On B^*-clopen continuity, oscillation and convergence

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Abstract

In this paper a new generalization of B^*-continuity has been defined, to be called as B^*-clopen continuity and many of its properties have been studied including the convergence behaviour of sequences of such functions. Also a new type of oscillation function, called cl-oscillation has been defined, which characterizes the B^*-clopen continuity.

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1 Introduction

Continuous functions have been generalized by various weaker notions over the years and the hunt for more weaker versions is still on. Some of these continuities include semi-continuity [14], quasi-continuity [11], slightly continuity [9], clopen continuity [21], B-continuity [16] and B^*-continuity [5]. Ganguly and Mitra [5], [6] introduced and studied the concept of B^*-continuity, which is a generalization of both quasi-continuity and B-continuity. B^*-continuity deals with Baire property [20] and denseness of the space. In our earlier paper [10], we have further generalized B^*-continuity by defining the notion of contra B^*-continuity, slightly B^*-continuity and weakly B^*-continuity. Several properties of these functions as well as their inter-relationship were studied there.

In the present paper, we suppliment the notions defined in [10] by defining a new generalized B^*-continuity, to be called B^*-clopen continuity and study its relationship with other continuities.

The concept of oscillation function ω_f corresponding to a function f is a classical notion [1] which is related with the continuity of f. It is known that f is continuous at a point if and only if ω_f is zero at that point [1]. Subsequently, different oscillation functions were introduced and studied corresponding to different types of continuities, e.g., Borsik [2], [3] defined oscillation functions for quasi continuity and almost continuity. Ganguly and Mitra [7] did it for B^*-continuity. These new functions play a similar role for quasi continuity, cliquishness, almost continuity, B^*-continuity points as oscillation does for continuity points.

Next aim of this paper is to define a new oscillation function, to be called as cl-oscillation function and characterize the B^*-clopen continuity through it.

The final aim of this paper is to study the convergence behaviour of ordinary as well as transfinite sequence of B^*-clopen continuous functions. The concept of the pointwise convergence of a transfinite sequence of real functions of real variable was given by Sierpinski [23]. Given a set of functions, which are continuous in some generalized sense, may or may not be closed with respect to the transfinite convergence. Various criteria are given by many authors for such sets to be closed [12], [15], [17], [22]. We shall show that B^*-clopen continuity is not preserved under pointwise convergence of ordinary sequences whereas in the case of transfinite sequences, it is so. Moreover, we shall prove that for ordinary sequences, uniform convergence preserves the B^*-clopen continuity

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and also we shall prove that the uniform convergence of B^*-clopen continuous functions imply the uniform convergence of the corresponding sequence of oscillation functions.

2 Some known B^*-continuities

Throughout the paper, X and Y will denote topological spaces, unless specified otherwise. By $\text{int}(A)$ and $\text{cl}(A)$ we shall denote the interior and closure of the set A.

First let us recall few definitions:

Definition 2.1. [21] A function $f : X \to Y$ is said to be clopen continuous if for each point $x \in X$ and for every open set V containing $f(x)$, there exists a clopen set U containing x such that $f(U) \subset V$.

Definition 2.2. [5] A subset A of X is said to be a B^*-set if it is not nowhere dense having the property of Baire.

Definition 2.3. [8] Let $P \subset X$. A point $x \in X$ is said to be a B^*-cluster point of P if for every B^*-set B containing x, $P \cap B \neq \varnothing$. The set of all B^*-cluster points of P is called B^*-cluster derived set of P and denoted by B^*-cls-d-P.
A set P is said to be B^*-closed if B^*-cls-d-$P \subset P$. The B^*-closure of $P = P \cup (B^*\text{-cls-d-}P)$ is denoted by $B^*_{\text{cl}}(P)$.
The complement of a B^*-closed set is known as B^*-open set.

Now, we give below the notion of B^*-continuity [5] and its various generalizations introduced and studied very recently in [10]:

Definition 2.4. [5] A function $f : X \to Y$ is said to be B^*-continuous at $x \in X$ if for every open set U containing x and for every open set V containing $f(x)$, there is a B^*-set $B \subset U$ containing x such that $f(B) \subset V$.

Definition 2.5. [10] A function $f : X \to Y$ is said to be weakly B^*-continuous at $x \in X$ if for every open set U containing x and for every open set V containing $f(x)$, there is a B^*-set $B \subset U$ containing x such that $f(B) \subset \text{cl}(V)$.

