(1-\boldsymbol{\delta})}{\boldsymbol{\delta}+1}\mathcal{F}_1(\varnothing_1)\mathcal{F}_2(\varnothing_1)\\
&+\frac{1}{4}\Big[\frac{2\boldsymbol{\delta}}{(\boldsymbol{\delta}+1)(\boldsymbol{\delta}+2)}(M(\mathcal{F}_1,\mathcal{F}_2;\varnothing_1,\varnothing_2)+N(\mathcal{F}_1,\mathcal{F}_2;\varnothing_1,\varnothing_2))\\
&+N(\mathcal{F}_1,\mathcal{F}_2;\varnothing_1,\varnothing_2)\Big],
\end{aligned}$$

where

$$\begin{aligned}
M(\mathcal{F}_1,\mathcal{F}_2;\varnothing_1,\varnothing_2) &= \mathcal{F}_1(\varnothing_1)\mathcal{F}_2(\varnothing_1)+\mathcal{F}_1(\varnothing_2)\mathcal{F}_2(\varnothing_2),\\
N(\mathcal{F}_1,\mathcal{F}_2;\varnothing_1,\varnothing_2) &= \mathcal{F}_1(\varnothing_1)\mathcal{F}_2(\varnothing_2)+\mathcal{F}_1(\varnothing_2)\mathcal{F}_2(\varnothing_1).
\end{aligned}$$

Proof. Since $\mathcal{F}_1, \mathcal{F}_2$ are generalized φ-convex functions, we have

$$
\begin{aligned}
& \mathcal{F}_1\left(\frac{2\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1)}{2}\right)\mathcal{F}_2\left(\frac{2\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1)}{2}\right)\\
= \;& \mathcal{F}_1\left(\frac{\varnothing_1+\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1)}{2}\right)\mathcal{F}_2\left(\frac{\varnothing_1+\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1)}{2}\right)\\
= \;& \mathcal{F}_1\left(\frac{(1-\imath)\varnothing_1+\imath(\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1))+\imath\varnothing_1+(1-\imath)(\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1))}{2}\right)\\
& \times\mathcal{F}_2\left(\frac{(1-\imath)\varnothing_1+\imath(\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1))+\imath\varnothing_1+(1-\imath)(\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1))}{2}\right)\\
\leq \;& \frac{1}{2}\left[\mathcal{F}_1((1-\imath)\varnothing_1+\imath(\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1)))+\mathcal{F}_1(\imath\varnothing_1+(1-\imath)(\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1)))\right]\\
& \times\frac{1}{2}\left[\mathcal{F}_2((1-\imath)\varnothing_1+\imath(\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1)))+\mathcal{F}_2(\imath\varnothing_1+(1-\imath)(\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1)))\right]\\
= \;& \frac{1}{4}\left[(\mathcal{F}_1((1-\imath)\varnothing_1+\imath(\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1))))(\mathcal{F}_2((1-\imath)\varnothing_1+\imath(\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1))))\right.\\
& \left.+(\mathcal{F}_1(\imath\varnothing_1+(1-\imath)(\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1))))(\mathcal{F}_2(\imath\varnothing_1+(1-\imath)(\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1))))\right]\\
& +\frac{1}{4}\left[(\mathcal{F}_1((1-\imath)\varnothing_1+\imath(\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1))))(\mathcal{F}_2(\imath\varnothing_1+(1-\imath)(\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1))))\right.\\
& \left.+(\mathcal{F}_1(\imath\varnothing_1+(1-\imath)(\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1))))(\mathcal{F}_2((1-\imath)\varnothing_1+\imath(\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1))))\right]\\
\leq \;& \frac{1}{4}\left[(\mathcal{F}_1((1-\imath)\varnothing_1+\imath(\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1))))(\mathcal{F}_2((1-\imath)\varnothing_1+\imath(\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1))))\right.\\
& \left.+(\mathcal{F}_1(\imath\varnothing_1+(1-\imath)(\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1))))(\mathcal{F}_2(\imath\varnothing_1+(1-\imath)(\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1))))\right]\\
& +\frac{1}{4}\left[((1-\imath)\mathcal{F}_1(\varnothing_1)+\imath\mathcal{F}_1(\varnothing_2))(\imath\mathcal{F}_2(\varnothing_1)+(1-\imath)\mathcal{F}_2(\varnothing_2))\right.\\
& \left.+(\imath\mathcal{F}_1(\varnothing_1)+(1-\imath)\mathcal{F}_1(\varnothing_2))((1-\imath)\mathcal{F}_2(\varnothing_1)+\imath\mathcal{F}_2(\varnothing_2))\right]
\end{aligned}
$$

$$
\begin{aligned}
= \;& \frac{1}{4}\left[(\mathcal{F}_1((1-\imath)\varnothing_1+\imath(\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1))))(\mathcal{F}_2((1-\imath)\varnothing_1+\imath(\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1))))\right.\\
+ \;& \left.(\mathcal{F}_1(\imath\varnothing_1+(1-\imath)(\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1))))(\mathcal{F}_2(\imath\varnothing_1+(1-\imath)(\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1))))\right]\\
+ \;& \frac{1}{4}\left[2\imath(1-\imath)(\mathcal{F}_1(\varnothing_1)\mathcal{F}_2(\varnothing_1)+\mathcal{F}_1(\varnothing_2)\mathcal{F}_2(\varnothing_2))\right.\\
+ \;& \left.(\imath^2+(1-\imath)^2)(\mathcal{F}_1(\varnothing_2)\mathcal{F}_2(\varnothing_1)+\mathcal{F}_1(\varnothing_1)\mathcal{F}_2(\varnothing_2))\right].
\end{aligned}
$$

Multiplying both sides of the above inequality with $\imath^{\boldsymbol{\delta}-1}$ and integrating with respect to $\imath$ on $[0,1]$,

we have

$$\begin{aligned}
&\int_0^1 \imath^{\boldsymbol{\delta}-1}\mathcal{F}_1\left(\frac{2\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1)}{2}\right)\mathcal{F}_2\left(\frac{2\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1)}{2}\right)d\imath\\
\leq\ & \frac{\Gamma(\boldsymbol{\delta})}{4[\mathbf{E}_\alpha(\varnothing_2-\varnothing_1)]^{\boldsymbol{\delta}}}\Big[J^{\boldsymbol{\delta}}_{\varnothing_1^+}\mathcal{F}_1((\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1))\mathcal{F}_2((\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1))\\
&+J^{\boldsymbol{\delta}}_{(\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1))^-}\mathcal{F}_1(\varnothing_1)\mathcal{F}_2(\varnothing_1)\Big]\\
+\ & \frac{1}{4}\left[\frac{2}{(\boldsymbol{\delta}+1)(\boldsymbol{\delta}+2)}(M(\mathcal{F}_1,\mathcal{F}_2;\varnothing_1,\varnothing_2)+N(\mathcal{F}_1,\mathcal{F}_2;\varnothing_1,\varnothing_2))\right.\\
&\left.+\frac{1}{\boldsymbol{\delta}}N(\mathcal{F}_1,\mathcal{F}_2;\varnothing_1,\varnothing_2)\right],
\end{aligned}$$

which implies that

$$\begin{aligned}
&\mathcal{F}_1\left(\frac{2\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1)}{2}\right)\mathcal{F}_2\left(\frac{2\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1)}{2}\right)\\
\leq\ & \frac{\Gamma(\boldsymbol{\delta}+1)}{4[\mathbf{E}_\alpha(\varnothing_2-\varnothing_1)]^{\boldsymbol{\delta}}}\Big[J^{\boldsymbol{\delta}}_{\varnothing_1^+}\mathcal{F}_1((\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1))\mathcal{F}_2((\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1))\\
&+J^{\boldsymbol{\delta}}_{(\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1))^-}\mathcal{F}_1(\varnothing_1)\mathcal{F}_2(\varnothing_1)\Big]\\
+\ & \frac{1}{4}\left[\frac{2\boldsymbol{\delta}}{(\boldsymbol{\delta}+1)(\boldsymbol{\delta}+2)}(M(\mathcal{F}_1,\mathcal{F}_2;\varnothing_1,\varnothing_2)+N(\mathcal{F}_1,\mathcal{F}_2;\varnothing_1,\varnothing_2))\right.\\
&\left.+N(\mathcal{F}_1,\mathcal{F}_2;\varnothing_1,\varnothing_2)\right].
\end{aligned}$$

