Third order differential subordination and superordination results for analytic functions involving the Hohlov operator

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Abstract

The problems of third order differential subordination as well as superordination for functions analytic in the open unit disk seem to be new and have been of interest after the seminal work of Antonino and Miller [1], and Tang [29, 30, 31]. In the present paper by considering suitable classes of admissible functions associated with Hohlov operator, various third order differential subordination as well as differential superordination results are obtained. As a consequence, the dual problems which gives the sandwich-type relations are presented. Upon suitable choice of the parameters, the results obtained in this paper include some classical as well as recently studied results. An attempt has also been made to illustrate the applications of the new results in the context of electromagnetic cloaking problem.

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1 Introduction and preliminaries

1.1 Differential subordination

The notion of differential subordination in the complex plane $\mathbb{C}$ is a generalization of differential inequalities in the real line $\mathbb{R}$. Research in this area was pioneered by the works of Miller, Mocanu, and Reade in 1981. The monograph by Miller and Mocanu [21] also see [18] gives a good introduction to the theory of differential subordination. In 2003, Miller and Mocanu [22] investigated the dual problem of differential superordination, whereas Bulboaca [2] investigated both subordination and superordination related results. The theory of first and second order differential subordination and superordination have been used by numerous authors to solve various problems in this field (see [5, 32, 10, 9, 28, 24, 4, 3, 19, 20]). Recently, Antonino and Miller [1] extended the theory of second order differential subordination in the open unit disk $\mathbb{U}$ introduced by Miller and Mocanu [21] to the third order case. However, the concept of third oder differential subordination have originally found in the work of Ponnusamy and Juneja [24], (also see [21]). In 2014, Tang et al. [29] introduced the concept of the third order differential superordination, which is a generalization of the second order differential superordination. In the recent years, some work have been carried out on results related to the third order differential subordination and superordination in the different context. For example see ([8, 14, 15, 25, 29, 30, 31]).

Let $\mathcal{H}$ be the class of functions analytic in the *open* unit disk $\mathbb{U} := \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$, and $\mathcal{H}[a, n]$ $(n \in \mathbb{N} := \{1, 2, 3, \cdots\})$ be the subclass of $\mathcal{H}$ consisting of functions of $f(z) =$

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$a + a_n z^n + a_{n+1} z^{n+1} + \cdots, \quad (z \in \mathbb{U})$. Let $\mathcal{A}(\subset \mathcal{H})$ be the class of functions analytic in $\mathbb{U}$ and has the Taylor-Maclaurin series of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad (z \in \mathbb{U}). \tag{1.1}$$

Suppose that f and g are in $\mathcal{H}$. We say that f is *subordinate* to g, (or g is *superordinate* to f), write as $f \prec g$ in $\mathbb{U}$ or $f(z) \prec g(z) \quad (z \in \mathbb{U})$, if there exists a function $\omega \in \mathcal{H}$, satisfying the conditions of the Schwarz lemma (i.e. $\omega(0) = 0$ and $|\omega(z)| < 1$) such that $f(z) = g(\omega(z)) \quad (z \in \mathbb{U})$. It follows that $f(z) \prec g(z)$ $(z \in \mathbb{U}) \Longrightarrow f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$. In particular, if g is *univalent* in $\mathbb{U}$, then the reverse implication also holds (cf.[21]).

Definition 1. [[1], Definition 1, p.440]. Let $\psi : \mathbb{C}^4 \times \mathbb{U} \longrightarrow \mathbb{C}$ and the function $h(z)$ be univalent in $\mathbb{U}$. If the function $p(z)$ is analytic in $\mathbb{U}$ and satisfies the following third-order differential subordination

$$\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z) \prec h(z), \tag{1.2}$$

then $p(z)$ is called a *solution* of the differential subordination.

A univalent function $q(z)$ is called a *dominant* of the solutions of the differential subordination,or, more simply, a dominant if $p(z) \prec q(z)$ for all $p(z)$ satisfying (1.2). A dominant $\tilde{q}(z)$ that satisfies $\tilde{q}(z) \prec q(z)$ for all dominants $q(z)$ of (1.2) is said to be the *best dominant*.

Definition 2. [29] Let $\psi : \mathbb{C}^4 \times \overline{\mathbb{U}} \longrightarrow \mathbb{C}$ and the function $h(z)$ be univalent in $\mathbb{U}$. If the function $p(z)$ and $\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z)$ are univalent in $\mathbb{U}$ and satisfies the following third-order differential superordination

$$h(z) \prec \psi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z), \tag{1.3}$$

then $p(z)$ is called a *solution* of the differential superordination. An analytic function $q(z)$ is called a *subordinant* of the solutions of the differential superordination, or more simply a subordinant, if $q(z) \prec p(z)$ for all $p(z)$ satisfying (1.3).

A univalent subordinant $\tilde{q}(z)$ that satisfies $q(z) \prec \tilde{q}(z)$ for all subordinant $q(z)$ of (1.3) is said to be the *best subordinant*. We note that both the best dominant and best subordinant are unique up to rotation of $\mathbb{U}$.

Definition 3. [[1], Definition 2, p.441]. Let $\mathcal{Q}$ denote the set of functions q that are analytic and univalent on the set $\overline{\mathbb{U}} \setminus E(q)$, where $E(q) = \{\xi : \xi \in \partial\mathbb{U} : \lim_{z\to\xi} q(z) = \infty\}$, and are such that $\min \mid q'(\xi) \mid = \rho > 0$ for $\xi \in \partial\mathbb{U} \setminus E(q)$. Further, let the subclass of $\mathcal{Q}$ for which $q(0) = a$ be denoted by $\mathcal{Q}(a)$, $\mathcal{Q}(0) = \mathcal{Q}_0$ and $\mathcal{Q}(1) = \mathcal{Q}_1$.

The subordination methodology is applied to an appropriate class of admissible functions. The following class of admissible functions is due to Antonino and Miller.

Definition 4. [[1], Definition 3, p.449]. Let Ω be a set in $\mathbb{C}$ and $q \in \mathcal{Q}$ and $n \in \mathbb{N} \setminus \{1\}$. The class of admissible functions $\Psi_n[\Omega, q]$ consists of those functions $\psi : \mathbb{C}^4 \times \mathbb{U} \longrightarrow \mathbb{C}$ achieving the following admissibility conditions: $\psi(r, s, t, u; z) \notin \Omega$ whenever

$$r = q(\zeta), s = k\zeta q'(\zeta), \Re\left(\frac{t}{s} + 1\right) \geq k\Re\left(\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right) \text{ and } \Re\left(\frac{u}{s}\right) \geq k^2\Re\left(\frac{\zeta^2 q'''(\zeta)}{q'(\zeta)}\right),$$

where $z \in \mathbb{U}, \zeta \in \partial\mathbb{U} \setminus E(q)$, and $k \geq n$.

The next lemma is the foundation result in the theory of third-order differential subordination.

Lemma 1. [[1], Theorem 1, p.449]. Let $p \in \mathcal{H}[a,n]$ with $n \geq 2$, and $q \in \mathcal{Q}(a)$ achieving the following conditions: $\Re\left(\frac{\zeta q''(\zeta)}{q'(\zeta)}\right) \geq 0$, and $\left|\frac{zp'(z)}{q'(\zeta)}\right| \leq k$, where $z \in \mathbb{U}, \zeta \in \partial\mathbb{U}\backslash E(q)$, and $k \geq n \geq 2$. If Ω is a set in $\mathbb{C}, \psi \in \Psi_n[\Omega, q]$ and $\psi\left(p(z), zp'(z), z^2p''(z), z^3p'''(z); z\right) \subset \Omega$, then $p(z) \prec q(z) \qquad (z \in \mathbb{U})$.