Definition 2.6. [10] A function $f : X \to Y$ is said to be slightly B^*-continuous at $x \in X$ if for every open set U containing x and for every clopen set V containing $f(x)$, there exists a B^*-set $B \subset U$ containing x such that $f(B) \subset V$.

Definition 2.7. [10] A function $f : X \to Y$ is said to be contra B^*-continuous at $x \in X$ if for every open set U containing x and for every closed set V containing $f(x)$, there is a B^*-set $B \subset U$ containing x such that $f(B) \subset V$.

Definition 2.8. [10] A function $f : X \to Y$ is said to be B^*-irresolute at a point $x \in X$ if for every open set U in X containing x and for every B^*-set V containing $f(x)$, there exists a B^*-set $B \subset U$ containing x such that $f(B) \subset V$.

Remark 2.9. Among various generalizations of B^*-continuities, the inter-relationship is given by the following diagram, see [10]:

$$\begin{array}{ccccc} B^*\text{−Irresolute} & \Longrightarrow & B^*\text{−Continuity} & & \\ & & \Downarrow & & \\ \text{Contra } B^*\text{−Continuity} & \Longrightarrow & \text{Weakly } B^*\text{−Continuity} & \Longrightarrow & \text{Slightly } B^*\text{−Continuity.} \end{array}$$

3 B^*-clopen continuity

We begin by introducing the B^*-clopen continuity as follows:

Definition 3.1. A function $f : X \to Y$ is said to be B^*-clopen continuous at $x \in X$ if for every clopen set U containing x and for every open set V containing $f(x)$, there is a B^*-set $B \subset U$ containing x such that $f(B) \subset V$.

Theorem 3.2. Every B^*-continuous function is B^*-clopen continuous but not conversely.

Proof. Let U be a clopen set containing x and V be an open set containing $f(x)$. Since f is B^*-continuous at x, U being open in X, there is a B^*-set $B \subset U$ such that $f(B) \subset V$.
For the converse, let $X = \{a, b, c, d\}$ be a space with topologies

$$\tau = \Big\{\varnothing, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, b, c\}, X\Big\} \text{ and } \sigma = \{\varnothing, \{a, c\}, X\}.$$

The identity function $f : (X, \tau) \to (X, \sigma)$ is B^*-clopen continuous on X. However, f is not B^*-continuous at $a \in X$. Indeed, consider the open sets $U = \{a, b\}$ in (X, τ) and $V = \{a, c\}$ in (X, σ). Then, there does not exist any B^*-set $B \subset U$ such that $f(B) \subset V$. Q.E.D.

Remark 3.3. Neither slightly B^*-continuity implies B^*-clopen continuity nor vice-versa.
Let $X = \{a, b, c, d\}$ with topologies

$$\tau = \Big\{ \varnothing, \{c\}, \{d\}, \{b, c\}, \{c, d\}, \{a, b, c\}, \{b, c, d\}, X\Big\}$$

and

$$\sigma = \Big\{\varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}\{a, c\}, \{a, b, c\}, X\Big\}.$$

The identity function $f : (X, \tau) \to (X, \sigma)$ is slightly B^*-continuous on X but not B^*-clopen continuous at $a \in X$.
Let us now consider $X = \{a, b, c, d\}$ with topologies

$$\tau = \Big\{ \varnothing, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, b, c\}, X\Big\}$$

and

$$\sigma = \{\varnothing, \{a, c\}, \{b, d\}, X\}.$$

The identity function $f : (X, \tau) \to (X, \sigma)$ is B^*-clopen continuous but not slightly B^*-continuous at $a \in X$.

Remark 3.4. Weakly B^*-continuity and B^*-clopen continuity have no relation. Let $X = \{a, b, c, d\}$ with the topologies

$$\tau = \{\varnothing, X\} \quad \text{and} \quad \sigma = \{\varnothing, \{a, c\}, X\}.$$

The identity function $f : (X, \tau) \to (X, \sigma)$ is weakly B^*-continuous on X but not B^*-clopen continuous at $a \in X$.

The fact that B^*-clopen continuity does not imply weakly B^*-continuity follows from Remarks 2.9 and 3.3.

Remark 3.5. It is clear from Remarks 2.9 and 3.3 that B^*-clopen continuity does not imply contra B^*-continuity.