From generalized φ-convexity of $\mathcal{F}_1$ and $\mathcal{F}_2$, we have

$$\begin{aligned}
&(\mathcal{F}_1(\varnothing_1+\imath\mathbf{E}_\alpha(\varnothing_2-\varnothing_1)))(\mathcal{F}_2(\varnothing_1+\imath\mathbf{E}_\alpha(\varnothing_2-\varnothing_1)))\\
\leq\ & ((1-\imath)\mathcal{F}_1(\varnothing_1)+\imath\mathcal{F}_1(\varnothing_2))((1-\imath)\mathcal{F}_2(\varnothing_1)+\imath\mathcal{F}_2(\varnothing_2))
\end{aligned}$$

and

$$\begin{aligned}
&(\mathcal{F}_1(\varnothing_1+(1-\imath)\mathbf{E}_\alpha(\varnothing_2-\varnothing_1)))(\mathcal{F}_2(\varnothing_1+(1-\imath)\mathbf{E}_\alpha(\varnothing_2-\varnothing_1)))\\
\leq\ & (\imath\mathcal{F}_1(\varnothing_1)+(1-\imath)\mathcal{F}_1(\varnothing_2))(\imath\mathcal{F}_2(\varnothing_1)+(1-\imath)\mathcal{F}_2(\varnothing_2)).
\end{aligned}$$

Multiplying both sides of the above inequalities with $\imath^{\boldsymbol{\delta}-1}$ and integrating with respect to $\imath$ on $[0,1]$, we have

$$\begin{aligned}
&\int_0^1 \imath^{\boldsymbol{\delta}-1}(\mathcal{F}_1(\varnothing_1+\imath\mathbf{E}_\alpha(\varnothing_2-\varnothing_1)))(\mathcal{F}_2(\varnothing_1+\imath\mathbf{E}_\alpha(\varnothing_2-\varnothing_1)))d\imath\\
+\ & \int_0^1 \imath^{\boldsymbol{\delta}-1}(\mathcal{F}_1(\varnothing_1+(1-\imath)\mathbf{E}_\alpha(\varnothing_2-\varnothing_1)))(\mathcal{F}_2(\varnothing_1+(1-\imath)\mathbf{E}_\alpha(\varnothing_2-\varnothing_1)))d\imath\\
\leq\ & \int_0^1 \imath^{\boldsymbol{\delta}-1}((1-\imath)\mathcal{F}_1(\varnothing_1)+\imath\mathcal{F}_1(\varnothing_2))((1-\imath)\mathcal{F}_2(\varnothing_1)+\imath\mathcal{F}_2(\varnothing_2))d\imath\\
+\ & \int_0^1 \imath^{\boldsymbol{\delta}-1}(\imath\mathcal{F}_1(\varnothing_1)+(1-\imath)\mathcal{F}_1(\varnothing_2))(\imath\mathcal{F}_2(\varnothing_1)+(1-\imath)\mathcal{F}_2(\varnothing_2))d\imath.
\end{aligned}$$

After simple calculations we get the right hand side inequality. This completes the proof. ∎

Corollary 2.9. Let $\mathcal{F}_1$,$\mathcal{F}_2 : W = \left[\varnothing_1, \varnothing_1 + e^{i\varphi}(\varnothing_2 - \varnothing_1)\right] \to (0,\infty)$ be φ-convex functions on the interval W° (the interior of W) and $\varnothing_1$,$\varnothing_2 \in W^\circ$ with $e^{i\varphi}(\varnothing_2 - \varnothing_1) > 0$. Then for $\boldsymbol{\delta} > 0$, the following inequalities holds:

$$\begin{aligned}
&\mathcal{F}_1\left(\frac{2\varnothing_1 + e^{i\varphi}(\varnothing_2 - \varnothing_1)}{2}\right)\mathcal{F}_2\left(\frac{2\varnothing_1 + e^{i\varphi}(\varnothing_2 - \varnothing_1)}{2}\right)\\
\leq\ & \frac{\Gamma(\boldsymbol{\delta}+1)}{4[e^{i\varphi}(\varnothing_2 - \varnothing_1)]^{\boldsymbol{\delta}}}\Big[J^{\boldsymbol{\delta}}_{\varnothing_1^+}\mathcal{F}_1((\varnothing_1 + e^{i\varphi}(\varnothing_2 - \varnothing_1))\mathcal{F}_2((\varnothing_1 + e^{i\varphi}(\varnothing_2 - \varnothing_1))\\
&+J^{\boldsymbol{\delta}}_{(\varnothing_1 + e^{i\varphi}(\varnothing_2 - \varnothing_1))^-}\mathcal{F}_1(\varnothing_1)\mathcal{F}_2(\varnothing_1)\Big]\\
+\ & \frac{1}{4}\Big[\frac{2\boldsymbol{\delta}}{(\boldsymbol{\delta}+1)(\boldsymbol{\delta}+2)}(M(\mathcal{F}_1,\mathcal{F}_2;\varnothing_1,\varnothing_2) + N(\mathcal{F}_1,\mathcal{F}_2;\varnothing_1,\varnothing_2))\\
&+N(\mathcal{F}_1,\mathcal{F}_2;\varnothing_1,\varnothing_2)\Big]\\
\leq\ & \frac{\boldsymbol{\delta}}{(\boldsymbol{\delta}+1)(\boldsymbol{\delta}+2)}((\boldsymbol{\delta}+1)M(\mathcal{F}_1,\mathcal{F}_2;\varnothing_1,\varnothing_2) + N(\mathcal{F}_1,\mathcal{F}_2;\varnothing_1,\varnothing_2))\\
&+\frac{(1-\boldsymbol{\delta})}{\boldsymbol{\delta}+1}\mathcal{F}_1(\varnothing_1)\mathcal{F}_2(\varnothing_1)\\
+\ & \frac{1}{4}\Big[\frac{2\boldsymbol{\delta}}{(\boldsymbol{\delta}+1)(\boldsymbol{\delta}+2)}(M(\mathcal{F}_1,\mathcal{F}_2;\varnothing_1,\varnothing_2) + N(\mathcal{F}_1,\mathcal{F}_2;\varnothing_1,\varnothing_2))\\
&+N(\mathcal{F}_1,\mathcal{F}_2;\varnothing_1,\varnothing_2)\Big],
\end{aligned}$$

where

$$\begin{aligned}
M(\mathcal{F}_1,\mathcal{F}_2;\varnothing_1,\varnothing_2) &= \mathcal{F}_1(\varnothing_1)\mathcal{F}_2(\varnothing_1) + \mathcal{F}_1(\varnothing_2)\mathcal{F}_2(\varnothing_2),\\
N(\mathcal{F}_1,\mathcal{F}_2;\varnothing_1,\varnothing_2) &= \mathcal{F}_1(\varnothing_1)\mathcal{F}_2(\varnothing_2) + \mathcal{F}_1(\varnothing_2)\mathcal{F}_2(\varnothing_1).
\end{aligned}$$

Proof. We can easily prove our this result by using the idea of Theorem 2.8 and particular case (3) of Mittag-Lefler function. ∎

