Some curvature properties on Lorentzian β-Kenmotsu manifold with a semi-symmetric semi-metric connection

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Abstract

The object of the present paper is to study quasi conformal curvature tensor and conformal curvature tensor on Lorentzian $\beta-$Kenmotsu manifold with a semi-symmetric semi-metric connection. Moreover, we consider quasi conformally flat, $\xi-$quasi conformally flat, $\phi-$ conformally semisymmetric and conformally flat Lorentzian $\beta-$Kenmotsu manifold with a semi-symmetric semi-metric connection and obtain the scalar curvature r in each case.

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1 Introduction

The idea of a semi-symmetric linear connection on a differentiable manifold was introduced for the first time by A. Friedmann and J. A. Schouten [1] in 1924. A linear connection ∇ is said to be a semi-symmetric, if its torsion tensor T of the connection $\bar{\nabla}$

$$T(X,Y) = \bar{\nabla}_X Y - \bar{\nabla}_Y X - [X,Y],$$

satisfies

$$T(X,Y) = \eta(Y)X - \eta(X)Y,$$

where η is a 1-form. The connection ∇ is symmetric if the torsion tensor T vanishes, otherwise it is non-symmetric. The connection ∇ is metric connection if there is a Riemannian metric g in M such that $\nabla g = 0$, otherwise it is non-metric. It is well known that a linear connection is symmetric and metric if and only if it is the Levi-Civita connection. If moreover, a semi-symmetric connection $\bar{\nabla}$ satisfies the condition

$$(\bar{\nabla}_X g)(Y,Z) = 2\eta(X)g(Y,Z) - \eta(Y)g(X,Z) - \eta(Z)g(X,Y),$$

for all $X, Y, Z \in \varkappa(\bar{M})$, where $\varkappa(\bar{M})$ is the Lie algebra of vector fields of the manifold $\bar{M}$, then $\bar{\nabla}$ is said to be a semi-symmetric semi-metric connection [4].

In 1932, Hayden [8] introduced the notion of a semi-symmetric connection. Yano [14] studied some properties of a semi-symmetric metric connection on Riemannian manifold in 1970. Semi-symmetric metric connection has been studied by Imai [20] and also considered a hypersurface with a semi-symmetric metric connection [21] in 1972. Agashe and Chafle [17] introduced a semi-symmetric non-metric connection in a Riemannian manifold in 1992. Barua and Mukhopadhayay [5] studied a sequence of semi-metric connections on a Riemannian manifold in 1992. Submanifold of a Riemannian manifold admits a semi-symmetric semi-metric connection introduced by Barua

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[6] in 1998. In 2007, Khan [16] defined and studied a quarter-symmetric semi-metric connection on the Sasakian manifold.

In 1969, S. Tanno classified connected almost contact metric manifolds whose automorphism possess the maximum dimension [19]. For such a manifold, the sectional curvature of plane sections containing ξ is a constant , say c. He showed that they can be divided into three classes:

(1) Homogeneous normal contact Riemannian manifolds with $c > 0$.

(2) Global Riemannian products of a line or a circle with a Kaehler manifold of constant holomorphic sectional curvatureif $c = 0$.

(3) A wraped product space $R \times_f C$ if $c > 0$.

It is known that the manifolds of class (1) are characterized by admitting a Sasakian structure. Kenmotsu [11] characterized the differential geometric properties of the manifolds of class (3); the structure so obtained is now known as Kenmotsu structure. In general, these structures are not Sasakian [11].