Remark 3.6. In view of Remarks 3.3, 3.4 and 3.5, the relationship diagram in Remark 2.9 is extended as follows:

$$\begin{array}{c} B^*-\text{Irresolute} \Longrightarrow B^*-\text{Continuity} \Longrightarrow B^*-\text{Clopen Continuity} \\ \Downarrow \\ \text{Contra } B^*-\text{Continuity} \Longrightarrow \text{Weakly } B^*-\text{Continuity} \Longrightarrow \text{Slightly } B^*-\text{Continuity}. \end{array}$$

The relationship diagram in Remark 3.6 indicates that in general, the implications among various continuities are only one sided. We provide additional conditions so that the implications hold in the other direction as well. In order to do it, we first recall the following definitions:

Definition 3.7. [18] Let X be a topological space. Then X is said to be locally indiscrete if every open set of X is closed in X.

Definition 3.8. [4] Let (X, τ) be a topological space. Since the intersection of two clopen sets in (X, τ) is clopen, the clopen subsets of (X, τ) may be used as a base for a topology on X. The topology so generated is called the ultra-regularization of τ and is denoted by τ_u. A topological space (X, τ) is said to be ultra-regular if $\tau = \tau_u$.

Remark 3.9. (a) Let X be a locally indiscrete space. If $f : X \to Y$ is B^*-clopen continuous then f is B^*-continuous. Hence f is slightly B^*-continuous.

(b) Let Y be a locally indiscrete space. Then $f : X \to Y$ is B^*-continuous and contra B^*-continuous whenever f is slightly B^*-continuous. Hence f is also B^*-clopen continuous.

(c) Let X and Y be locally indiscrete spaces. Then B^*-clopen continuity, slightly B^*-continuity, contra B^*-continuity and B^*-continuity all are same.

(d) Let X be ultra regular and Y be locally indiscrete. Then $f : X \to Y$ is contra B^*-continuous, whenever it is B^*-clopen continuous.

Theorem 3.10. Consider the following statements:

(a) A function $f : (X, \tau) \to (Y, \sigma)$ is slightly B^*-continuous.

(b) $f : (X, \tau) \to (Y, \sigma_u)$ is B^*-clopen continuous.

(c) $f : (X, \tau_u) \to (Y, \sigma_u)$ is B^*-continuous.

It holds that $(a) \Longrightarrow (b) \Longrightarrow (c)$.

Proof. $(a) \Longrightarrow (b)$ Let $x \in X$ and $f : (X, \tau) \to (Y, \sigma)$ be slightly B^*-continuous. Let U be any clopen set containing x and V be an open set of (Y, σ_u) containing $f(x)$. Then, there exists a $W \subset V$, where $W \in \sigma_u$, i.e., W is a clopen set in (Y, σ) containing $f(x)$.
Now, f being slightly B^*-continuous at $x \in X$, for each clopen set $U \subset X$ with $x \in U$, there exists a B^*-set $B \subset X$ such that

$$f(B) \subset W \subset V,$$

so that, $f : (x, \tau) \to (Y, \sigma_u)$ is B^*-clopen continuous.
$(b) \Longrightarrow (c)$ Let U containing x be any open set in (X, τ_u) and V be any open set in (Y, σ_u) such that $f(x) \in V$. Then, U is a clopen set in (X, τ). f being B^*-clopen continuous, there exists a B^*-set $B \subset U$ such that $f(B) \subset V$, so that, $f : (X, \tau_u) \to (Y, \sigma_u)$ is B^*-continuous. Q.E.D.

Remark 3.11. It could neither be proved nor disproved that $(c) \Longrightarrow (a)$. However, if the continuities in (a), (b) and (c) are considered, respectively, as slightly continuous, clopen continuous and continuous then these statements are equivalent, see [19].

Theorem 3.12. If the graph of a function $f : X \to Y$ is B^*-clopen continuous, then the function f itself is also B^*-clopen continuous.

Proof. Let $g : X \to X \times Y$ defined by $g(x) = (x, f(x))$ denote the graph of f. Let $x \in X$, U be any clopen set containing x and V be an open set containing $f(x)$. Then $U \times V$ is open in $X \times Y$ containing $g(x)$ and g being B^*-clopen continuous at x, there exists a B^*-set $B \subset U$ containing x such that $g(B) = \{(b, f(b)) : b \in B\} \subset U \times V$. This means $f(B) \subset V$. Hence f is B^*-clopen continuous at x. Q.E.D.