Theorem 2.10. Let $\mathcal{F}_1 : W = [\varnothing_1, \varnothing_1 + \mathbf{E}_\alpha(\varnothing_2 - \varnothing_1)] \to (0,\infty)$ be generalized logarithmic φ-convex function on the interval W° (the interior of W) and $\varnothing_1$,$\varnothing_2 \in W^\circ$ with $\mathbf{E}_\alpha(\varnothing_2 - \varnothing_1) > 0$. Then for $\boldsymbol{\delta} > 0$, the following inequalities holds:

$$\begin{aligned}
&\mathcal{F}_1\left(A(2\varnothing_1, \mathbf{E}_\alpha(\varnothing_2 - \varnothing_1))\right)\\
\leq\ & \exp\left(\frac{\Gamma(\boldsymbol{\delta}+1)}{2[\mathbf{E}_\alpha(\varnothing_2 - \varnothing_1)]^{\boldsymbol{\delta}}}\Big[J^{\boldsymbol{\delta}}_{\varnothing_1^+}(\ln\mathcal{F}_1(\varnothing_1 + \mathbf{E}_\alpha(\varnothing_2 - \varnothing_1)))\right.\\
&\left.+J^{\boldsymbol{\delta}}_{(\varnothing_1 + \mathbf{E}_\alpha(\varnothing_2 - \varnothing_1))^-}(\ln\mathcal{F}_1(\varnothing_1))\Big]\right)\\
\leq\ & G\left(\mathcal{F}_1(\varnothing_1), \mathcal{F}_1(\varnothing_2)\right),
\end{aligned}$$

where $A(.,.)$ is arithmetic mean and $G(.,.)$ is the geometric mean.

Proof. Note that

$$\begin{aligned}
& \mathcal{F}_1\left(A(2\varnothing_1, \mathbf{E}_\alpha(\varnothing_2-\varnothing_1))\right) \\
= \; & \mathcal{F}_1\left(\frac{2\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1)}{2}\right) \\
= \; & \mathcal{F}_1\left(\frac{\imath\varnothing_1+(1-\imath)(\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1))+(1-\imath)\varnothing_1+\imath(\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1))}{2}\right) \\
\leq \; & \left(\mathcal{F}_1\left(\imath\varnothing_1+(1-\imath)(\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1))\right)\right)^{\frac{1}{2}} \\
& \times\left(\mathcal{F}_1((1-\imath)\varnothing_1+\imath(\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1)))\right)^{\frac{1}{2}}.
\end{aligned}$$

That is we have

$$\begin{aligned}
& \ln\left(\mathcal{F}_1\left(A(2\varnothing_1, \mathbf{E}_\alpha(\varnothing_2-\varnothing_1))\right)\right) \\
\leq \; & \frac{1}{2}\ln\left(\mathcal{F}_1\left(\imath\varnothing_1+(1-\imath)(\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1))\right)\right) \\
& +\frac{1}{2}\ln\left(\mathcal{F}_1((1-\imath)\varnothing_1+\imath(\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1)))\right).
\end{aligned}$$

Multiplying both sides of the above inequality with $\imath^{\boldsymbol{\delta}-1}$ and integrating with respect to $\imath$ on $[0,1]$, we have

$$\begin{aligned}
& \int_0^1 \imath^{\boldsymbol{\delta}-1}\ln\left(\mathcal{F}_1\left(A(2\varnothing_1, \mathbf{E}_\alpha(\varnothing_2-\varnothing_1))\right)\right)d\imath \\
\leq \; & \frac{1}{2}\int_0^1 \imath^{\boldsymbol{\delta}-1}\ln\left(\mathcal{F}_1\left(\imath\varnothing_1+(1-\imath)(\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1))\right)\right)d\imath \\
+ \; & \frac{1}{2}\int_0^1 \imath^{\boldsymbol{\delta}-1}\ln\left(\mathcal{F}_1((1-\imath)\varnothing_1+\imath(\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1)))\right)d\imath.
\end{aligned}$$

Hence

$$\begin{aligned}
& \ln\left(\mathcal{F}_1\left(A(2\varnothing_1, \mathbf{E}_\alpha(\varnothing_2-\varnothing_1))\right)\right) \\
\leq \; & \frac{\Gamma(\boldsymbol{\delta}+1)}{2[\mathbf{E}_\alpha(\varnothing_2-\varnothing_1)]^{\boldsymbol{\delta}}}\Big[J^{\boldsymbol{\delta}}_{\varnothing_1^+}(\ln\mathcal{F}_1((\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1))) \\
& +J^{\boldsymbol{\delta}}_{(\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1))^-}(\ln\mathcal{F}_1(\varnothing_1))\Big].
\end{aligned}$$

Then

$$\begin{aligned}
& \mathcal{F}_1\left(A(2\varnothing_1, \mathbf{E}_\alpha(\varnothing_2-\varnothing_1))\right) \\
\leq \; & \exp\left(\frac{\Gamma(\boldsymbol{\delta}+1)}{2[\mathbf{E}_\alpha(\varnothing_2-\varnothing_1)]^{\boldsymbol{\delta}}}\Big[J^{\boldsymbol{\delta}}_{\varnothing_1^+}(\ln\mathcal{F}_1((\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1))) \right. \\
& \left. +J^{\boldsymbol{\delta}}_{(\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1))^-}(\ln\mathcal{F}_1(\varnothing_1))\Big]\right).
\end{aligned}$$

Also

$$\begin{aligned}
&\frac{\Gamma(\boldsymbol{\delta})}{2[\mathbf{E}_\alpha(\varnothing_2-\varnothing_1)]^{\boldsymbol{\delta}}}\Big[J^{\boldsymbol{\delta}}_{\varnothing_1^+}(\ln\mathcal{F}_1((\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1)))\\
&+J^{\boldsymbol{\delta}}_{(\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1))^-}(\ln\mathcal{F}_1(\varnothing_1))\Big]\\
=\;&\frac{1}{2}\int_0^1 \imath^{\boldsymbol{\delta}-1}\ln\left(\mathcal{F}_1\left(\imath\varnothing_1+(1-\imath)(\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1))\right)\right)d\imath\\
&+\frac{1}{2}\int_0^1 \imath^{\boldsymbol{\delta}-1}\ln\left(\mathcal{F}_1((1-\imath)\varnothing_1+\imath(\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1)))\right)d\imath\\
\leq\;&\frac{1}{2}\int_0^1 \imath^{\boldsymbol{\delta}-1}\ln((\mathcal{F}_1(\varnothing_1))^{1-\imath}(\mathcal{F}_1(\varnothing_2))^{\imath})d\imath\\
&+\frac{1}{2}\int_0^1 \imath^{\boldsymbol{\delta}-1}\ln((\mathcal{F}_1(\varnothing_1))^{\imath}(\mathcal{F}_1(\varnothing_2))^{1-\imath})d\imath\\
=\;&\frac{1}{2}\int_0^1 \imath^{\boldsymbol{\delta}-1}\left((1-\imath)\ln\mathcal{F}_1(\varnothing_1)+\imath\ln\mathcal{F}_1(\varnothing_2)\right)d\imath\\
&+\frac{1}{2}\int_0^1 \imath^{\boldsymbol{\delta}-1}\left(\imath\ln\mathcal{F}_1(\varnothing_1)+(1-\imath)\ln\mathcal{F}_1(\varnothing_2)\right)d\imath\\
=\;&\frac{\ln\sqrt{\mathcal{F}_1(\varnothing_1)\mathcal{F}_1(\varnothing_2)}}{\boldsymbol{\delta}}
\end{aligned}$$