In the Gray-Hervella classification of almost Hermitian manifolds [3], there appers a class W_4 of Hermitian manifolds, which are closely related to locally conformal Kaehler manifolds. An almost contact metric structure (ϕ, ξ, η, g) on a manifold $\bar{M}$ is called trans-Sasakian structure [9], if the product manifold $(\bar{M} \times R, J, G)$ belongs to the class W_4, where J is the almost structure on $\bar{M} \times R$ defined by

$$J(X, \frac{ad}{dt}) = (\phi X - a\xi, \eta(X)\frac{ad}{dt}),$$

for all vector fields X on $\bar{M}$ and smoth function a on $\bar{M} \times R$ and G is the product metric on $\bar{M} \times R$. This may be expressed by the condition [7]

$$(\nabla_X \phi)Y = \alpha[g(X, Y)\xi - \eta(Y)X] + \beta[g(\phi X, Y)\xi - \eta(Y)\phi X],$$

for some smooth functions α and β on $\bar{M}$ and we say that the trans-Sasakian structure is of type (α, β). If $\alpha = 0$ then a trans-Sasakian structure of type (α, β) with β a non-zero constant is always $\beta-$Kenmotsu manifold.

Motivated by the above studies, in this paper, we study some new results on a semi-symmetric semi-metric connection in an Lorentzian $\beta-$Kenmotsu manifold. The paper is organized as follows: In Section 2, we give a brief account of a Lorentzian $\beta-$Kenmotsu manifold. In Section 3, we study quasi conformally flat Lorentzian $\beta-$Kenmotsu manifold with a semi-symmetric semi-metric connection and obtain scalar curvature r. In Section 4, we show that Lorentzian $\beta-$Kenmotsu manifold with a semi-symmetric semi-metric connection is $\xi-$quasi conformally flat, then $\bar{R}(X, Y)\xi = 0$ and obtain scalar scalar curvature r. Section 5 is devoted to the study of $\phi-$ Conformally semi-symmetric Lorentzian $\beta-$Kenmotsu manifold with a semi-symmetric semi-metric connection and obtain scalar scalar curvature r. In section 6, we study conformally flat Lorentzian $\beta-$Kenmotsu manifold with a semi-symmetric semi-metric connection and obtain scalar scalar curvature r.

2 Preliminaries

A differentiable manifold $\bar{M}$ of dimension $(2n + 1)$ is called Lorentzian $\beta-$Kenmotsu manifold if it admits a $(1, 1)-$tensor field ϕ, a contravariant vector field ξ, a covariant vector field η and a Lorentzian metric g which satisfy

$$\phi^2 X = X + \eta(X)\xi, \quad \eta(\xi) = -1, \quad \phi\xi = 0, \quad \eta(\phi X) = 0, \tag{2.1}$$

$$\begin{aligned} g(\phi X, \phi Y) &= g(X,Y) + \eta(X)\eta(Y), \\ g(X,\xi) &= \eta(X), \quad g(\phi X, Y) = -g(X, \phi Y), \end{aligned} \tag{2.2}$$

for all $X, Y \in \varkappa(\bar{M})$. Then such a structure (ϕ, ξ, η, g) is termed as Lorentzian para-contact structure and the manifold $\bar{M}$ with a Lorentzian para-contact structure is called a Lorentzian para-contact manifold [12]. On a Lorentzian para-contact manifold, we also have

$$(\nabla_X \phi)(Y) = \beta[g(\phi X, Y) + \eta(Y)\phi X], \tag{2.3}$$

for any $X, Y \in \varkappa(\bar{M})$, where ∇ is the Levi-Civita connection with respect to the Lorentzian metric g. Thus a Lorentzian para-contact manifold satisfying (2.3) is called a Lorentzian $\beta-$Kenmotsu manifold [2].