Definition 5. [29] Let Ω be a set in $\mathbb{C}$ and $q \in \mathcal{H}[a,n]$ and $q'(z) \neq 0$. The class of admissible functions $\Psi'_n[\Omega, q]$ consists of those functions $\psi : \mathbb{C}^4 \times \overline{\mathbb{U}} \longrightarrow \mathbb{C}$ that satisfy the following admissibility conditions: $\psi(r,s,t,u;\zeta) \in \Omega$ whenever

$$r = q(z), s = \frac{zq'(z)}{m}, \Re\left(\frac{t}{s}+1\right) \leq \frac{1}{m}\Re\left(\frac{zq''(z)}{q'(z)}+1\right), \quad \text{and} \quad \Re\left(\frac{u}{s}\right) \leq \frac{1}{m^2}\Re\left(\frac{z^2q'''(z)}{q'(z)}\right),$$

where $z \in \mathbb{U}, \zeta \in \partial\mathbb{U}$, and $m \geq n \geq 2$.

Lemma 2. [29] Let $p \in \mathcal{H}[a,n]$ with $\psi \in \Psi'_n[\Omega, q]$. If $\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z)$ is univalent in $\mathbb{U}$ and $p \in \mathcal{Q}(a)$ satisfying the following conditions: $\Re\left(\frac{zq''(z)}{q'(z)}\right) \geq 0, and \left|\frac{zp'(z)}{q'(z)}\right| \leq m$, where $z \in \mathbb{U}, \zeta \in \partial\mathbb{U}$, and $m \geq n \geq 2$, then $\Omega \subset \{\psi\left(p(z), zp'(z), z^2p''(z), z^3p'''(z); z\right) : z \in \mathbb{U}\}$ implies that $q(z) \prec p(z) \qquad (z \in \mathbb{U})$.

1.2 Hohlov operator

(cf.[12, 13]) In this section we recall the linear operator, $\mathcal{I}^c_{a,b}$ defined in terms of convolution as follows.

$$\mathcal{I}^c_{a,b}f(z) := z_2F_1(a,b;c;z) * f(z) \qquad (z \in \mathbb{U}), \tag{1.4}$$

where ${}_2F_1(a,b,c;z)$ is the Gauss hypergeometric function given by

$${}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!} \qquad (z \in \mathbb{U}),$$

for the complex parameters a, b and c with $c \neq 0, -1, -2, -3, \cdots$, $(\alpha)_n$ denotes the Pochhammer symbol (or shifted factorial) and $*$ denote the convolution or Hadamard product. Note that convolutions of two analytic functions is also analytic. From (1.4) it is clear that

$$z(\mathcal{I}^c_{a,b}f(z))' = a(\mathcal{I}^c_{a+1,b}f(z)) - (a-1)\mathcal{I}^c_{a,b}f(z). \tag{1.5}$$

Upon suitable choice of the parameters, the above defined three-parameter family $\mathcal{I}^c_{a,b}$, unifies various other linear operators which are introduced and studied earlier have been pointed out in [11] and the reference therein. In [11], Gochhayat et al. discussed various coefficient related problems of a newly defined family associated with $\mathcal{I}^c_{a,b}$. In the sequel to earlier investigations, by considering suitable classes of admissible functions, we state and prove some interesting inclusion results on third order differential subordination and superordination involving $\mathcal{I}^c_{a,b}$. More precisely, following third order differential *sandwich-type* relations are obtained:

$$q_1(z) \prec \mathcal{I}^c_{a,b}f(z) \prec q_2(z), \quad q_1(z) \prec \frac{\mathcal{I}^c_{a,b}f(z)}{z} \prec q_2(z) \text{ and } q_1(z) \prec \frac{\mathcal{I}^c_{a+1,b}f(z)}{\mathcal{I}^c_{a,b}f(z)} \prec q_2(z).$$

2 Third order differential subordination of the Hohlov operator $\mathcal{I}^c_{a,b}$

In this section the following class of admissible functions is introduced which are required to prove the main results.

Definition 6. Let Ω be a set in $\mathbb{C}, a, a+1$ and $a+2 \in \mathbb{C} \setminus \{0\}$ and $q \in \mathcal{Q}_0 \bigcap \mathcal{H}_0$. The class of admissible function $\Phi_{\mathcal{I}}[\Omega, q]$ consist of those functions $\varphi : \mathbb{C}^4 \times \mathbb{U} \longrightarrow \mathbb{C}$ that satisfy the following admissibility conditions: $\varphi(v, w, x, y; z) \notin \Omega$ whenever

$$v = q(\zeta), w = \frac{k\zeta q'(\zeta) + (a-1)q(\zeta)}{a}, \Re\left(\frac{a(a+1)x - a(a-1)v}{aw - (a-1)v} + (1-2a)\right) \geq k\Re\left(\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right),$$

and

$$\Re\left(\frac{a(a+1)(a+2)y - (3a+3)a(a+1)x + a(a-1)v(2a+2)}{aw - (a-1)v} + (3a^2+3a)\right) \geq k^2\Re\left(\frac{\zeta^2 q'''(\zeta)}{q'(\zeta)}\right),$$

where $z \in \mathbb{U}, \zeta \in \partial\mathbb{U} \setminus E(q)$, and $k \geq 2$.

Theorem 1. Let $\varphi \in \Phi_{\mathcal{I}}[\Omega, q]$. If the function $f \in \mathcal{A}$ and $q \in \mathcal{Q}_0$ satisfy the following conditions:

$$\Re\left(\frac{\zeta q''(\zeta)}{q'(\zeta)}\right) \geq 0, \left|\frac{\mathcal{I}^c_{a+1,b} f(z)}{q'(\zeta)}\right| \leq k, \tag{2.1}$$

and

$$\left\{\varphi(\mathcal{I}^c_{a,b} f(z), \mathcal{I}^c_{a+1,b} f(z), \mathcal{I}^c_{a+2,b} f(z), \mathcal{I}^c_{a+3,b} f(z); z) : z \in \mathbb{U}\right\} \subset \Omega, \tag{2.2}$$

then

$$\mathcal{I}^c_{a,b} f(z) \prec q(z), \qquad (z \in \mathbb{U}).$$

Proof. Define the analytic function $p(z)$ in $\mathbb{U}$ by

$$p(z) = \mathcal{I}^c_{a,b} f(z). \tag{2.3}$$

From equation (1.5) and (2.3), we have $\mathcal{I}^c_{a+1,b} f(z) = \frac{zp'(z) + (a-1)p(z)}{a}$. Similar arguments yields,

$$\mathcal{I}^c_{a+2,b} f(z) = \frac{z^2 p''(z) + 2azp'(z) + a(a-1)p(z)}{a(a+1)}, \tag{2.4}$$

and

$$\mathcal{I}^c_{a+3,b} f(z) = \frac{z^3 p'''(z) + (3a+3)z^2 p''(z) + (3a^2+3a)zp'(z) + a(a-1)(a+1)p(z)}{a(a+1)(a+2)}. \tag{2.5}$$

Define the transformation from $\mathbb{C}^4$ to $\mathbb{C}$ by

$$v(r, s, t, u) = r, \quad w(r, s, t, u) = \frac{s + (a-1)r}{a}, \quad x(r, s, t, u) = \frac{t + 2as + a(a-1)r}{a(a+1)}, \tag{2.6}$$

and

$$y(r, s, t, u) = \frac{u + (3a+3)t + (3a^2+3a)s + a(a-1)(a+1)r}{a(a+1)(a+2)}. \tag{2.7}$$

Let

$$\psi(r,s,t,u)=\varphi(v,w,x,y;z)=\varphi\Bigg(r,\frac{s+(a-1)r}{a},\frac{t+2as+a(a-1)r}{a(a+1)},\\ \frac{u+(3a+3)t+(3a^2+3a)s+a(a-1)(a+1)r}{a(a+1)(a+2)};z\Bigg). \quad (2.8)$$