This completes the proof. ∎

Corollary 2.11. Let $\mathcal{F}_1 : W = \left[\varnothing_1, \varnothing_1 + e^{i\varphi}(\varnothing_2 - \varnothing_1)\right] \to (0,\infty)$ be logarithmically φ-convex function on the interval W° (the interior of W) and $\varnothing_1$,$\varnothing_2 \in W^\circ$ with $e^{i\varphi}(\varnothing_2-\varnothing_1) > 0$. Then for $\boldsymbol{\delta} > 0$, the following inequalities holds:

$$\begin{aligned}
&\mathcal{F}_1\left(A(2\varnothing_1, e^{i\varphi}(\varnothing_2-\varnothing_1))\right)\\
\leq\;&\exp\left(\frac{\Gamma(\boldsymbol{\delta}+1)}{2[e^{i\varphi}(\varnothing_2-\varnothing_1)]^{\boldsymbol{\delta}}}\Big[J^{\boldsymbol{\delta}}_{\varnothing_1^+}(\ln\mathcal{F}_1(\varnothing_1+e^{i\varphi}(\varnothing_2-\varnothing_1)))\right.\\
&\left.+J^{\boldsymbol{\delta}}_{(\varnothing_1+e^{i\varphi}(\varnothing_2-\varnothing_1))^-}(\ln\mathcal{F}_1(\varnothing_1))\Big]\right)\\
\leq\;&G\left(\mathcal{F}_1(\varnothing_1),\mathcal{F}_1(\varnothing_2)\right).
\end{aligned}$$

Proof. We can easily prove our this result by using the idea of Theorem 2.10 and particular case (3) of Mittag-Lefler function. ∎

Theorem 2.12. Let $\mathcal{F}_1 : W = [\varnothing_1, \varnothing_1 + \mathbf{E}_\alpha(\varnothing_2 - \varnothing_1)] \to (0,\infty)$ be generalized logarithmic φ-convex function on the interval W° (the interior of W) and $\varnothing_1$,$\varnothing_2 \in W^\circ$ with $\mathbf{E}_\alpha(\varnothing_2-\varnothing_1) > 0$.

Then for $\boldsymbol{\delta} > 0$, the following inequalities holds:

$$\begin{aligned}
&\int_{\varnothing_1}^{\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1)} (x-\varnothing_1)^{\boldsymbol{\delta}-1} G(\mathcal{F}_1(x), \mathcal{F}_1(2\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1)-x))dx \\
\leq\ & \frac{\Gamma(\boldsymbol{\delta})}{2[\mathbf{E}_\alpha(\varnothing_2-\varnothing_1)]^{\boldsymbol{\delta}}}\Big[J^{\boldsymbol{\delta}}_{\varnothing_1^+}(\ln \mathcal{F}_1((\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1))) \\
&+J^{\boldsymbol{\delta}}_{(\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1))^-}(\ln \mathcal{F}_1(\varnothing_1))\Big] \\
\leq\ & \frac{[\mathbf{E}_\alpha(\varnothing_2-\varnothing_1)]^{\boldsymbol{\delta}}}{2}\big[\mathcal{F}_1(\varnothing_1)H(\mathcal{F}_1;\varnothing_1,\varnothing_2,\boldsymbol{\delta})+\mathcal{F}_1(\varnothing_2)K(\mathcal{F}_1;\varnothing_1,\varnothing_2,\boldsymbol{\delta})\big],
\end{aligned}$$

where

$$\begin{aligned}
H(\mathcal{F}_1;\varnothing_1,\varnothing_2,\boldsymbol{\delta}) &= \int_0^1 \imath^{\boldsymbol{\delta}-1}\left[\frac{\mathcal{F}_1(\varnothing_2)}{\mathcal{F}_1(\varnothing_1)}\right]^{\imath} d\imath, \\
K(\mathcal{F}_1;\varnothing_1,\varnothing_2,\boldsymbol{\delta}) &= \int_0^1 (1-\imath)^{\boldsymbol{\delta}-1}\left[\frac{\mathcal{F}_1(\varnothing_2)}{\mathcal{F}_1(\varnothing_1)}\right]^{\imath} d\imath,
\end{aligned}$$

and $G(.,.)$ is geometric mean.

Proof. To prove the first inequality, from AM-GM inequality, we obtain

$$G(\mathcal{F}_1(x), \mathcal{F}_1(2\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1)-x)) \leq \frac{\mathcal{F}_1(x)+\mathcal{F}_1(2\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1)-x)}{2}.$$

Multiplying both sides of the above inequality with $(x-\varnothing_1)^{\boldsymbol{\delta}-1}$ and integrating with respect x on $[\varnothing_1,\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1)]$, gives that

$$\begin{aligned}
&\int_{\varnothing_1}^{\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1)} (x-\varnothing_1)^{\boldsymbol{\delta}-1} G(\mathcal{F}_1(x), \mathcal{F}_1(2\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1)-x))dx \\
\leq\ & \int_{\varnothing_1}^{\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1)} (x-\varnothing_1)^{\boldsymbol{\delta}-1} \frac{\mathcal{F}_1(x)+\mathcal{F}_1(2\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1)-x)}{2}dx \\
=\ & \frac{\Gamma(\boldsymbol{\delta})}{2[\mathbf{E}_\alpha(\varnothing_2-\varnothing_1)]^{\boldsymbol{\delta}}}\Big[J^{\boldsymbol{\delta}}_{\varnothing_1^+}(\ln \mathcal{F}_1((\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1))) \\
&+J^{\boldsymbol{\delta}}_{(\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1))^-}(\ln \mathcal{F}_1(\varnothing_1))\Big].
\end{aligned}$$

As $\mathcal{F}_1$ is logarithmic generalized φ-convex function, so

$$\begin{aligned}
&\int_{\varnothing_1}^{\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1)}(x-\varnothing_1)^{\boldsymbol{\delta}-1}\frac{\mathcal{F}_1(x)+\mathcal{F}_1(2\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1)-x)}{2}dx\\
=\;&\frac{[\mathbf{E}_\alpha(\varnothing_2-\varnothing_1)]^{\boldsymbol{\delta}}}{2}\Big\{\int_0^1 \imath^{\boldsymbol{\delta}-1}\mathcal{F}_1(\varnothing_1+\imath\mathbf{E}_\alpha(\varnothing_2-\varnothing_1))d\imath\\
+\;&\int_0^1(1-\imath)^{\boldsymbol{\delta}-1}\mathcal{F}_1(\varnothing_1+(1-\imath)\mathbf{E}_\alpha(\varnothing_2-\varnothing_1))d\imath\Big\}\\
\leq\;&\frac{[\mathbf{E}_\alpha(\varnothing_2-\varnothing_1)]^{\boldsymbol{\delta}}}{2}\Big\{\int_0^1 \imath^{\boldsymbol{\delta}-1}\left(\mathcal{F}_1(\varnothing_1)\right)^{1-\imath}\left(\mathcal{F}_1(\varnothing_2)\right)^{\imath}d\imath\\
&+\int_0^1(1-\imath)^{\boldsymbol{\delta}-1}\left(\mathcal{F}_1(\varnothing_1)\right)^{\imath}\left(\mathcal{F}_1(\varnothing_2)\right)^{1-\imath}d\imath\Big\}\\
=\;&\frac{[\mathbf{E}_\alpha(\varnothing_2-\varnothing_1)]^{\boldsymbol{\delta}}}{2}\Bigg\{\mathcal{F}_1(\varnothing_1)\int_0^1\imath^{\boldsymbol{\delta}-1}\left[\frac{\mathcal{F}_1(\varnothing_2)}{\mathcal{F}_1(\varnothing_1)}\right]^{\imath}d\imath\\
&+\mathcal{F}_1(\varnothing_2)\int_0^1(1-\imath)^{\boldsymbol{\delta}-1}\left[\frac{\mathcal{F}_1(\varnothing_1)}{\mathcal{F}_1(\varnothing_2)}\right]^{\imath}d\imath\Bigg\}\\
=\;&\frac{[\mathbf{E}_\alpha(\varnothing_2-\varnothing_1)]^{\boldsymbol{\delta}}}{2}\big[\mathcal{F}_1(\varnothing_1)H(\mathcal{F}_1;\varnothing_1,\varnothing_2,\boldsymbol{\delta})+\mathcal{F}_1(\varnothing_2)K(\mathcal{F}_1;\varnothing_1,\varnothing_2,\boldsymbol{\delta})\big].
\end{aligned}$$