Theorem 3.13. If for each clopen set U in X and each open set V in Y, $U \cap f^{-1}(V)$ is dense in U and if f is B^*-clopen continuous on a dense set, it will be so in the whole of X.

Proof. Let f be B^*-clopen continuous on E, where $\mathrm{cl}(E) = X$ and $y \in X \setminus E$. Let U be a clopen set in X containing y and V be an open set in Y containing $f(y)$. Let, if possible, $U \cap f^{-1}(V)$ does not contain any B^*-set. Then two cases may arise:
Case I $f^{-1}(V)$ is nowhere dense in U. As $U \cap f^{-1}(V)$ is dense in U, this contradicts that $y \notin E$. Hence, $E = X$.
Case II $U \cap f^{-1}(V)$ is not nowhere dense in U, but does not have a subset having the property of Baire. Then for

$$B \subset U \cap f^{-1}(V),\ B \neq (O \setminus I) \cup J,$$

where, O is open and I, J are sets of first category. Then, B does not intersect any open set. This contradicts $U \cap f^{-1}(V)$ is not nowhere dense. Hence, $E = X$. Q.E.D.

The proof of the following theorem is simple and thus we omit it.

Theorem 3.14. Let $\{Y_\lambda : \lambda \in \Lambda\}$ be a family of topological spaces with the index set Λ. By $\prod_{\lambda \in \Lambda} Y_\lambda$, or simply $\prod Y_\lambda$, we denote the product space. If a function $f : X \to \prod Y_\lambda$ is B^*-clopen continuous, then $P_\lambda o f : X \to Y_\lambda$ is B^*-clopen continuous for each $\lambda \in \Lambda$, where P_λ is projection of $\prod Y_\lambda$ onto Y_λ.

Remark 3.15. The composition of two B^*-clopen continuous functions need not be B^*-clopen continuous.

Example 3.16. Let $X = \{a, b, c, d\}$ with topologies

$$\tau = \{\ \varnothing, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{c, d\}, \{a, b, c\}, \{a, c, d\}, X\},$$

$$\sigma = \{\varnothing, \{c\}, \{c, d\}, \{a, b, c\}, X\} \ \text{ and } \ \theta = \{\varnothing, \{b, c\}, X\}.$$

The identity functions $f : (X, \tau) \to (X, \sigma)$ and $g : (X, \sigma) \to (X, \theta)$ are B^*-clopen continuous on X. But the identity function $gof : (X, \tau) \to (X, \theta)$ is not B^*-clopen continuous at $b \in X$. Indeed, consider the clopen set $U = \{a, b\}$ and open set $V = \{b, c\}$. Then, there does not exist any B^*-set $B \subset U$ such that $f(B) \subset V$.

The following theorem gives the sufficient conditions for the composition of two functions to be B^*-clopen continuous.

Theorem 3.17. (a) Let $f : X \to Y$ is B^*-clopen continuous and $g : Y \to Z$ is clopen continuous then $gof : X \to Z$ is B^*-clopen continuous.

(b) If f is B^*-irresolute and g is B^*-clopen continuous, then gof is B^*-clopen continuous.

Proof. (a) Let $x \in X$, U be a clopen set in X containing x and W be open in Z containing $(gof)(x)$. Now, g being clopen continuous, there exists a clopen set V in Y containing $f(x)$ such that

$$g(V) \subset W.$$

Hence, by B^*-clopen continuity of f, there exists a B^*-set $B \subset U$ such that

$$f(B) \subset V \subset g^{-1}(W),$$

which implies,

$$g(f(B)) \subset W.$$

(b) Let $x \in X$ and U be a clopen set containing x in X and W be any open set in Z containing $(gof)(x)$. Let V be any clopen set in Y containing $f(x)$. Now, g being B^*-clopen continuous, there exists a B^*-set $B_Y \subseteq V$ containing $f(x)$ such that $g(B_Y) \subseteq W$. Also, f being B^*-irresolute, there exists a B^*-set B in X such that $x \in B \subseteq U$ such that

$$f(B) \subseteq B_Y,$$

which implies,

$$g\big(f(B)\big) \subseteq g(B_Y) \subseteq W.$$

Therefore, gof is B^*-clopen continuous. Q.E.D.