The proof will make use of Lemma 1. Using equations (2.3) to (2.5), and from (2.8), we have

$$\psi\left(p(z),zp'(z),z^2p''(z),z^3p'''(z);z\right) = \varphi\left(\mathcal{I}^c_{a,b}f(z),\mathcal{I}^c_{a+1,b}f(z),\mathcal{I}^c_{a+2,b}f(z),\mathcal{I}^c_{a+3,b}f(z);z\right). \quad (2.9)$$

Hence,(2.2) becomes

$$\psi\left(p(z),zp'(z),z^2p''(z),z^3p'''(z);z\right)\in\Omega.$$

Note that

$$\frac{t}{s}+1=\frac{a(a+1)x-a(a-1)v}{aw-(a-1)v}+(1-2a)$$

and

$$\frac{u}{s}=\frac{a(a+1)(a+2)y-(3a+3)a(a+1)x+a(a-1)v(2a+2)}{aw-(a-1)v}+(3a^2+3a).$$

Thus, the admissibility condition for $\varphi\in\Phi_{\mathcal{I}}[\Omega,q]$ in Definition 6 is equivalent to the admissibility condition for $\psi\in\Psi_2[\Omega,q]$ as given in Definition 4 with $n=2$. Therefore, by using (2.1) and Lemma 1, we have $\mathcal{I}^c_{a,b}f(z)\prec q(z)$. This completes the proof. Q.E.D.

The next result is an extension of Theorem 1 to the case where behavior of $q(z)$ on $\partial\mathbb{U}$ is not known.

Corollary 1. Let $\Omega\subset\mathbb{C}$ and let the function q be univalent in $\mathbb{U}$ with $q(0)=0$. Let $\varphi\in\Phi_{\mathcal{I}}[\Omega,q_\rho]$ for some $\rho\in(0,1)$, where $q_\rho(z)=q(\rho z)$. If the function $f\in\mathcal{A}$ and q_ρ satisfy the following conditions

$$\Re\left(\frac{\zeta q''_\rho(\zeta)}{q'_\rho(\zeta)}\right)\geq 0,\qquad \left|\frac{\mathcal{I}^c_{a+1,b}f(z)}{q'_\rho(\zeta)}\right|\leq k\quad (z\in\mathbb{U},\zeta\in\partial\mathbb{U}\setminus E(q_\rho)), \quad (2.10)$$

and $\varphi(\mathcal{I}^c_{a,b}f(z),\mathcal{I}^c_{a+1,b}f(z),\mathcal{I}^c_{a+2,b}f(z),\mathcal{I}^c_{a+3,b}f(z);z)\in\Omega$, then $\mathcal{I}^c_{a,b}f(z)\prec q(z)\qquad (z\in\mathbb{U})$.

Proof. From Theorem 1, then $\mathcal{I}^c_{a,b}f(z)\prec q_\rho(z)$. The result asserted by Corollary 1 is now deduced from the following subordination property $q_\rho(z)\prec q(z),\qquad (z\in\mathbb{U})$. Q.E.D.

If $\Omega\neq\mathbb{C}$ is a simply connected domain, then $\Omega=h(\mathbb{U})$ for some conformal mapping $h(z)$ of $\mathbb{U}$ onto Ω. In this case, the class $\Phi_{\mathcal{I}}[h(\mathbb{U}),q]$ is written as $\Phi_{\mathcal{I}}[h,q]$. The following result follows immediately as a consequence of Theorem 1.

Theorem 2. Let $\varphi\in\Phi_{\mathcal{I}}[h,q]$. If the function $f\in\mathcal{A}$ and $q\in\mathcal{Q}_0$ satisfy the conditions (2.1) and

$$\varphi(\mathcal{I}^c_{a,b}f(z),\mathcal{I}^c_{a+1,b}f(z),\mathcal{I}^c_{a+2,b}f(z),\mathcal{I}^c_{a+3,b}f(z);z)\prec h(z), \quad (2.11)$$

then $\mathcal{I}^c_{a,b}f(z)\prec q(z)\qquad (z\in\mathbb{U})$.

The next result is an immediate consequence of Corollary 1.

Corollary 2. Let $\Omega \subset \mathbb{C}$ and let the function q be univalent in $\mathbb{U}$ with $q(0) = 0$. Let $\varphi \in \Phi_{\mathcal{I}}[h, q_\rho]$ for some $\rho \in (0, 1)$, where $q_\rho(z) = q(\rho z)$. If the function $f \in \mathcal{A}$ and q_ρ satisfy the conditions (2.10) and $\varphi(\mathcal{I}^c_{a,b}f(z), \mathcal{I}^c_{a+1,b}f(z), \mathcal{I}^c_{a+2,b}f(z), \mathcal{I}^c_{a+3,b}f(z); z) \prec h(z)$, then $\mathcal{I}^c_{a,b}f(z) \prec q(z)$ $(z \in \mathbb{U})$.

The following result yields the best dominant of the differential subordination (2.11).

Theorem 3. Let the function h be univalent in $\mathbb{U}$ and let $\varphi : \mathbb{C}^4 \times \mathbb{U} \longrightarrow \mathbb{C}$ and ψ be given by (2.8). Suppose that the differential equation

$$\psi(q(z), zq'(z), z^2q''(z), z^3q'''(z); z) = h(z), \tag{2.12}$$

has a solution $q(z)$ with $q(0) = 0$, which satisfy condition (2.1). If the function $f \in \mathcal{A}$ satisfies condition (2.11) and $\varphi(\mathcal{I}^c_{a,b}f(z), \mathcal{I}^c_{a+1,b}f(z), \mathcal{I}^c_{a+2,b}f(z), \mathcal{I}^c_{a+3,b}f(z); z)$ is analytic in $\mathbb{U}$, then $\mathcal{I}^c_{a,b}f(z) \prec q(z)$ and $q(z)$ is the best dominant.

Proof. From Theorem 1, we have q is a dominant of (2.11). Since q satisfies (2.12), it is also a solution of (2.11) and therefore q will be dominated by all dominants. Hence q is the best dominant. This completes the proof. Q.E.D.

In view of Definition 6, and in the special case $q(z) = Mz, \quad M > 0$, the class of admissible functions $\Phi_{\mathcal{I}}[\Omega, q]$, denoted by $\Phi_{\mathcal{I}}[\Omega, M]$, is expressed as follows.

Definition 7. Let Ω be a set in $\mathbb{C}, a, a+1$ and $a+2 \in \mathbb{C} \setminus \{0\}$ and $M > 0$. The class of admissible function $\Phi_{\mathcal{I}}[\Omega, M]$ consists of those functions $\varphi : \mathbb{C}^4 \times \mathbb{U} \longrightarrow \mathbb{C}$ such that

$$\varphi\Bigg(Me^{i\theta}, \frac{(k+(a-1))Me^{i\theta}}{a}, \frac{L+[2ak+a(a-1)]Me^{i\theta}}{a(a+1)}, \frac{N+(3a+3)L+[(3a^2+3a)k+a(a-1)(a+1)]Me^{i\theta}}{a(a+1)(a+2)}; z\Bigg) \notin \Omega, \tag{2.13}$$

whenever $z \in \mathbb{U}$, $\Re(Le^{-i\theta}) \geq (k-1)kM$ and $\Re(Ne^{-i\theta}) \geq 0$ for all $\theta \in \mathbb{R}$ and $k \geq 2$.