This completes the proof. ∎

Corollary 2.13. Let $\mathcal{F}_1 : W=\left[\varnothing_1,\varnothing_1+e^{i\varphi}(\varnothing_2-\varnothing_1)\right]\to(0,\infty)$ be logarithmically φ-convex function on the interval W° (the interior of W) and $\varnothing_1$,$\varnothing_2\in W^\circ$ with $e^{i\varphi}(\varnothing_2-\varnothing_1)>0$. Then for $\boldsymbol{\delta}>0$, the following inequalities holds:

$$\begin{aligned}
&\int_{\varnothing_1}^{\varnothing_1+e^{i\varphi}(\varnothing_2-\varnothing_1)}(x-\varnothing_1)^{\boldsymbol{\delta}-1}G(\mathcal{F}_1(x),\mathcal{F}_1(2\varnothing_1+e^{i\varphi}(\varnothing_2-\varnothing_1)-x))dx\\
\leq\;&\frac{\Gamma(\boldsymbol{\delta})}{2[e^{i\varphi}(\varnothing_2-\varnothing_1)]^{\boldsymbol{\delta}}}\Big[J^{\boldsymbol{\delta}}_{\varnothing_1^+}(\ln\mathcal{F}_1((\varnothing_1+e^{i\varphi}(\varnothing_2-\varnothing_1)))\\
&+J^{\boldsymbol{\delta}}_{(\varnothing_1+e^{i\varphi}(\varnothing_2-\varnothing_1))^-}(\ln\mathcal{F}_1(\varnothing_1))\Big]\\
\leq\;&\frac{[e^{i\varphi}(\varnothing_2-\varnothing_1)]^{\boldsymbol{\delta}}}{2}\big[\mathcal{F}_1(\varnothing_1)H(\mathcal{F}_1;\varnothing_1,\varnothing_2,\boldsymbol{\delta})+\mathcal{F}_1(\varnothing_2)K(\mathcal{F}_1;\varnothing_1,\varnothing_2,\boldsymbol{\delta})\big].
\end{aligned}$$

Proof. We can easily prove our this result by using the idea of Theorem 2.12 and particular case (3) of Mittag-Lefler function. ∎

Theorem 2.14. Let $\mathcal{F}_1,\mathcal{F}_2 : W=[\varnothing_1,\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1)]\to(0,\infty)$ be generalized logarithmic φ-convex functions on the interval W° (the interior of W) and $\varnothing_1$,$\varnothing_2\in W^\circ$ with $\mathbf{E}_\alpha(\varnothing_2-\varnothing_1)>0$.

Then for $\boldsymbol{\delta} > 0$, the following inequalities holds:

$$\begin{aligned}
&\frac{1}{[\mathbf{E}_\alpha(\varnothing_2-\varnothing_1)]^{\boldsymbol{\delta}}}\int_{\varnothing_1}^{\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1)}(x-\varnothing_1)^{\boldsymbol{\delta}-1}\mathcal{F}_1(x)\mathcal{F}_2(x)dx\\
\leq\ &\Big\{\mathcal{F}_1(\varnothing_1)\mathcal{F}_2(\varnothing_1)\mathcal{F}_1(\mathcal{F}_1,\mathcal{F}_2;\varnothing_1,\varnothing_2,\boldsymbol{\delta});\\
&\frac{1}{2^{\boldsymbol{\delta}+1}}\Big[[\mathcal{F}_1(\varnothing_1)]^2S(\mathcal{F}_1;\varnothing_1,\varnothing_2,\boldsymbol{\delta})+[\mathcal{F}_2(\varnothing_1)]^2R(\mathcal{F}_2;\varnothing_1,\varnothing_2,\boldsymbol{\delta})\Big]\Big\},
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{F}_1(\mathcal{F}_1,\mathcal{F}_2;\varnothing_1,\varnothing_2,\boldsymbol{\delta}) &= \int_0^1 \imath^{\boldsymbol{\delta}-1}\left[\frac{\mathcal{F}_1(\varnothing_2)\mathcal{F}_2(\varnothing_2)}{\mathcal{F}_1(\varnothing_1)\mathcal{F}_2(\varnothing_1)}\right]^{\imath}d\imath,\\
S(\mathcal{F}_1;\varnothing_1,\varnothing_2,\boldsymbol{\delta}) &= \int_0^2 \imath^{\boldsymbol{\delta}-1}\left[\frac{\mathcal{F}_1(\varnothing_2)}{\mathcal{F}_1(\varnothing_1)}\right]^{\imath}d\imath,\\
R(\mathcal{F}_2;\varnothing_1,\varnothing_2,\boldsymbol{\delta}) &= \int_0^2 \imath^{\boldsymbol{\delta}-1}\left[\frac{\mathcal{F}_2(\varnothing_2)}{\mathcal{F}_2(\varnothing_1)}\right]^{\imath}d\imath.
\end{aligned}$$

Proof. As $\mathcal{F}_1,\mathcal{F}_2$ are logarithmic generalized φ-convex functions, we have

$$\begin{aligned}
\mathcal{F}_1(\varnothing_1+\imath\mathbf{E}_\alpha(\varnothing_2-\varnothing_1)) &\leq [\mathcal{F}_1(\varnothing_1)]^{1-\imath}[\mathcal{F}_1(\varnothing_2)]^{\imath} \text{ and}\\
\mathcal{F}_2(\varnothing_1+\imath\mathbf{E}_\alpha(\varnothing_2-\varnothing_1)) &\leq [\mathcal{F}_2(\varnothing_1)]^{1-\imath}[\mathcal{F}_2(\varnothing_2)]^{\imath}.
\end{aligned}$$

Hence

$$\begin{aligned}
&\frac{1}{[\mathbf{E}_\alpha(\varnothing_2-\varnothing_1)]^{\boldsymbol{\delta}}}\int_{\varnothing_1}^{\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1)}(x-\varnothing_1)^{\boldsymbol{\delta}-1}\mathcal{F}_1(x)\mathcal{F}_2(x)dx\\
=\ &\int_0^1 \imath^{\boldsymbol{\delta}-1}\mathcal{F}_1(\varnothing_1+\imath\mathbf{E}_\alpha(\varnothing_2-\varnothing_1))\mathcal{F}_2(\varnothing_1+\imath\mathbf{E}_\alpha(\varnothing_2-\varnothing_1))d\imath\\
\leq\ &\int_0^1 \imath^{\boldsymbol{\delta}-1}\left(\mathcal{F}_1(\varnothing_1)\mathcal{F}_2(\varnothing_1)\right)^{1-\imath}\left(\mathcal{F}_1(\varnothing_2)\mathcal{F}_2(\varnothing_2)\right)^{\imath}d\imath\\
=\ &\mathcal{F}_1(\varnothing_1)\mathcal{F}_2(\varnothing_1)\int_0^1 \imath^{\boldsymbol{\delta}-1}\left[\frac{\mathcal{F}_1(\varnothing_2)\mathcal{F}_2(\varnothing_2)}{\mathcal{F}_1(\varnothing_1)\mathcal{F}_2(\varnothing_1)}\right]^{\imath}d\imath\\
=\ &\mathcal{F}_1(\varnothing_1)\mathcal{F}_2(\varnothing_1)\mathcal{F}_1(\mathcal{F}_1,\mathcal{F}_2;\varnothing_1,\varnothing_2,\boldsymbol{\delta}).
\end{aligned}$$