4 cl-Oscillation function

The concept of oscillation function ω_f corresponding to a function f is a classical notion [1] which is related with the continuity of f. It is known that f is continuous at a point if and only if ω_f is zero at that point [1]. Subsequently, different oscillation functions were introduced and studied corresponding to different types of continuities, e.g., Borsik [2], [3] defined oscillation functions for quasi continuity and almost continuity. Ganguly and Mitra [7] did it for B^*-continuity.
Here, we introduce a new oscillation function, to be called cl-oscillation function and characterize the B^*-clopen continuity through it.
Now onwards throughout the paper, X will be a topological space and Y a metric space with metric d.

Definition 4.1. Let $x \in X$. Let $\mathcal{U}_x$ denote the family of all clopen neighborhoods of x and for $A \subset X$, $\mathcal{B}_A$ denote the family of all B^*-sets $B \subset A$ containing x. Let $f : X \to Y$ and $U \in \mathcal{U}_x$ be fixed. We define

$$\Omega_f^c(x, U) = \inf_B \left\{ \sup_{z \in B} \left\{ d\Big(f(x), f(z)\Big) : B \in \mathcal{B}_U \right\} \right\}$$

and call it to be the cl-oscillation of f at x over U.
The number $\Omega_f^c(x)$ defined by

$$\Omega_f^c(x) = \sup_{U \in \mathcal{U}_x} \Big\{ \Omega_f^c(x, U) \Big\}.$$

is called the cl-oscillation of f at x. This defines an extended real valued function on X and is called the cl-oscillation function of f.

Below we characterize the B^*-clopen continuity in terms of cl-oscillation:

Theorem 4.2. A function $f : X \to Y$ is B^*-clopen continuous at $x \in X$ if and only if $\Omega_f^c(x) = 0$.

Proof. Let $\Omega_f^c(x) = 0$ and $\varepsilon > 0$ be given. It follows from the definition of $\Omega_f^c(x)$ that for every clopen set U containing the point x, we have

$$\Omega_f^c(x, U) < \varepsilon$$

for any arbitrary clopen set U containing x,
i.e.,

$$\inf_B \left\{ \sup_{z \in B} \left\{ d\Big(f(x), f(z)\Big) : B \in \mathcal{B}_U \right\} \right\} < \varepsilon$$

so that, there exists a B^*-set $B \in \mathcal{B}_U$ such that

$$\sup_{z \in B} \left\{ d\Big(f(x), f(z)\Big) \right\} < \varepsilon$$

which implies that for all $z \in B$

$$d(f(x), f(z)) < \varepsilon.$$

Since ε is arbitrary, B^*-clopen continuity of f at x follows.
Conversely, let f be B^*-clopen continuous at $x \in X$. Let U be a clopen set containing x and $V = S\Big(f(x), \frac{\eta}{2}\Big)$ be an open set containing $f(x)$. Then there exists a B^*-set $B \subset U \cap f^{-1}(V)$. Since $f(B) \subset V$, we have for all $z \in B$

$$d\Big(f(x), f(z)\Big) < \frac{\eta}{2}$$

which gives that

$$\sup_{z \in B} \left\{ d\Big(f(x), f(z)\Big) \right\} \leq \frac{\eta}{2}$$

and this further implies that

$$\inf_B \left\{ \sup_{z \in B} \left\{ d\Big(f(x), f(z)\Big) \right\} \right\} \leq \frac{\eta}{2}$$

i.e., for every clopen set U containing the point x

$$\Omega_f^c(x,U) \leq \frac{\eta}{2}.$$

Therefore,

$$\Omega_f^c(x) = \sup_U \Big(\Omega_f^c(x,U)\Big) \leq \frac{\eta}{2} < \eta.$$

Since $\eta > 0$ is arbitrary, $\Omega_f^c(x) = 0$. Q.E.D.

Remark 4.3. The countable union of B^*-closed sets need not be B^*-closed. Indeed, consider the space $X = [0,1]$ with usual topology. Let A and B denote respectively, the rationals and irrationals in $(0,1)$. Clearly A and B are both B^*-closed. However, $A \cup B = (0,1)$ is not B^*-closed, since 0 and 1 are the two B^*-cluster points of $(0,1)$ not contained in this set.

Remark 4.4. It follows from Remark 4.3 that the countable intersection of B^*-open sets need not be B^*-open.