Corollary 3. Let $\varphi \in \Phi_{\mathcal{I}}[\Omega, M]$. If the function $f \in \mathcal{A}$ satisfies $\left|\mathcal{I}^c_{a+1,b}f(z)\right| \leq kM \quad (k \geq 2; M > 0)$, and $\varphi\left(\mathcal{I}^c_{a,b}f(z), \mathcal{I}^c_{a+1,b}f(z), \mathcal{I}^c_{a+2,b}f(z), \mathcal{I}^c_{a+3,b}f(z); z\right) \in \Omega$, then $\left|\mathcal{I}^c_{a,b}f(z)\right| < M$.

In this special case $\Omega = q(\mathbb{U}) = \{w : |w| < M\}$, the class $\Phi_{\mathcal{I}}[\Omega, M]$ is simply denoted by $\Phi_{\mathcal{I}}[M]$. Corollary 3 can now be written in the following form

Corollary 4. Let $\varphi \in \Phi_{\mathcal{I}}[M]$. If the function $f \in \mathcal{A}$ satisfies $\left|\mathcal{I}^c_{a+1,b}f(z)\right| \leq kM \quad (k \geq 2; M > 0)$, and $\left|\varphi\big(\mathcal{I}^c_{a,b}f(z), \mathcal{I}^c_{a+1,b}f(z), \mathcal{I}^c_{a+2,b}f(z), \mathcal{I}^c_{a+3,b}f(z); z\big)\right| < M$, then $\left|\mathcal{I}^c_{a,b}f(z)\right| < M$.

Corollary 5. Let $\Re(a) \geq \frac{1-k}{2}, \quad 0 \neq a \in \mathbb{C}, k \geq 2$ and $M > 0$. If the function $f \in \mathcal{A}$ satisfies $|\mathcal{I}^c_{a+1,b}f(z)| \leq M$, then $|\mathcal{I}^c_{a,b}f(z)| < M$.

Proof. By taking $\varphi(v, w, x, y; z) = w$ in Corollary 4 which follows the required results. Q.E.D.

Corollary 6. Let $k \geq 2, \quad 0 \neq a \in \mathbb{C}$ and $M > 0$. If the function $f \in \mathcal{A}$ satisfies $|\mathcal{I}^c_{a+1,b}f(z)| \leq kM$, and $|\mathcal{I}^c_{a+1,b}f(z) - \mathcal{I}^c_{a,b}f(z)| < \frac{M}{|a|}$, then $|\mathcal{I}^c_{a,b}f(z)| < M$.

Definition 8. Let Ω be a set in $\mathbb{C}, q \in \mathcal{Q}_1 \cap \mathcal{H}_1$ and $a, a+1, a+2 \in \mathbb{C} \setminus \{0\}$. The class of admissible functions $\Phi_{\mathcal{I},1}[\Omega, q]$ consists of those functions $\varphi : \mathbb{C}^4 \times \mathbb{U} \longrightarrow \mathbb{C}$ that satisfy the following admissibility condition $\varphi(v, w, x, y; z) \notin \Omega$ whenever

$$v = q(\zeta), w = \frac{k\zeta q'(\zeta) + aq(\zeta)}{a}, \Re\left(\frac{(a+1)(x-v)}{w-v} - (1+2a)\right) \geq k\Re\left(\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right),$$

and

$$\Re\left(\frac{(a+1)(a+2)(y-3x-3w-v)}{w-v}\right) \geq k^2\Re\left(\frac{\zeta^2 q'''(\zeta)}{q'(\zeta)}\right),$$

where $z \in \mathbb{U}, \zeta \in \partial\mathbb{U} \setminus E(q)$, and $k \geq 2$.

The proof of the following theorem run parallel to that of Theorem 1 and we choose to omit the details.

Theorem 4. Let $\varphi \in \Phi_{\mathcal{I},1}[\Omega, q]$.If the function $f \in \mathcal{A}$ and $q \in \mathcal{Q}_1$ satisfy the following conditions:

$$\Re\left(\frac{\zeta q''(\zeta)}{q'(\zeta)}\right) \geq 0, \left|\frac{\mathcal{I}^c_{a+1,b}f(z)}{zq'(\zeta)}\right| \leq k, \tag{2.14}$$

and $\left\{\varphi\left(\frac{\mathcal{I}^c_{a,b}f(z)}{z}, \frac{\mathcal{I}^c_{a+1,b}f(z)}{z}, \frac{\mathcal{I}^c_{a+2,b}f(z)}{z}, \frac{\mathcal{I}^c_{a+3,b}f(z)}{z}; z\right) : z \in \mathbb{U}\right\} \subset \Omega$, then $\frac{\mathcal{I}^c_{a,b}f(z)}{z} \prec q(z) \qquad (z \in \mathbb{U})$.

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(\mathbb{U})$ for some conformal mapping $h(z)$ of $\mathbb{U}$ onto Ω. In this case, the class $\Phi_{\mathcal{I},1}[h(\mathbb{U}), q]$ is written as $\Phi_{\mathcal{I},1}[h, q]$. Proceeding similarly as in the previous theorem, the following result is an immediate consequence of Theorem 4.

Theorem 5. Let $\varphi \in \Phi_{\mathcal{I},1}[h, q]$.If the function $f \in \mathcal{A}$ and $q \in \mathcal{Q}_1$ satisfy the following conditions (2.14) and $\varphi\left(\frac{\mathcal{I}^c_{a,b}f(z)}{z}, \frac{\mathcal{I}^c_{a+1,b}f(z)}{z}, \frac{\mathcal{I}^c_{a+2,b}f(z)}{z}, \frac{\mathcal{I}^c_{a+3,b}f(z)}{z}; z\right) \prec h(z)$, then $\frac{\mathcal{I}^c_{a,b}f(z)}{z} \prec q(z) \qquad (z \in \mathbb{U})$.

In the particular case $q(z) = Mz, \quad M > 0$, and in view of Definition 8, the class of admissible functions $\Phi_{\mathcal{I},1}[\Omega, q]$, denoted by $\Phi_{\mathcal{I},1}[\Omega, M]$, is expressed as follows.

Definition 9. Let Ω be a set in $\mathbb{C}, a, a+1$ and $a+2 \in \mathbb{C} \setminus \{0\}$ and $M > 0$. The class of admissible function $\Phi_{\mathcal{I},1}[\Omega, M]$ consist of those functions $\varphi : \mathbb{C}^4 \times \mathbb{U} \longrightarrow \mathbb{C}$ such that

$$\varphi\Bigg(Me^{i\theta}, \frac{(k+a)Me^{i\theta}}{a}, \frac{L + [2(1+a)k + a(a+1)]Me^{i\theta}}{a(a+1)}, \frac{N + 3(a+2)L + [3(a+1)(a+2)k + a(a+1)(a+2)]Me^{i\theta}}{a(a+1)(a+2)}; z\Bigg) \notin \Omega. \tag{2.15}$$

whenever $z \in \mathbb{U}, \Re(Le^{-i\theta}) \geq (k-1)kM$, and $\Re(Ne^{-i\theta}) \geq 0$ for all $\theta \in \mathbb{R}$ and $k \geq 2$.

Corollary 7. Let $\varphi \in \Phi_{\mathcal{I},1}[\Omega, M]$. If the function $f \in \mathcal{A}$ satisfies $\left|\frac{\mathcal{I}^c_{a+1,b}f(z)}{z}\right| \leq kM \quad (k \geq 2; M > 0; z \in \mathbb{U})$, and $\varphi\left(\frac{\mathcal{I}^c_{a,b}f(z)}{z}, \frac{\mathcal{I}^c_{a+1,b}f(z)}{z}, \frac{\mathcal{I}^c_{a+2,b}f(z)}{z}, \frac{\mathcal{I}^c_{a+3,b}f(z)}{z}; z\right) \in \Omega$, then $\left|\frac{\mathcal{I}^c_{a,b}f(z)}{z}\right| < M$.