From (2.3), It is easy to obtain that

$$\nabla_X \xi = -\beta\phi^2 X = -\beta[X + \eta(X)\xi], \tag{2.4}$$

$$(\nabla_X \eta)Y = \beta g(\phi X, \phi Y) = \beta[g(X,Y) + \eta(X)\eta(Y)].$$

Moreover the Riemannian curvature tensor R, the Ricci tensor S and the Ricci operator Q on a Lorentzian $\beta-$Kenmotsu manifold $\bar{M}$ with respect to the Levi-Civita connection satisfy the following equations [2]:

$$R(X,Y)\xi = \beta^2[\eta(Y)X - \eta(X)Y], \tag{2.5}$$

$$R(\xi, X)Y = \beta^2[g(X,Y)\xi - \eta(Y)X], \tag{2.6}$$

$$R(\xi, X)\xi = \beta^2[\eta(X)\xi + X], \tag{2.7}$$

$$S(X,\xi) = 2n\beta^2\eta(X), \quad S(\xi,\xi) = -2n\beta^2, \tag{2.8}$$

$$Q\xi = 2n\beta^2\xi, \tag{2.9}$$

$$S(\phi X, \phi Y) = S(X,Y) - 2n\beta^2\eta(X)\eta(Y), \tag{2.10}$$

where $X, Y \in \varkappa(\bar{M})$ and $S(X,Y) = g(QX,Y)$. The relation between the semi-symmetric semi-metric connection $\bar{\nabla}$ and the Levi-Civita connection ∇ on M is given by

$$\bar{\nabla}_X Y = \nabla_X Y - \eta(X)Y + g(X,Y)\xi. \tag{2.11}$$

Let $\bar{M}$ be an $(2n+1)-$dimensional Lorentzian $\beta-$Kenmotsu manifold. The curvature tensor $\bar{R}$ of $\bar{M}$ with respect to a semi-symmetric semi-metric connection $\bar{\nabla}$ is defined by

$$\bar{R}(X,Y)Z = \bar{\nabla}_X\bar{\nabla}_Y Z - \bar{\nabla}_Y\bar{\nabla}_X Z - \bar{\nabla}_{[X,Y]}Z. \tag{2.12}$$

From (2.2), (2.11) and (2.12), we obtain

$$\begin{aligned} \bar{R}(X,Y)Z &= R(X,Y)Z - (\beta-1)[\eta(X)g(Y,Z)\xi \\ &\quad -\eta(Y)g(X,Z)\xi] - \beta[g(Y,Z)X - g(X,Z)Y], \end{aligned} \tag{2.13}$$

where $X, Y, Z \in \varkappa(\bar{M})$. On contracting X in (2.13), we get

$$\bar{S}(X,Y) = S(X,Y) + (2\beta - (2n+1)\beta + 1)g(X,Y) + (\beta-1)\eta(X)\eta(Y), \tag{2.14}$$

where $\bar{S}$ and S are the Ricci tensors of the connections $\bar{\nabla}$ and ∇, respectively on $\bar{M}$. This gives

$$\bar{Q}X = QX + (2\beta - (2n+1)\beta + 1)X + (\beta - 1)\eta(Y)\xi. \tag{2.15}$$

where $\bar{Q}$ and Q are the Ricci operators of the connections $\bar{\nabla}$ and ∇, respectively on $\bar{M}$.

Contracting again X and Y in (2.14), it follows that

$$\bar{r} = r - 2n[2n\beta + 1]. \tag{2.16}$$

where $\bar{r}$ and r are the scalar curvatures of the connections $\bar{\nabla}$ and ∇, respectively on $\bar{M}$.

From (2.14) , it follows that

$$\bar{S}(X,Y) = \bar{S}(Y,X). \tag{2.17}$$

Thus the Ricci tensor $\bar{S}$ in a Lorentzian $\beta-$Kenmotsu manifold with a semi-symmetric semi-metric connection is symmetric.