In view of Remarks 4.3 and 4.4 and motivated by the constructions of the classical $\mathcal{F}_\sigma$ and $\mathcal{G}_\delta$ sets [13], we introduce the notion of $\mathcal{F}_\sigma^*$ and $\mathcal{G}_\delta^*$ sets as follows:

Definition 4.5. A set, which is a countable union of B^*-closed sets is called an $\mathcal{F}_\sigma^*$-set. Similarly, a set, which is a countable intersection of B^*-open sets is called a $\mathcal{G}_\delta^*$-set.

We prove the following:

Theorem 4.6. For a function $f : X \to Y$, the following hold:

(i) The set of points where f is not B^*-clopen continuous forms an $\mathcal{F}_\sigma^*$-set.

(ii) The set of points where f is B^*-clopen continuous is a $\mathcal{G}_\delta^*$-set.

Proof. (i) In view of Theorem 4.2, f is B^*-clopen continuous at a point $x \in X$, if and only if $\Omega_f^c(x) = 0$. Let us define, for a fixed $n \in \mathbb{N}$,

$$F_n = \{x \in X : \Omega_f^c(x) \geq 1/n\}.$$

Let x_0 be a B^*-cluster point of F_n. If $x_0 \notin F_n$, then $\Omega_f^c(x_0) < \frac{1}{n}$, so that for every clopen set U containing x_0, we have

$$\Omega_f^c(x_0,U) \leq \Omega_f^c(x_0) < \frac{1}{n}$$

i.e.,

$$\inf_B \left\{ \sup_{y\in B} \Big\{ d\big(f(y), f(x_0)\big) : B \in \mathcal{B}_U \Big\} \right\} < \frac{1}{n}.$$

Therefore, there exists a B^*-set B such that

$$\sup_{y\in B} \Big\{ d\big(f(y), f(x_0)\big) \Big\} < \frac{1}{n}$$

i.e., for all $y \in B$

$$d\big(f(y), f(x_0)\big) < \frac{1}{n}.$$

This implies that, there exists a B^*-set B containing x_0 such that $F_n \cap B = \varnothing$, which contradicts the fact that x_0 is a B^*-cluster point of F_n. Hence, $x_0 \in F_n$ and F_n is B^*-closed. Now, the set of all points where f is not B^*-clopen continuous can be written as

$$F = \bigcup_{n=1}^{\infty} F_n.$$

Since each F_n is B^*-closed, it follows that F is an $\mathcal{F}_\sigma^*$-set and we are done.

(ii) It is an immediate consequence of *(i)* above. Q.E.D.

5 Convergence

In this section, we study the behaviour of the limit functions of the sequences of B^*-clopen continuous functions under pointwise as well as uniform convergence. We begin with an example showing that B^*-clopen continuity is not preserved under pointwise convergence.

Example 5.1. Consider the sequence of functions $f_n(x) = x^n, x \in [0,1]$. Clearly, the pointwise limit function f of $\{f_n\}$ is given by

$$f(x) = \begin{cases} 0 & if\ x \neq 1 \\ 1 & if\ x = 1. \end{cases}$$

Each f_n being continuous is B^*-clopen continuous. But f is not B^*-clopen continuous.

However, we shall prove that the pointwise limit of a transfinite sequence of B^*-clopen continuous functions remains B^*-clopen continuous. The concept of the pointwise convergence of a transfinite sequence of real functions of real variable was given by Sierpinski [23] (see also [12]).

Definition 5.2. [12] Let X be a T_1-topological space satisfying the first axiom of countability and β be an ordinal number. A sequence $\{x_\alpha\}_{\alpha<\beta}$, in X is called the transfinite sequence of type β (written, t.s.t. β).

Definition 5.3. [12] Let X be a T_1-topological space satisfying the first axiom of countability, Ω be first uncountable ordinal number and $\{x_\alpha\}_{\alpha<\Omega}$ be the t.s.t. Ω. The sequence $\{x_\alpha\}_{\alpha<\Omega}$ is said to converge to a point $x \in X$, written $x_\alpha \to x$, or $\lim\limits_{\alpha\to\Omega} x_\alpha = x$, if for every neighbourhood V of x, there exists an ordinal number $\eta < \Omega$ such that $x_\alpha \in V$ for every $\eta < \alpha < \Omega$.