In this special case $\Omega = q(\mathbb{U}) = \{w : |w| < M\}$, the class $\Phi_{\mathcal{I},1}[\Omega, M]$ is simply denoted by $\Phi_{\mathcal{I},1}[M]$. Corollary 7 can now be written in the following form

Corollary 8. Let $\varphi \in \Phi_{\mathcal{I},1}[M]$. If the function $f \in \mathcal{A}$ satisfies $\left|\frac{\mathcal{I}^c_{a+1,b}f(z)}{z}\right| \leq kM \quad (k \geq 2; M > 0)$, and $\left|\varphi\left(\frac{\mathcal{I}^c_{a,b}f(z)}{z}, \frac{\mathcal{I}^c_{a+1,b}f(z)}{z}, \frac{\mathcal{I}^c_{a+2,b}f(z)}{z}, \frac{\mathcal{I}^c_{a+3,b}f(z)}{z}\right); z\right| < M$, then $\left|\frac{\mathcal{I}^c_{a,b}f(z)}{z}\right| < M$.

Corollary 9. Let $a \in \mathbb{C}$ with $\Re(a) \geq \frac{-k}{2}$, $k \geq 2$ and $M > 0$. If the function $f \in \mathcal{A}$ satisfies $\left|\frac{\mathcal{I}^c_{a+1,b}f(z)}{z}\right| \leq M$ then $\left|\frac{\mathcal{I}^c_{a,b}f(z)}{z}\right| < M$.

Definition 10. Let Ω be a set in $\mathbb{C}, a, a+1, a+2$ and $a+3 \in \mathbb{C} \setminus \{0\}$ and $q \in \mathcal{Q}_1 \cap \mathcal{H}_1$. The class of admissible functions $\Phi_{\mathcal{I},2}[\Omega, q]$ consists of those functions $\varphi : \mathbb{C}^4 \times \mathbb{U} \longrightarrow \mathbb{C}$ that satisfy the following admissibility conditions $\varphi(v, w, x, y, ; z) \notin \Omega$ whenever

$$v = q(\zeta), w = \frac{1}{(a+1)}\left(\frac{k\zeta q'(\zeta)}{q(\zeta)} + aq(\zeta) + 1\right),$$

$$\Re\left(\frac{((a+2)x - (a+1)w - 1)(a+1)wv}{(a+1)vw - v(av+1)} + (a+1)w - 2av - 1\right) \geq k\Re\left(\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right),$$

and

$$\begin{aligned}
&\Re\left[(a+1)(a+2)(a+3)xywv - 7(a+1)(a+2)xwv - x^2wv(a+1)(a+2)^2 + \right.\\
&2xw^2v(a+1)^2(a+2) - vx^3(a+1)^3 - 2w^2v(a+1)^2 + xv(a+2) + 16wv(a+1)\\
&-12v - 30av^2 - 27a^2v^3 + 25av^2w(a+1) + 3v(av+1)^2 - 3av^2w^2(a+1)^2 -\\
&4av^2xw(a+1)(a+2) + 2awv^2(a+1)(av+1) - 8a^3v^4 + 4a^2v^3w(a+1) +\\
&\left.2v(av+1)^3\right] \times [(a+1)vw - v(av+1)]^{-1} \geq k^2\Re\left(\frac{\zeta^2 q'''(\zeta)}{q'(\zeta)}\right),
\end{aligned}$$

where $z \in \mathbb{U}, \zeta \in \partial\mathbb{U} \setminus E(q)$ and $k \geq 2$.

Following the same technique as in the proof of Theorem 1, we arrive the following results:

Theorem 6. Let $\varphi \in \Phi_{\mathcal{I},2}[\Omega, q]$. If the function $f \in \mathcal{A}$ and $q \in \mathcal{Q}_1$ satisfy the following conditions

$$\Re\left(\frac{\zeta q''(\zeta)}{q'(\zeta)}\right) \geq 0, \left|\frac{\mathcal{I}_{a+2}f(z)}{\mathcal{I}_{a+1}f(z)q'(\zeta)}\right| \leq k, \tag{2.16}$$

$\left\{\varphi\left(\frac{\mathcal{I}^c_{a+1,b}f(z)}{\mathcal{I}^c_{a,b}f(z)}, \frac{\mathcal{I}^c_{a+2,b}f(z)}{\mathcal{I}^c_{a+1,b}f(z)}, \frac{\mathcal{I}^c_{a+3,b}f(z)}{\mathcal{I}^c_{a+2,b}f(z)}, \frac{\mathcal{I}^c_{a+4,b}f(z)}{\mathcal{I}^c_{a+3,b}f(z)}; z\right) : z \in \mathbb{U}\right\} \subset \Omega$, then $\frac{\mathcal{I}^c_{a+1,b}f(z)}{\mathcal{I}^c_{a,b}f(z)} \prec q(z) \quad (z \in \mathbb{U})$.

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(\mathbb{U})$ for some conformal mapping $h(z)$ of $\mathbb{U}$ onto Ω. In this case, the class $\Phi_{\mathcal{I},1}[h(\mathbb{U}), q]$ is written as $\Phi_{\mathcal{I},2}[h, q]$. The following result follows immediately as a consequence of Theorem 6 and hence we omit the proof.

Theorem 7. Let $\varphi \in \Phi_{\mathcal{I},2}[h, q]$. If the function $f \in \mathcal{A}$ and $q \in \mathcal{Q}_1$ satisfy the conditions (2.16) and $\varphi\left(\frac{\mathcal{I}^c_{a+1,b}f(z)}{\mathcal{I}^c_{a,b}f(z)}, \frac{\mathcal{I}^c_{a+2,b}f(z)}{\mathcal{I}^c_{a+1,b}f(z)}, \frac{\mathcal{I}^c_{a+3,b}f(z)}{\mathcal{I}^c_{a+2,b}f(z)}, \frac{\mathcal{I}^c_{a+4,b}f(z)}{\mathcal{I}^c_{a+3,b}f(z)}; z\right) \prec h(z)$, then $\frac{\mathcal{I}^c_{a+1,b}f(z)}{\mathcal{I}^c_{a,b}f(z)} \prec q(z)$.

3 Third order differential superordination of the operator $\mathcal{I}_{a,b}^{c}$

In this section, the third-order differential superordination theorems for the operator $\mathcal{I}_{a,b}^{c}f(z)$ is investigated. For this, the class of admissible functions is given in the following definition.

Definition 11. Let Ω be a set in $\mathbb{C}, a, a+1, a+2 \in \mathbb{C}\setminus\{0\}$ and $q \in \mathcal{H}_0$ with $q'(z) \neq 0$. The class of admissible function $\Phi'_{\mathcal{I}}[\Omega, q]$ consists of those functions $\varphi : \mathbb{C}^4 \times \overline{\mathbb{U}} \longrightarrow \mathbb{C}$ that satisfy the following admissibility conditions: $\varphi(v, w, x, y; \zeta) \in \Omega$ whenever

$$v = q(z), w = \frac{zq'(z) + m(a-1)q(z)}{ma}, \Re\left(\frac{a(a+1)x - a(a-1)v}{aw - (a-1)v} + (1-2a)\right) \leq \frac{1}{m}\Re\left(\frac{zq''(z)}{q'(z)} + 1\right),$$

and

$$\Re\left(\frac{a(a+1)(a+2)y - (3a+3)a(a+1)x + a(a-1)v(2a+2)}{aw - (a-1)v} + (3a^2+3a)\right) \leq \frac{1}{m^2}\Re\left(\frac{z^2q'''(z)}{q'(z)}\right),$$

where $z \in \mathbb{U}, \zeta \in \partial\mathbb{U}$, and $m \geq 2$.