From (2.12) and using (2.1), (2.2), (2.5), (2.6) & (2.7) we conclude that

$$\bar{R}(X,Y)\xi = \beta(\beta-1)[\eta(Y)X - \eta(X)Y], \tag{2.18}$$

$$\bar{R}(\xi,X)Y = (\beta^2-1)g(X,Y)\xi - \beta(\beta-1)\eta(X)Y + (\beta-1)\eta(X)\eta(Y)\xi, \tag{2.19}$$

$$\bar{R}(\xi,X)\xi = \beta(\beta-1)[\eta(X)\xi + X]. \tag{2.20}$$

Now from (2.14), we get

$$\bar{S}(X,\xi) = [2n\beta^2 - 2(n+1)\beta + 4n + 2]\eta(X), \tag{2.21}$$

and from (2.15), we get

$$\bar{Q}\xi = [2n\beta(\beta-1) + 2]\xi. \tag{2.22}$$

3 Quasi conformally flat Lorentzian $\beta-$Kenmotsu manifold with a semi-symmetric semi-metric connection

Definition 3.1. A Quasi conformal curvature tensor $\bar{\mathbb{C}}$ of a Lorentzian $\beta-$Kenmotsu manifold $\bar{M}$ with a semi-symmetric semi-metric connection $\bar{\nabla}$ is given by [18]

$$\begin{aligned}\bar{\mathbb{C}}(X,Y)Z &= \bar{R}(X,Y)Z + a[\bar{S}(Y,Z)X - \bar{S}(X,Z)Y] + b[g(Y,Z)\bar{Q}X - g(X,Z)\bar{Q}Y] \\ &\quad - \frac{c\bar{r}}{2n+1}(\frac{1}{2n} + a + b)[g(Y,Z)X - g(X,Z)Y].\end{aligned} \tag{3.1}$$

where a and b are constants such that $a, b \neq 0$ and $\bar{R}$, $\bar{S}$ and $\bar{r}$ are the curvature tensor, the Ricci tensor and the scalar curvature with a semi-symmetric semi-metric connection on $\bar{M}$.

Theorem 3.2. If a Lorentzian $\beta-$Kenmotsu manifold with a semi-symmetric semi-metric connection is quasi conformally flat, then the scalar curvature r is given by

$$\begin{aligned}r &= (\frac{2n(2n+1)}{c(1+2n(a+b))})\{\beta(\beta-1) \\ &\quad + a(2n\beta^2 - 2(n+1)\beta + 4n + 2) + 2nb\beta^2 \\ &\quad + b(2\beta - (2n+1)\beta + 1) + \frac{2n(2n\beta+1)c}{2n+1}(\frac{1}{2n} + a + b)\}.\end{aligned}$$

Proof. Let us assume that the manifold $\bar{M}$ with a semi-symmetric semi-metric connection is quasi conformally flat, then

$$g(\bar{\mathbb{C}}(X,Y)Z,\phi W)=0, \tag{3.2}$$

and hence

$$\begin{aligned}&g(\bar{R}(X,Y)Z,\phi W)+a[\bar{S}(Y,Z)g(X,\phi W)-\bar{S}(X,Z)g(Y,\phi W)]\\&+b[g(Y,Z)g(\bar{Q}X,\phi W)-g(X,Z)g(\bar{Q}Y,\phi W)]\\&-\frac{c\bar{r}}{2n+1}(\frac{1}{2n}+a+b)[g(Y,Z)g(X,\phi W)-g(X,Z)g(Y,\phi W)]=0.\end{aligned} \tag{3.3}$$

Putting $Y=Z=\xi$ and using (2.1), (2.2), (2.20) and (2.21) in (3.3), we have

$$\begin{aligned}&\{-\beta(\beta-1)-a(2n\beta^2-2(n+1)\beta+4n+2)\\&-2nb\beta^2-b(2\beta-(2n+1)\beta+1)\\&+\frac{c\bar{r}}{2n+1}(\frac{1}{2n}+a+b)\}g(X,\phi W)=0.\end{aligned} \tag{3.4}$$

which on using (2.16) gives

$$\begin{aligned}r &= (\frac{2n(2n+1)}{c(1+2n(a+b))})\{\beta(\beta-1)\\&+a(2n\beta^2-2(n+1)\beta+4n+2)+2nb\beta^2\\&+b(2\beta-(2n+1)\beta+1)+\frac{2n(2n\beta+1)c}{2n+1}(\frac{1}{2n}+a+b)\},\end{aligned}$$

as $g(X,\phi W)\neq 0$. ∎

4 $\xi-$Quasi conformally flat Lorentzian $\beta-$Kenmotsu manifold with a semi-symmetric semi-metric connection