Also for the second inequality, we have

$$\begin{aligned}
&\frac{1}{[\mathbf{E}_\alpha(\varnothing_2-\varnothing_1)]^{\boldsymbol{\delta}}}\int_{\varnothing_1}^{\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1)}(x-\varnothing_1)^{\boldsymbol{\delta}-1}\mathcal{F}_1(x)\mathcal{F}_2(x)dx\\
=\ &\int_0^1 \imath^{\boldsymbol{\delta}-1}\mathcal{F}_1(\varnothing_1+\imath\mathbf{E}_\alpha(\varnothing_2-\varnothing_1))\mathcal{F}_2(\varnothing_1+\imath\mathbf{E}_\alpha(\varnothing_2-\varnothing_1))d\imath\\
\leq\ &\frac{1}{2}\int_0^1 \imath^{\boldsymbol{\delta}-1}\left[\{\mathcal{F}_1(\varnothing_1+\imath\mathbf{E}_\alpha(\varnothing_2-\varnothing_1))\}^2+\{\mathcal{F}_2(\varnothing_1+\imath\mathbf{E}_\alpha(\varnothing_2-\varnothing_1))\}^2\right]d\imath\\
\leq\ &\frac{1}{2}\int_0^1 \imath^{\boldsymbol{\delta}-1}\left\{\left[\left[[\mathcal{F}_1(\varnothing_1)]^{1-\imath}[\mathcal{F}_1(\varnothing_2)]^{\imath}\right]^2+\left[[\mathcal{F}_2(\varnothing_1)]^{1-\imath}[\mathcal{F}_2(\varnothing_2)]^{\imath}\right]^2\right]\right\}d\imath
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\left\{[\mathcal{F}_1(\varnothing_1)]^2\int_0^2 \imath^{\boldsymbol{\delta}-1}\left[\frac{\mathcal{F}_1(\varnothing_2)}{\mathcal{F}_1(\varnothing_1)}\right]^{\imath} d\imath + [\mathcal{F}_2(\varnothing_1)]^2\int_0^2 \imath^{\boldsymbol{\delta}-1}\left[\frac{\mathcal{F}_2(\varnothing_2)}{\mathcal{F}_2(\varnothing_1)}\right]^{\imath} d\imath\right\}\\
&= \frac{1}{2^{\boldsymbol{\delta}+1}}\Big\{[\mathcal{F}_1(\varnothing_1)]^2 S(\mathcal{F}_1;\varnothing_1,\varnothing_2,\boldsymbol{\delta}) + [\mathcal{F}_2(\varnothing_1)]^2 R(\mathcal{F}_2;\varnothing_1,\varnothing_2,\boldsymbol{\delta})\Big\}.
\end{aligned}
$$

This completes the proof. ∎

Corollary 2.15. Let $\mathcal{F}_1, \mathcal{F}_2 : W = \big[\varnothing_1, \varnothing_1 + e^{i\varphi}(\varnothing_2 - \varnothing_1)\big] \to (0,\infty)$ be logarithmically φ-convex functions on the interval W° (the interior of W) and $\varnothing_1$,$\varnothing_2 \in W^\circ$ with $e^{i\varphi}(\varnothing_2-\varnothing_1) > 0$. Then for $\boldsymbol{\delta} > 0$, the following inequalities holds:

$$
\begin{aligned}
&\frac{1}{[e^{i\varphi}(\varnothing_2-\varnothing_1)]^{\boldsymbol{\delta}}}\int_{\varnothing_1}^{\varnothing_1+e^{i\varphi}(\varnothing_2-\varnothing_1)} (x-\varnothing_1)^{\boldsymbol{\delta}-1}\mathcal{F}_1(x)\mathcal{F}_2(x)dx\\
\leq\ &\Big\{\mathcal{F}_1(\varnothing_1)\mathcal{F}_2(\varnothing_1)\mathcal{F}_1(\mathcal{F}_1,\mathcal{F}_2;\varnothing_1,\varnothing_2,\boldsymbol{\delta});\\
&\frac{1}{2^{\boldsymbol{\delta}+1}}\Big[[\mathcal{F}_1(\varnothing_1)]^2 S(\mathcal{F}_1;\varnothing_1,\varnothing_2,\boldsymbol{\delta}) + [\mathcal{F}_2(\varnothing_1)]^2 R(\mathcal{F}_2;\varnothing_1,\varnothing_2,\boldsymbol{\delta})\Big]\Big\}.
\end{aligned}
$$

Proof. We can easily prove our this result by using the idea of Theorem 2.14 and particular case (3) of Mittag-Lefler function. ∎

Theorem 2.16. Let $\mathcal{F}_1, \mathcal{F}_2 : W = [\varnothing_1, \varnothing_1 + \mathbf{E}_\alpha(\varnothing_2 - \varnothing_1)] \to (0,\infty)$ be generalized logarithmic φ-convex functions on the interval W° (the interior of W) and $\varnothing_1$,$\varnothing_2 \in W^\circ$ with $\mathbf{E}_\alpha(\varnothing_2-\varnothing_1) > 0$. Then for $\boldsymbol{\delta} > 0$, the following inequality hold:

$$
\begin{aligned}
&\ln\left(\mathcal{F}_1\left(\frac{2\varnothing_1 + \mathbf{E}_\alpha(\varnothing_2-\varnothing_1)}{2}\right)\mathcal{F}_2\left(\frac{2\varnothing_1 + \mathbf{E}_\alpha(\varnothing_2-\varnothing_1)}{2}\right)\right)\\
\leq\ &\frac{\Gamma(\boldsymbol{\delta}+1)}{2[\mathbf{E}_\alpha(\varnothing_2-\varnothing_1)]^{\boldsymbol{\delta}}}\\
\times\ &\Big\{\Big[J^{\boldsymbol{\delta}}_{\varnothing_1^+}(\ln\mathcal{F}_1((\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1))) + J^{\boldsymbol{\delta}}_{(\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1))^-}(\ln\mathcal{F}_1(\varnothing_1))\Big]\\
+\ &\Big[J^{\boldsymbol{\delta}}_{\varnothing_1^+}(\ln\mathcal{F}_2((\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1))) + J^{\boldsymbol{\delta}}_{(\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1))^-}(\ln\mathcal{F}_2(\varnothing_1))\Big]\Big\}.
\end{aligned}
$$

Proof. As $\mathcal{F}_1$ and $\mathcal{F}_2$ are logarithmic generalized φ-convex functions, we have

$$
\begin{aligned}
&\mathcal{F}_1\left(\frac{2\varnothing_1 + \mathbf{E}_\alpha(\varnothing_2-\varnothing_1)}{2}\right)\mathcal{F}_2\left(\frac{2\varnothing_1 + \mathbf{E}_\alpha(\varnothing_2-\varnothing_1)}{2}\right)\\
=\ &\mathcal{F}_1\left(\frac{(1-\imath)\varnothing_1 + \imath(\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1)) + \imath\varnothing_1 + (1-\imath)(\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1))}{2}\right)\\
\times\ &\mathcal{F}_2\left(\frac{(1-\imath)\varnothing_1 + \imath(\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1)) + \imath\varnothing_1 + (1-\imath)(\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1))}{2}\right)\\
\leq\ &[(\mathcal{F}_1((1-\imath)\varnothing_1 + \imath(\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1))))(\mathcal{F}_1(\imath\varnothing_1 + (1-\imath)(\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1))))]^{\frac{1}{2}}\\
\times\ &[(\mathcal{F}_2((1-\imath)\varnothing_1 + \imath(\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1))))(\mathcal{F}_2(\imath\varnothing_1 + (1-\imath)(\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1))))]^{\frac{1}{2}}.
\end{aligned}
$$