Theorem 8. Let $\varphi \in \Phi'_{\mathcal{I}}[\Omega, q]$. If the function $f \in \mathcal{A}$ and $\mathcal{I}_{a,b}^{c}f(z) \in \mathcal{Q}_0$ and $q \in \mathcal{H}_0$ with $q'(z) \neq 0$ satisfy the following conditions:

$$\Re\left(\frac{zq''(z)}{q'(z)}\right) \geq 0, \left|\frac{\mathcal{I}_{a+1,b}^{c}f(z)}{q'(z)}\right| \leq m, \tag{3.1}$$

and $\varphi(\mathcal{I}_{a,b}^{c}f(z), \mathcal{I}_{a+1,b}^{c}f(z), \mathcal{I}_{a+2,b}^{c}f(z), \mathcal{I}_{a+3,b}^{c}f(z); z)$ is univalent in $\mathbb{U}$,then

$$\Omega \subset \left\{\varphi(\mathcal{I}_{a,b}^{c}f(z), \mathcal{I}_{a+1,b}^{c}f(z), \mathcal{I}_{a+2,b}^{c}f(z), \mathcal{I}_{a+3,b}^{c}f(z); z)z \in \mathbb{U}\right\}, \tag{3.2}$$

implies that $q(z) \prec \mathcal{I}_{a,b}^{c}f(z) \qquad (z \in \mathbb{U})$.

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(\mathbb{U})$ for some conformal mapping $h(z)$ of $\mathbb{U}$ onto Ω. In this case, the class $\Phi'_{\mathcal{I}}[h(\mathbb{U}), q]$ is written as $\Phi'_{\mathcal{I}}[h, q]$. The following result is an immediate consequence of Theorem 8.

Theorem 9. Let $\varphi \in \Phi'_{\mathcal{I}}[h, q]$ and h be analytic in $\mathbb{U}$.If the function $f \in \mathcal{A}$ and $\mathcal{I}_{a}f(z) \in \mathcal{Q}_0$ and $q \in \mathcal{H}_0$ with $q'(z) \neq 0$ satisfy the conditions (3.1) and $\varphi(\mathcal{I}_{a,b}^{c}f(z), \mathcal{I}_{a+1,b}^{c}f(z), \mathcal{I}_{a+2,b}^{c}f(z),$
$\mathcal{I}_{a+3,b}^{c}f(z); z)$, is univalent in $\mathbb{U}$, then

$$h(z) \prec \varphi\left(\mathcal{I}_{a,b}^{c}f(z), \mathcal{I}_{a+1,b}^{c}f(z), \mathcal{I}_{a+2,b}^{c}f(z), \mathcal{I}_{a+3,b}^{c}f(z); z\right) \tag{3.3}$$

implies that $q(z) \prec \mathcal{I}_{a,b}^{c}f(z) \quad (z \in \mathbb{U})$.

Theorem 8 and 9 can only be used to obtain subordinants of the third-order differential superordination of the forms (3.2) or (3.3). The next result shows the existence of best subordinant of (3.3) for a suitable φ.

Theorem 10. Let the function h be univalent in $\mathbb{U}$ and let $\varphi : \mathbb{C}^4 \times \overline{\mathbb{U}} \longrightarrow \mathbb{C}$ and ψ be given by (2.8). Suppose that the differential equation

$$\psi(q(z), zq'(z), z^2q''(z), z^3q'''(z); z) = h(z), \tag{3.4}$$

has a solution $q(z) \in \mathcal{Q}_0$. If the function $f \in \mathcal{A}, \mathcal{I}^c_{a,b}f(z) \in \mathcal{Q}_0$ and $q \in \mathcal{H}_0$ with $q'(z) \neq 0$, which satisfy the condition (3.1) and $\varphi\left(\mathcal{I}^c_{a,b}f(z), \mathcal{I}^c_{a+1,b}f(z), \mathcal{I}^c_{a+2,b}f(z), \mathcal{I}^c_{a+3,b}f(z); z\right)$ is analytic in $\mathbb{U}$, then $h(z) \prec \varphi\left(\mathcal{I}^c_{a,b}f(z), \mathcal{I}^c_{a+1,b}f(z), \mathcal{I}^c_{a+2,b}f(z), \mathcal{I}^c_{a+3,b}f(z); z\right)$ implies that $q(z) \prec \mathcal{I}^c_{a,b}f(z) \quad (z \in \mathbb{U})$ and $q(z)$ is the best dominant.

Definition 12. Let Ω be a set in $\mathbb{C}, q \in \mathcal{H}_1$ and $a, a+1, a+2 \in \mathbb{C} \setminus \{0\}$. The class of admissible functions $\Phi'_{\mathcal{I},1}[\Omega, q]$ consists of those functions $\varphi : \mathbb{C}^4 \times \overline{\mathbb{U}} \longrightarrow \mathbb{C}$ that satisfy the admissibility condition $\varphi(v, w, x, y; \zeta) \in \Omega$ whenever

$$v = q(z), w = \frac{zq'(z) + amq(z)}{am}, \Re\left(\frac{(a+1)(x-v)}{w-v} - (1+2a)\right) \leq \frac{1}{m}\Re\left(\frac{zq''(z)}{q'(z)} + 1\right),$$

and $\Re\left(\frac{(a+1)(a+2)(y-3x-3w-v)}{w-v}\right) \leq \frac{1}{m^2}\Re\left(\frac{z^2q'''(z)}{q'(z)}\right)$, where $z \in \mathbb{U}, \zeta \in \partial\mathbb{U}$, and $m \geq 2$.

Theorem 11. Let $\varphi \in \Phi'_{\mathcal{I},1}[\Omega, q]$. If the function $f \in \mathcal{A}, \frac{\mathcal{I}^c_{a,b}f(z)}{z} \in \mathcal{Q}_1$ and $q \in \mathcal{H}_1$ with $q'(z) \neq 0$ satisfy the following conditions:

$$\Re\left(\frac{zq''(z)}{q'(z)}\right) \geq 0, \left|\frac{\mathcal{I}^c_{a+1,b}f(z)}{zq'(z)}\right| \leq m, \tag{3.5}$$

and $\varphi\left(\frac{\mathcal{I}^c_{a,b}f(z)}{z}, \frac{\mathcal{I}^c_{a+1,b}f(z)}{z}, \frac{\mathcal{I}^c_{a+2,b}f(z)}{z}, \frac{\mathcal{I}^c_{a+3,b}f(z)}{z}; z\right)$ is univalent in $\mathbb{U}$, then

$$\Omega \subset \left\{\varphi\left(\frac{\mathcal{I}^c_{a,b}f(z)}{z}, \frac{\mathcal{I}^c_{a+1,b}f(z)}{z}, \frac{\mathcal{I}^c_{a+2,b}f(z)}{z}, \frac{\mathcal{I}^c_{a+3,b}f(z)}{z}; z\right) : z \in \mathbb{U}\right\}, \tag{3.6}$$

implies that $q(z) \prec \frac{\mathcal{I}^c_{a,b}f(z)}{z} \qquad (z \in \mathbb{U})$.

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(\mathbb{U})$ for some conformal mapping $h(z)$ of $\mathbb{U}$ onto Ω. In this case, the class $\Phi'_{\mathcal{I},1}[h(\mathbb{U}), q]$ is written as $\Phi'_{\mathcal{I},1}[h, q]$. The following result is an immediate consequence of Theorem 11.