Theorem 4.1. If a Lorentzian $\beta-$Kenmotsu manifold with a semi-symmetric semi-metric connection is $\xi-$quasi conformally flat, then $\bar{R}(X,Y)\xi=0$ and the scalar curvature r is given by

$$\begin{aligned}r &= (\frac{2n(2n+1)}{c(1+2n(a+b))})\{\beta(\beta-1)+a(2n\beta^2-2(n+1)\beta+4n+2)\\&+b(2n\beta(\beta-1)+2)+\frac{2n(2n\beta+1)c}{2n+1}(\frac{1}{2n}+a+b)\}.\end{aligned}$$

Proof. If a Lorentzian $\beta-$Kenmotsu manifold with a semi-symmetric semi-metric connection is $\xi-$quasi conformally flat, then

$$\bar{\mathbb{C}}(X,Y)\xi=0, \tag{4.1}$$

and hence

$$\begin{aligned}&\bar{R}(X,Y)\xi+a[\bar{S}(Y,\xi)X-\bar{S}(X,\xi)Y]+b[g(Y,\xi)\bar{Q}X-g(X,\xi)\bar{Q}Y]\\&-\frac{c\bar{r}}{2n+1}(\frac{1}{2n}+a+b)[g(Y,\xi)X-g(X,\xi)Y]=0.\end{aligned} \tag{4.2}$$

Taking inner product of (4.2) with U, we have

$$\begin{aligned}&g(\bar{R}(X,Y)\xi,U)+a[\bar{S}(Y,\xi)g(X,U)-\bar{S}(X,\xi)g(Y,U)]\\&+b[g(Y,\xi)g(\bar{Q}X,U)-g(X,\xi)g(\bar{Q}Y,U)]\\&-\frac{c\bar{r}}{2n+1}(\frac{1}{2n}+a+b)[g(Y,\xi)g(X,U)-g(X,\xi)g(Y,U)]=0.\end{aligned}\tag{4.3}$$

Taking $U=\xi$ and using (2.1), (2.2),(2.20) and (2.21) in (4.3), we get

$$g(\bar{R}(X,Y)\xi,\xi)=0,$$

which implies

$$\bar{R}(X,Y)\xi=0.\tag{4.4}$$

Using (2.2), (2.21) and (4.4) in (4.2), it follows that

$$\begin{aligned}&\frac{c\bar{r}}{2n+1}(\frac{1}{2n}+a+b)\}[\eta(Y)X-\eta(X)Y]\\=\;&\{\beta(\beta-1)+a(2n\beta^2-2(n+1)\beta+4n+2)\\&+b(2n\beta(\beta-1)+2).\end{aligned}\tag{4.5}$$

which on using (2.16) gives

$$\begin{aligned}r\;=\;&(\frac{2n(2n+1)}{c(1+2n(a+b))})\{\beta(\beta-1)+a(2n\beta^2-2(n+1)\beta+4n+2)\\&+b(2n\beta(\beta-1)+2)+\frac{2n(2n\beta+1)c}{2n+1}(\frac{1}{2n}+a+b)\}.\end{aligned}$$

Hence proved. ∎

5 $\phi-$ Conformally semi-symmetric Lorentzian $\beta-$Kenmotsu manifold with a semi-symmetric semi-metric connection

Definition 5.1. The conformal curvature tensor $\mathbb{N}$ of a Lorentzian $\beta-$Kenmotsu manifold $\bar{M}$ with a semi-symmetric semi-metric connection $\bar{\nabla}$ is given by [18]

$$\begin{aligned}\bar{\mathbb{N}}(X,Y)Z\;=\;&\bar{R}(X,Y)Z+\frac{1}{2n-1}[\bar{S}(X,Z)Y-\bar{S}(Y,Z)X+g(X,Z)\bar{Q}Y\\&-g(Y,Z)\bar{Q}X]-\frac{\bar{r}}{2n(2n-1)}[g(X,Z)Y-g(Y,Z)X],\end{aligned}\tag{5.1}$$

where $\bar{R}$, $\bar{S}$ and $\bar{r}$ are the curvature tensor, the Ricci tensor and the scalar curvature with a semi-symmetric semi-metric connection on $\bar{M}$.