Taking logarithm on the both sides of above inequality, we obtain that

$$\begin{aligned}
&\ln\left(\mathcal{F}_1\left(\frac{2\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1)}{2}\right)\mathcal{F}_2\left(\frac{2\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1)}{2}\right)\right)\\
\leq\ &\frac{1}{2}[\ln(\mathcal{F}_1\left((1-\imath)\varnothing_1+\imath(\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1)\right)))\\
+\ &\ln(\mathcal{F}_1(\imath\varnothing_1+(1-\imath)(\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1))))]\\
+\ &\frac{1}{2}[\ln(\mathcal{F}_2\left((1-\imath)\varnothing_1+\imath(\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1)\right)))\\
+\ &\ln(\mathcal{F}_2(\imath\varnothing_1+(1-\imath)(\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1))))].
\end{aligned}$$

Multiplying both sides of the above inequality with $\imath^{\boldsymbol{\delta}-1}$ and integrating with respect to $\imath$ on $[0,1]$, we have

$$\begin{aligned}
&\int_0^1 \imath^{\boldsymbol{\delta}-1}\ln\left(\mathcal{F}_1\left(\frac{2\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1)}{2}\right)\mathcal{F}_2\left(\frac{2\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1)}{2}\right)\right)d\imath\\
\leq\ &\frac{1}{2}\int_0^1 \imath^{\boldsymbol{\delta}-1}[\ln(\mathcal{F}_1\left((1-\imath)\varnothing_1+\imath(\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1)\right)))\\
+\ &\ln(\mathcal{F}_1(\imath\varnothing_1+(1-\imath)(\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1))))]d\imath\\
+\ &\frac{1}{2}\int_0^1 \imath^{\boldsymbol{\delta}-1}[\ln(\mathcal{F}_2\left((1-\imath)\varnothing_1+\imath(\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1)\right)))\\
+\ &\ln(\mathcal{F}_2(\imath\varnothing_1+(1-\imath)(\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1))))]d\imath.
\end{aligned}$$

Using the definition of Riemann-Liouville fractional integrals the proof is completed. ∎

Corollary 2.17. Let $\mathcal{F}_1$, $\mathcal{F}_2 : W = \left[\varnothing_1, \varnothing_1 + e^{i\varphi}(\varnothing_2-\varnothing_1)\right] \to (0,\infty)$ be logarithmically φ-convex functions on the interval W° (the interior of W) and $\varnothing_1$,$\varnothing_2 \in W^\circ$ with $e^{i\varphi}(\varnothing_2-\varnothing_1) > 0$. Then for $\boldsymbol{\delta} > 0$, the following inequalities holds:

$$\begin{aligned}
&\ln\left(\mathcal{F}_1\left(\frac{2\varnothing_1+e^{i\varphi}(\varnothing_2-\varnothing_1)}{2}\right)\mathcal{F}_2\left(\frac{2\varnothing_1+e^{i\varphi}(\varnothing_2-\varnothing_1)}{2}\right)\right)\\
\leq\ &\frac{\Gamma(\boldsymbol{\delta}+1)}{2[e^{i\varphi}(\varnothing_2-\varnothing_1)]^{\boldsymbol{\delta}}}\\
\times\ &\left\{\left[J^{\boldsymbol{\delta}}_{\varnothing_1^+}(\ln\mathcal{F}_1((\varnothing_1+e^{i\varphi}(\varnothing_2-\varnothing_1)))+J^{\boldsymbol{\delta}}_{(\varnothing_1+e^{i\varphi}(\varnothing_2-\varnothing_1))^-}(\ln\mathcal{F}_1(\varnothing_1))\right]\right.\\
+\ &\left.\left[J^{\boldsymbol{\delta}}_{\varnothing_1^+}(\ln\mathcal{F}_2((\varnothing_1+e^{i\varphi}(\varnothing_2-\varnothing_1)))+J^{\boldsymbol{\delta}}_{(\varnothing_1+e^{i\varphi}(\varnothing_2-\varnothing_1))^-}(\ln\mathcal{F}_2(\varnothing_1))\right]\right\}.
\end{aligned}$$

Proof. We can easily prove our this result by using the idea of Theorem 2.16 and particular case (3) of Mittag-Lefler function. ∎

Theorem 2.18. Let $\mathcal{F}_1, \mathcal{F}_2 : W = [\varnothing_1, \varnothing_1 + \mathbf{E}_\alpha(\varnothing_2-\varnothing_1)] \to (0,\infty)$ be generalized φ-convex functions on the interval W° (the interior of W) and $\varnothing_1$,$\varnothing_2 \in W^\circ$ with $\mathbf{E}_\alpha(\varnothing_2-\varnothing_1) > 0$. Then for

$\boldsymbol{\delta} > 0$, the following inequalities holds:

$$\begin{aligned}
& \frac{1}{[\mathbf{E}_\alpha(\varnothing_2-\varnothing_1)]^{\boldsymbol{\delta}}}\int_{\varnothing_1}^{\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1)}(x-\varnothing_1)^{\boldsymbol{\delta}-1}\mathcal{F}_1(x)\mathcal{F}_2(x)dx \\
\leq\ & \frac{(\boldsymbol{\delta}+1)M(\mathcal{F}_1,\mathcal{F}_2;\varnothing_1,\varnothing_2)+N(\mathcal{F}_1,\mathcal{F}_2;\varnothing_1,\varnothing_2)}{(\boldsymbol{\delta}+1)(\boldsymbol{\delta}+2)} \\
& +\frac{(1-\boldsymbol{\delta})}{\boldsymbol{\delta}(\boldsymbol{\delta}+1)}\mathcal{F}_1(\varnothing_1)\mathcal{F}_2(\varnothing_1) \\
\leq\ & \frac{1}{\boldsymbol{\delta}+2}W(\mathcal{F}_1,\mathcal{F}_2;\varnothing_1,\varnothing_2)+\frac{2}{(\boldsymbol{\delta}+1)(\boldsymbol{\delta}+2)}\mathbf{T}(\mathcal{F}_1,\mathcal{F}_2;\varnothing_1,\varnothing_2) \\
& +\frac{(1-\boldsymbol{\delta})}{\boldsymbol{\delta}(\boldsymbol{\delta}+1)}U(\mathcal{F}_1,\mathcal{F}_2;\varnothing_1),
\end{aligned}$$

where

$$\begin{aligned}
U(\mathcal{F}_1;\mathcal{F}_2,\varnothing_1) &= \mathcal{F}_1^2(\varnothing_1)+\mathcal{F}_2^2(\varnothing_1), \\
\mathbf{T}(\mathcal{F}_1,\mathcal{F}_2;\varnothing_1,\varnothing_2) &= \mathcal{F}_1(\varnothing_1)\mathcal{F}_1(\varnothing_2)+\mathcal{F}_2(\varnothing_1)\mathcal{F}_2(\varnothing_2), \\
W(\mathcal{F}_1,\mathcal{F}_2;\varnothing_1,\varnothing_2) &= U(\mathcal{F}_1;\mathcal{F}_2,\varnothing_1)+\mathcal{F}_1^2(\varnothing_2)+\mathcal{F}_2^2(\varnothing_2).
\end{aligned}$$

Proof. As $\mathcal{F}_1, \mathcal{F}_2$ are generalized φ-convex functions, we have

$$\begin{aligned}
\mathcal{F}_1(\varnothing_1+\imath\mathbf{E}_\alpha(\varnothing_2-\varnothing_1)) &\leq (1-\imath)\mathcal{F}_1(\varnothing_1)+\imath\mathcal{F}_1(\varnothing_2) \text{ and} \\
\mathcal{F}_2(\varnothing_1+\imath\mathbf{E}_\alpha(\varnothing_2-\varnothing_1)) &\leq (1-\imath)\mathcal{F}_2(\varnothing_1)+\imath\mathcal{F}_2(\varnothing_2).
\end{aligned}$$