Theorem 12. Let $\varphi \in \Phi'_{\mathcal{I},1}[h, q]$ and h be analytic in $\mathbb{U}$. If the function $f \in \mathcal{A}, \frac{\mathcal{I}^c_{a,b}f(z)}{z} \in \mathcal{Q}_1$ and $q \in \mathcal{H}_1$ with $q'(z) \neq 0$ satisfy the conditions (3.5) and

$$\varphi\left(\frac{\mathcal{I}^c_{a,b}f(z)}{z}, \frac{\mathcal{I}^c_{a+1,b}f(z)}{z}, \frac{\mathcal{I}^c_{a+2,b}f(z)}{z}, \frac{\mathcal{I}^c_{a+3,b}f(z)}{z}; z\right)$$

is univalent in $\mathbb{U}$, then $h(z) \prec \varphi\left(\frac{\mathcal{I}^c_{a,b}f(z)}{z}, \frac{\mathcal{I}^c_{a+1,b}f(z)}{z}, \frac{\mathcal{I}^c_{a+2,b}f(z)}{z}, \frac{\mathcal{I}^c_{a+3,b}f(z)}{z}; z\right)$, implies that

$$q(z) \prec \frac{\mathcal{I}^c_{a,b}f(z)}{z} \qquad (z \in \mathbb{U}).$$

Definition 13. Let Ω be a set in $\mathbb{C}, a, a+1, a+2, a+3 \in \mathbb{C} \setminus \{0\}$ and $q \in \mathcal{H}_1$ with $q'(z) \neq 0$. The class of admissible functions $\Phi'_{\mathcal{I},2}[\Omega, q]$ consists of those functions $\varphi : \mathbb{C}^4 \times \overline{\mathbb{U}} \longrightarrow \mathbb{C}$ that satisfy the admissibility conditions $\varphi(v, w, x, y; \zeta) \in \Omega$ whenever

$$v = q(z), w = \frac{1}{a+1}\left(\frac{zq'(z)}{mq(z)} + aq(z) + 1\right),$$

$$\Re\left(\frac{((a+2)x-(a+1)w-1)(a+1)wv}{(a+1)vw-v(av+1)}+(a+1)w-2av-1\right)\leq\frac{1}{m}\Re\left(\frac{zq''(z)}{q'(z)}+1\right),$$

and

$$\begin{aligned}&\Re\left[(a+1)(a+2)(a+3)xywv-7(a+1)(a+2)xwv-x^2wv(a+1)(a+2)^2+\right.\\&2xw^2v(a+1)^2(a+2)-vx^3(a+1)^3-2w^2v(a+1)^2+xv(a+2)+16wv(a+1)\\&-12v-30av^2-27a^2v^3+25av^2w(a+1)+3v(av+1)^2-3av^2w^2(a+1)^2-\\&4av^2xw(a+1)(a+2)+2awv^2(a+1)(av+1)-8a^3v^4+4a^2v^3w(a+1)+\\&\left.2v(av+1)^3\right]\times\left[(a+1)vw-v(av+1)\right]^{-1}\leq\frac{1}{m^2}\Re\left(\frac{z^2q'''(z)}{q'(z)}\right),\end{aligned}$$

where $z\in\mathbb{U},\zeta\in\partial\mathbb{U}$ and $m\geq 2$.

Theorem 13. Let $\varphi\in\Phi'_{\mathcal{I},2}[\Omega,q]$. If the function $f\in\mathcal{A}$ and $\frac{\mathcal{I}^c_{a+1,b}f(z)}{\mathcal{I}^c_{a,b}f(z)}\in\mathcal{Q}_1$ and $q\in\mathcal{H}_1$ with $q'(z)\neq 0$ satisfy the following conditions

$$\Re\left(\frac{zq''(z)}{q'(z)}\right)\geq 0,\left|\frac{\mathcal{I}^c_{a+2,b}f(z)}{\mathcal{I}^c_{a+1,b}q'(z)}\right|\leq m,\tag{3.7}$$

and $\varphi\left(\frac{\mathcal{I}^c_{a+1,b}f(z)}{\mathcal{I}^c_{a,b}f(z)},\frac{\mathcal{I}^c_{a+2,b}f(z)}{\mathcal{I}^c_{a+1,b}f(z)},\frac{\mathcal{I}^c_{a+3,b}f(z)}{\mathcal{I}^c_{a+2,b}f(z)},\frac{\mathcal{I}^c_{a+4,b}f(z)}{\mathcal{I}^c_{a+3,b}f(z)};z\right)$ is univalent in $\mathbb{U}$, then

$$\Omega\subset\left\{\varphi\left(\frac{\mathcal{I}^c_{a+1,b}f(z)}{\mathcal{I}^c_{a,b}f(z)},\frac{\mathcal{I}^c_{a+2,b}f(z)}{\mathcal{I}^c_{a+1,b}f(z)},\frac{\mathcal{I}^c_{a+3,b}f(z)}{\mathcal{I}^c_{a+2,b}f(z)},\frac{\mathcal{I}^c_{a+4,b}f(z)}{\mathcal{I}^c_{a+3,b}f(z)};z\right):z\in\mathbb{U}\right\},\tag{3.8}$$

then $q(z)\prec\frac{\mathcal{I}^c_{a+1,b}f(z)}{\mathcal{I}^c_{a,b}f(z)}\qquad(z\in\mathbb{U})$.

If $\Omega\neq\mathbb{C}$ is a simply connected domain, then $\Omega=h(\mathbb{U})$, for some conformal mapping h(z) of $\mathbb{U}$ on to Ω. In this case the class $\Phi'_{\mathcal{I},2}[h(\mathbb{U}),q]$ is written as $\Phi'_{\mathcal{I}}[h,q]$. The following result is a consequence of Theorem 13.

Theorem 14. Let $\varphi\in\Phi'_{\mathcal{I},2}[h,q]$. If the function $f\in\mathcal{A}$ and $\frac{\mathcal{I}^c_{a+1,b}f(z)}{\mathcal{I}^c_{a,b}f(z)}\in\mathcal{Q}_1$ and $q\in\mathcal{H}_1$ with $q'(z)\neq 0$ satisfy the conditions (3.7) and
$\varphi\left(\frac{\mathcal{I}^c_{a+1,b}f(z)}{\mathcal{I}^c_{a,b}f(z)},\frac{\mathcal{I}^c_{a+2,b}f(z)}{\mathcal{I}^c_{a+1,b}f(z)},\frac{\mathcal{I}^c_{a+3,b}f(z)}{\mathcal{I}^c_{a+2,b}f(z)},\frac{\mathcal{I}^c_{a+4,b}f(z)}{\mathcal{I}^c_{a+3,b}f(z)};z\right)$ is univalent in $\mathbb{U}$, then

$$h(z)\prec\varphi\left(\frac{\mathcal{I}^c_{a+1,b}f(z)}{\mathcal{I}^c_{a,b}f(z)},\frac{\mathcal{I}^c_{a+2,b}f(z)}{\mathcal{I}^c_{a+1,b}f(z)},\frac{\mathcal{I}^c_{a+3,b}f(z)}{\mathcal{I}^c_{a+2,b}f(z)},\frac{\mathcal{I}^c_{a+4,b}f(z)}{\mathcal{I}^c_{a+3,b}f(z)};z\right),\tag{3.9}$$

implies that $q(z)\prec\frac{\mathcal{I}^c_{a+1,b}f(z)}{\mathcal{I}^c_{a,b}f(z)}\qquad(z\in\mathbb{U})$.

4 Sandwich type results

Combining Theorem 2 and 9, we obtain the following sandwich-type results.