Theorem 5.2. For a $(2n+1)$ dimensional $\phi-$ Conformally semi-symmetric $(n>1)$ Lorentzian $\beta-$Kenmotsu manifold with a semi-symmetric semi-metric connection, the curvature tensor r is given by

$$r=2n(2n-1)(1-\beta^2)+2n(4n\beta^2-(4n-1)\beta+3)+2n(2n\beta+1).$$

Proof. Let $\bar{M}$ be a $(2n+1)$ dimensional $(n>1)$ $\phi-$ Conformally semi-symmetric Lorentzian $\beta-$Kenmotsu manifold with a semi-symmetric semi-metric connection. Therefore $\bar{\mathbb{N}}(X,Y)\cdot\phi=0$ turns into

$$(\bar{\mathbb{N}}(X,Y)\cdot\phi)Z=\bar{\mathbb{N}}(X,Y)\phi Z-\phi\bar{\mathbb{N}}(X,Y)Z=0, \tag{5.2}$$

for any vector fields $X,Y,Z\in\varkappa(\bar{M})$. Now from (5.1), we have

$$\begin{aligned}\bar{\mathbb{N}}(X,Y)\phi Z &= \bar{R}(X,Y)\phi Z+\frac{1}{2n-1}[\bar{S}(X,\phi Z)Y-\bar{S}(Y,\phi Z)X+g(X,\phi Z)\bar{Q}Y\\ &\quad -g(Y,\phi Z)\bar{Q}X]-\frac{\bar{r}}{2n(2n-1)}[g(X,\phi Z)Y-g(Y,\phi Z)X].\end{aligned} \tag{5.3}$$

In view of (2.1), (2.13), (2.14) and (2.16), (5.3) yields

$$\begin{aligned}\bar{\mathbb{N}}(X,Y)\phi Z &= \{\beta^2[g(Y,\phi Z)X-g(X,\phi Z)Y]--(\beta-1)[\eta(X)g(Y,\phi Z)\xi\\ &\quad -\eta(Y)g(X,\phi Z)\xi]-\beta[g(Y,\phi Z)X-g(X,\phi Z)Y]\}\\ &\quad +\frac{1}{2n-1}\{S(X,\phi Z)Y-S(Y,\phi Z)X\\ &\quad +(2\beta-(2n+1)\beta+1)(g(X,\phi Z)Y-g(Y,\phi Z)X)\\ &\quad +2n\beta^2(g(X,\phi Z)Y-g(Y,\phi Z)X)+(2\beta-(2n+1)\beta\\ &\quad +1)(g(X,\phi Z)Y-g(Y,\phi Z)X)+(\beta-1)(g(X,\phi Z)\eta(Y)\xi\\ &\quad -g(Y,\phi Z)\eta(X)\xi\}-\frac{r-2n(2n\beta+1)}{2n(2n-1)}[g(X,\phi Z)Y-g(Y,\phi Z)X].\end{aligned} \tag{5.4}$$