Note that

$$\begin{aligned}
& \frac{1}{[\mathbf{E}_\alpha(\varnothing_2-\varnothing_1)]^{\boldsymbol{\delta}}}\int_{\varnothing_1}^{\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1)}(x-\varnothing_1)^{\boldsymbol{\delta}-1}\mathcal{F}_1(x)\mathcal{F}_2(x)dx \\
=\ & \int_0^1 \imath^{\boldsymbol{\delta}-1}\mathcal{F}_1(\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1))\mathcal{F}_2(\varnothing_1+\mathbf{E}_\alpha(\varnothing_2-\varnothing_1))d\imath \\
\leq\ & \int_0^1 \imath^{\boldsymbol{\delta}-1}\left[(1-\imath)\mathcal{F}_1(\varnothing_1)+\imath\mathcal{F}_1(\varnothing_2)\right]\left[(1-\imath)\mathcal{F}_2(\varnothing_1)+\imath\mathcal{F}_2(\varnothing_2)\right]d\imath \\
& +\imath(1-\imath)\left[\mathcal{F}_1(\varnothing_1)\mathcal{F}_2(\varnothing_2)+\mathcal{F}_1(\varnothing_2)\mathcal{F}_2(\varnothing_1)\right]\Big\}d\imath \\
=\ & \frac{(\boldsymbol{\delta}+1)M(\mathcal{F}_1,\mathcal{F}_2;\varnothing_1,\varnothing_2)+N(\mathcal{F}_1,\mathcal{F}_2;\varnothing_1,\varnothing_2)}{(\boldsymbol{\delta}+1)(\boldsymbol{\delta}+2)}+\frac{(1-\boldsymbol{\delta})}{\boldsymbol{\delta}(\boldsymbol{\delta}+1)}\mathcal{F}_1(\varnothing_1)\mathcal{F}_2(\varnothing_1)
\end{aligned}$$

$$\begin{aligned}
& \leq \frac{1}{2}\int_0^1 \imath^{\boldsymbol{\delta}-1}\left\{\left[(1-\imath)\mathcal{F}_1(\varnothing_1)+\imath\mathcal{F}_1(\varnothing_2)\right]^2+\left[(1-\imath)\mathcal{F}_2(\varnothing_1)+\imath(\mathcal{F}_2(\varnothing_2))\right]^2\right\}d\imath \\
=\ & \frac{1}{\boldsymbol{\delta}+2}W(\mathcal{F}_1,\mathcal{F}_2;\varnothing_1,\varnothing_2)+\frac{2}{(\boldsymbol{\delta}+1)(\boldsymbol{\delta}+2)}\mathbf{T}(\mathcal{F}_1,\mathcal{F}_2;\varnothing_1,\varnothing_2) \\
& +\frac{(1-\boldsymbol{\delta})}{\boldsymbol{\delta}(\boldsymbol{\delta}+1)}U(\mathcal{F}_1,\mathcal{F}_2;\varnothing_1).
\end{aligned}$$

Hence, we get the desired results. ∎

Corollary 2.19. Let $\mathcal{F}_1, \mathcal{F}_2 : W = \left[\varnothing_1, \varnothing_1 + e^{i\varphi}(\varnothing_2 - \varnothing_1)\right] \to (0, \infty)$ be φ-convex functions on the interval W° (the interior of W) and $\varnothing_1$,$\varnothing_2 \in W^\circ$ with $e^{i\varphi}(\varnothing_2 - \varnothing_1) > 0$. Then for $\boldsymbol{\delta} > 0$, the following inequalities holds:

$$\begin{aligned}
&\frac{1}{[e^{i\varphi}(\varnothing_2 - \varnothing_1)]^{\boldsymbol{\delta}}} \int_{\varnothing_1}^{\varnothing_1 + e^{i\varphi}(\varnothing_2 - \varnothing_1)} (x - \varnothing_1)^{\boldsymbol{\delta}-1} \mathcal{F}_1(x)\mathcal{F}_2(x)dx \\
\leq\ & \frac{(\boldsymbol{\delta}+1)M(\mathcal{F}_1, \mathcal{F}_2; \varnothing_1, \varnothing_2) + N(\mathcal{F}_1, \mathcal{F}_2; \varnothing_1, \varnothing_2)}{(\boldsymbol{\delta}+1)(\boldsymbol{\delta}+2)} \\
&+ \frac{(1-\boldsymbol{\delta})}{\boldsymbol{\delta}(\boldsymbol{\delta}+1)} \mathcal{F}_1(\varnothing_1)\mathcal{F}_2(\varnothing_1) \\
\leq\ & \frac{1}{\boldsymbol{\delta}+2} W(\mathcal{F}_1, \mathcal{F}_2; \varnothing_1, \varnothing_2) + \frac{2}{(\boldsymbol{\delta}+1)(\boldsymbol{\delta}+2)} \mathbf{T}(\mathcal{F}_1, \mathcal{F}_2; \varnothing_1, \varnothing_2) \\
&+ \frac{(1-\boldsymbol{\delta})}{\boldsymbol{\delta}(\boldsymbol{\delta}+1)} U(\mathcal{F}_1, \mathcal{F}_2; \varnothing_1).
\end{aligned}$$

Proof. We can easily prove our this result by using the idea of Theorem 2.18 and particular case (3) of Mittag-Lefler function. ∎

3 Some special cases

1. If we use $\varphi = \mathbf{0}$ in Corollary 1, then we have Hermite-Hadamard Integral inequality for fractional integral that established by M. Z. Sarikaya et al in [21].

2. If we use $\boldsymbol{\delta} = \mathbf{1}$ and $\varphi = 0$ in Corollary 1, then we have Hermite-Hadamard integral inequality (1.1).

3. If we use If we use $\varphi = \mathbf{0}$ in Corollary 2, then we have proved Theorem 2.4 in [3].

4. If we put $\boldsymbol{\delta} = \mathbf{1}$ and $\varphi = 0$ in Corollary 2, then we have proved inequality in [19], that is

$$2h\left(\frac{\varnothing_1 + \varnothing_2}{2}\right)\mathcal{F}_2\left(\frac{\varnothing_1 + \varnothing_2}{2}\right) \leq \frac{1}{\varnothing_2 - \varnothing_1}\int_{\varnothing_1}^{\varnothing_2} f(x)dx + \frac{1}{6}M(\varnothing_1, \varnothing_2) + \frac{1}{3}N(\varnothing_1, \varnothing_2).$$

5. Let $\varphi = 0$ and $\delta = 1$ in Corollary 3, we have Theorem 1.1 of [4].

6. If we suppose $\varphi = 0$ and $\delta = 1$ in Corollary 4, then we obtain last inequality of Theorem 1.2 of [4].

7. If we suppose $\varphi = 0$ and $\delta = 1$ in corollary 6, then we get the following inequality

$$\ln\left(h\left(\frac{\varnothing_1 + \varnothing_2}{2}\right)\mathcal{F}_2\left(\frac{\varnothing_1 + \varnothing_2}{2}\right)\right) \leq \frac{1}{b-a}\int_a^b \ln(h(x)\mathcal{F}_2(x))dx.$$

8. If we choose $\varphi = 0$ and $\delta = 1$ in corollary 7, then we have following inequalities

$$\begin{aligned} & \frac{1}{\varnothing_2 - \varnothing_1} \int_a^b h(x)\mathcal{F}_2(x)dx \\ \leq \; & \frac{1}{3}M(h, \mathcal{F}_2; \varnothing_1, \varnothing_2) + \frac{1}{6}N(h, \mathcal{F}_2; \varnothing_1, \varnothing_2) \\ \leq \; & \frac{1}{3}W(h, \mathcal{F}_2; \varnothing_1, \varnothing_2) + \frac{1}{3}T(h, \mathcal{F}_2; \varnothing_1, \varnothing_2). \end{aligned}$$

4 Conclusion

Interested reader can obtain in a similar way new results for different operators such as the k-Riemann-Liouville fractional integral, Katugampola fractional integrals, the conformable fractional integral, Hadamard fractional integrals, etc., and this results can be applied in different areas of pure and applied sciences.

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