Corollary 10. Let h_1 and q_1 be analytic functions in $\mathbb{U}$, h_2 be univalent function in $\mathbb{U}$, $q_2 \in \mathcal{Q}_0$ with $q_1(0) = q_2(0) = 0$ and $\varphi \in \Phi_{\mathcal{I}}[h_2, q_2] \cap \Phi'_{\mathcal{I}}[h_1, q_1]$. If the function $f \in \mathcal{A}, \mathcal{I}^c_{a,b}f(z) \in \mathcal{Q}_0 \cap \mathcal{H}_0$, and $\varphi(\mathcal{I}^c_{a,b}f(z), \mathcal{I}^c_{a+1,b}f(z), \mathcal{I}^c_{a+2,b}f(z), \mathcal{I}^c_{a+3,b}f(z); z)$, is univalent in $\mathbb{U}$, and the condition (2.1) and (3.1) are satisfied, then

$$h_1(z) \prec \varphi(\mathcal{I}^c_{a,b}f(z), \mathcal{I}^c_{a+1,b}f(z), \mathcal{I}^c_{a+2,b}f(z), \mathcal{I}^c_{a+3,b}f(z); z) \prec h_2(z)$$

implies that $q_1(z) \prec \mathcal{I}^c_{a,b}f(z) \prec q_2(z) \qquad (z \in \mathbb{U})$.

Combing Theorems 5 and 12, we obtain the following sandwich-type result.

Corollary 11. Let h_1 and q_1 be analytic functions in $\mathbb{U}$, h_2 be univalent function in $\mathbb{U}$, $q_2 \in \mathcal{Q}_1$ with $q_1(0) = q_2(0) = 1$ and $\varphi \in \Phi_{\mathcal{I},1}[h_2, q_2] \cap \Phi'_{\mathcal{I},1}[h_1, q_1]$. If the function $f \in \mathcal{A}$, $\frac{\mathcal{I}^c_{a,b}f(z)}{z} \in \mathcal{Q}_1 \cap \mathcal{H}_1$, and $\varphi\left(\frac{\mathcal{I}^c_{a,b}f(z)}{z}, \frac{\mathcal{I}^c_{a+1,b}f(z)}{z}, \frac{\mathcal{I}^c_{a+2,b}f(z)}{z}, \frac{\mathcal{I}^c_{a+3,b}f(z)}{z}; z\right)$, is univalent in $\mathbb{U}$, and the condition (2.14) and (3.5) are satisfied, then

$$h_1(z) \prec \varphi\left(\frac{\mathcal{I}^c_{a,b}f(z)}{z}, \frac{\mathcal{I}^c_{a+1,b}f(z)}{z}, \frac{\mathcal{I}^c_{a+2,b}f(z)}{z}, \frac{\mathcal{I}^c_{a+3,b}f(z)}{z}; z\right) \prec h_2(z),$$

implies that $q_1(z) \prec \frac{\mathcal{I}^c_{a,b}f(z)}{z} \prec q_2(z) \qquad (z \in \mathbb{U})$.

Combining Theorems 7 and 14, we obtain the following sandwich-type result.

Corollary 12. Let h_1 and q_1 be analytic functions in $\mathbb{U}$, h_2 be univalent functions in $\mathbb{U}$, $q_2 \in \mathcal{Q}_1$ with $q_1(0) = q_2(0) = 1$ and $\varphi \in \Phi_{\mathcal{I},2}[h_2, q_2] \cap \Phi'_{\mathcal{I},2}[h_1, q_1]$. If the function $f \in \mathcal{A}$, $\frac{\mathcal{I}^c_{a+1,b}f(z)}{\mathcal{I}^c_{a,b}f(z)} \in \mathcal{Q}_1 \cap \mathcal{H}_1$, and $\varphi\left(\frac{\mathcal{I}^c_{a+1,b}f(z)}{\mathcal{I}^c_{a,b}f(z)}, \frac{\mathcal{I}^c_{a+2,b}f(z)}{\mathcal{I}^c_{a+1,b}f(z)}, \frac{\mathcal{I}^c_{a+3,b}f(z)}{\mathcal{I}^c_{a+2,b}f(z)}, \frac{\mathcal{I}^c_{a+4,b}f(z)}{\mathcal{I}^c_{a+3,b}f(z)}; z\right)$, is univalent in $\mathbb{U}$, and the condition (2.16) and (3.7) are satisfied, then

$$h(z) \prec \varphi\left(\frac{\mathcal{I}^c_{a+1,b}f(z)}{\mathcal{I}^c_{a,b}f(z)}, \frac{\mathcal{I}^c_{a+2,b}f(z)}{\mathcal{I}^c_{a+1,b}f(z)}, \frac{\mathcal{I}^c_{a+3,b}f(z)}{\mathcal{I}^c_{a+2,b}f(z)}, \frac{\mathcal{I}^c_{a+4,b}f(z)}{\mathcal{I}^c_{a+3,b}f(z)}; z\right) \prec h_2(z),$$

implies that $q_1(z) \prec \frac{\mathcal{I}^c_{a+1,b}f(z)}{\mathcal{I}^c_{a,b}f(z)} \prec q_2(z) \qquad (z \in \mathbb{U})$.

5 Applications and concluding remarks

It is pertinent that making objects invisible to humans, which has been long of subject of science fiction, has been materialized in the range of microwave radiation [17, 26]. The idea is to keep an object inside a cloak made up of material opaque to electromagnetic radiation so that an interrogating instrument can not view the object. This problem of electromagnetic cloaking has been recently considered in the view point of subordination by Mishra et al.[23]. In that paper they have studied the relation between the object, cloak and the viewer in electromagnetic field by considering cloaked object by function $f(z)$ and the cloak by the function $q(z)$. It is required that

$$f(z) \prec q(z) \qquad (z \in \mathbb{U}). \tag{5.1}$$

More recently, Zayed et al. [33] apply the notion of subordination to study the vortex inside an infinite circular cylinder by considering $f(z)$ the vortex and $q(z)$ as the surface of cylinder. For different $q(z)$, they also showed that (5.1) holds true. Note that, upon specializing the parameters $a = b = c = 1$ in Theorem 1 and taking $q(z) = \frac{1+z}{1-z}$, which is an analytic univalent convex map as the cloak function, (5.1) holds true. Furthermore in Corollary 1, by considering different values of $\rho \in (0, 1)$ in $q_\rho(z) = q(\rho z) = \frac{1+\rho z}{1-\rho z}$, we get the best dominant as the cloak function. Figure 1 illustrate the idea how the $q(\rho z)$ is dominated by $q(z)$. Here each best dominant $q(\rho z)$, which is a cloak function as well as a bounded circular function.

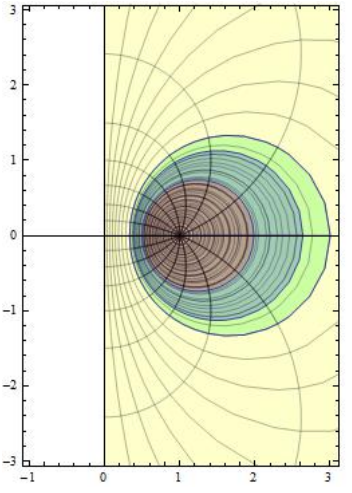


FIGURE 1. Images of the unit disk under $q(z) = \frac{1-z}{1+z}$ and $q(\rho z) = \frac{1+\rho z}{1-\rho z}$ for $\rho = 0.5,\ 0.45,\ 0.33,\ 0.25$.

Remark 1. In our present investigation, we have made use of the linear operator introduced and studied by Hohlov [12], to systematically investigate several suitable classes of admissible functions. The dual properties of the third-order differential subordinations are presented. The existence of much more general convolution operators than the Hohlov operator, such as the Dziok-Srivastava operator [6, 7] and the Srivastava-Wright operator [16, 27] are found in literature. Thus, it is an interest to study the dual problems by introducing the suitable families of admissible functions in the view point of these generalized operators. Furthermore, we also put it to somebody that the following functions:

$$\frac{1-z}{1+z},\ \frac{1}{1-z^2},\ \frac{1}{1-z},\ \text{etc}$$

need to be considered as the cloak function in Theorem 1.

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