Similarly, we have

$$\begin{aligned}\phi\bar{\mathbb{N}}(X,Y)Z &= \{\beta^2(g(Y,Z)\phi X-g(X,Z)\phi Y)-\beta(g(Y,Z)\phi X-g(X,Z)\phi Y)\}\\ &\quad +\frac{1}{2n-1}\{S(X,Z)\phi Y-S(Y,Z)\phi X\\ &\quad -(2\beta-(2n+1)\beta+1)(g(Y,Z)\phi X-g(X,Z)\phi Y)\\ &\quad +(\beta-1)(\eta(X)\eta(Z)\phi Y-\eta(Y)\eta(Z)\phi X)\\ &\quad -2n\beta^2(g(Y,Z)\phi X-g(X,Z)\phi Y)\\ &\quad -(2\beta-(2n+1)\beta+1)(g(Y,Z)\phi X-g(X,Z)\phi Y)\}\\ &\quad +\frac{r-2n(2n\beta+1)}{2n(2n-1)}(g(Y,Z)\phi X-g(X,Z)\phi Y).\end{aligned} \tag{5.5}$$

Putting (5.4) and (5.5) and then taking $Y=\xi$ in (5.2), we obtain

$$\begin{aligned}&[-\beta^2+1+\frac{1}{2n-1}(4n\beta^2-4n\beta+\beta+3)-\frac{r-2n(2n\beta+1)}{2n(2n-1)}]g(X,\phi Z)\xi\\ &+[-\beta^2-\beta-\frac{1}{2n-1}(-4n\beta^2+2n\beta-2)-\frac{r-2n(2n\beta+1)}{2n(2n-1)}]\eta(Z)\phi X=0.\end{aligned} \tag{5.6}$$

Now considering Z to be orthogonal to ξ, then $\eta(Z)=0$ and $g(X,\phi Z)\neq 0$, which implies that

$$[-\beta^2+1+\frac{1}{2n-1}(4n\beta^2-4n\beta+\beta+3)-\frac{r-2n(2n\beta+1)}{2n(2n-1)}]=0,$$

which on simplifying gives

$$r = 2n(2n-1)(1-\beta^2) + 2n(4n\beta^2 - (4n-1)\beta + 3) + 2n(2n\beta + 1).$$

Hence proved. ∎

6 Conformally flat Lorentzian $\beta-$Kenmotsu manifold with a semi-symmetric semi-metric connection

Theorem 6.1. If a Lorentzian $\beta-$Kenmotsu manifold with a semi-symmetric semi-metric connection is conformally flat, then the scalar curvature r is given by

$$\begin{aligned} r &= 2n(2n-1)\beta(1-\beta) + 2n(4n(\beta^2 - \beta + 1) \\ &\quad -\beta + 3) + 2n(2n\beta + 1). \end{aligned}$$

Proof. Let us assume that the manifold $\bar{M}$ with a semi-symmetric semi-metric connection is conformally flat, then

$$g(\bar{\mathbb{N}}(X,Y)Z, \phi W) = 0, \tag{6.1}$$

and hence

$$\begin{aligned} &g(\bar{R}(X,Y)Z, \phi W) + \frac{1}{2n-1}[\bar{S}(X,Z)g(Y,\phi W) \\ &-\bar{S}(Y,Z)g(X,\phi W) + g(X,Z)g(\bar{Q}Y,\phi W) - g(Y,Z)g(\bar{Q}X,\phi W)] \\ &-\frac{\bar{r}}{2n(2n-1)}[g(X,Z)g(Y,\phi W) - g(Y,Z)g(X,\phi W)] = 0. \end{aligned} \tag{6.2}$$

Putting $Y = Z = \xi$ and using (2.1), (2.2), (2.20) and (2.21) in (6.2), we have

$$\begin{aligned} &[\beta(1-\beta) + \frac{1}{2n-1}(4n(\beta^2 - \beta + 1) - \beta + 3) \\ &-\frac{\bar{r}}{2n(2n-1)}]g(X,\phi W) = 0, \end{aligned}$$

which on using (2.16) gives

$$\begin{aligned} r &= 2n(2n-1)\beta(1-\beta) + 2n(4n(\beta^2 - \beta + 1) \\ &\quad -\beta + 3) + 2n(2n\beta + 1), \end{aligned}$$

as $g(X, \phi W) \neq 0$. ∎

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