Remark 5.4. In order to define the sequence $\{x_\alpha\}_{\alpha<\beta}$ t.s.t. β, there is no requirement that X be a T_1-topological space satisfying the first axiom of countability. However, for convergence this assumption is required.

Theorem 5.5. Let Ω be the first uncountable ordinal number. Consider the sequence $\{f_\alpha\}_{\alpha<\Omega}$ of functions t.s.t. Ω, where $f_\alpha : X \to Y, \alpha < \Omega$ such that $f_\alpha(x) \to f(x)$ for each $x \in X$. If each f_α is B^*-clopen continuous, then f is also B^*-clopen continuous.

Proof. By the definition of pointwise convergence, to each $x \in X$ and every neighborhood V of $f(x)$, there exists an ordinal number $\alpha_1 < \Omega$ such that $f_\alpha(x) \in V$ for every $\alpha_1 < \alpha < \Omega$. Further, the convergence also implies that there exists $\mu < \Omega$ such that for $\alpha > \mu$ we have $f_\alpha(x) = f(x)$ for all $x \in X$.
Let $\gamma > max(\mu, \alpha_1)$ be any fixed ordinal number. Then, f_γ is B^*-clopen continuous at some point, say, x_0. So, for any clopen set U containing x_0 and open neighborhood V of $f_\gamma(x_0)$, there exists a B^*-set B containing x_0 such that for every $b \in B$, we have

$$f_\gamma(b) \in V.$$

Now since, $\gamma > \mu$, $f_\gamma(x_0) = f(x_0)$ and $f_\gamma(b) = f(b)$ for all $b \in B$. Consequently, f is B^*-clopen continuous at x_0. Q.E.D.

We have observed that B^*-clopen continuity is not preserved under pointwise convergence of ordinary sequences whereas in the case of transfinite sequences, it is so. We prove below that for ordinary sequences, uniform convergence preserves the B^*-clopen continuity.

Theorem 5.6. Let $\{f_n\}$ be a sequence of functions $f_n : X \to Y$, converging uniformly to a function $f : X \to Y$. If each f_n is B^*-clopen continuous, then f is also B^*-clopen continuous.

Proof. Since $f_n \to f$ uniformly on X, given $\varepsilon > 0$, there exists $N_0 \in \mathbb{N}$ such that for all $n \geq N_0$ and for all $x \in X$

$$d\Big(f_n(x), f(x)\Big) < \frac{\varepsilon}{3}.$$

Let $y \in X$ be an arbitrary point. Since f_n is B^*-clopen continuous, for any clopen set U containing y, there exists a B^*-set B, $y \in B \subset U$ such that for all $x \in B$,

$$f_n(x) \in S\Big(f_n(y), \frac{\varepsilon}{3}\Big),$$

i.e.,

$$d(f_n(x), f_n(y) < \frac{\varepsilon}{3}.$$

Let $x \in B$. Then for all $n > N_0$, we have

$$d\big(f(x), f(y)\big) \leq d\big(f(x), f_n(x)\big) + d\big(f_n(x), f_n(y)\big) + d\big(f_n(y), f(y)\big) < \varepsilon$$

so that,

$$B \subset U \cap f^{-1}\big(S(f(y), \varepsilon)\big)$$

and we are done. Q.E.D.

Theorem 5.7. Let $\{f_n\}$ be a sequence of B^*-clopen continuous functions $f_n : X \to Y$, converging uniformly to a function $f : X \to Y$. Then $\Omega^c_{f_n} \to \Omega^c_f$ uniformly on X.

Proof. Let $\varepsilon > 0$. Then, there exists $N_0 \in \mathbb{N}$, such that for all $n \geq N_0$, and for all $x \in X$

$$d\big(f_n(x), f(x)\big) \leq \varepsilon.$$

Let $x \in X$ and U be a clopen set containing x. Then, for any B^*-set $B \subset U$ with $x \in B$, we have for all $y \in B$ and $n \geq N_0$

$$\begin{aligned} d\big(f(y), f(x)\big) &\leq d\big(f(y), f_n(y)\big) + d\big(f_n(y), f_n(x)\big) + d\big(f_n(x), f(x)\big) \\ &\leq 2\varepsilon + d\big(f_n(y), f_n(x)\big). \end{aligned}$$

Similarly, we can establish that

$$d\big(f_n(y), f_n(x)\big) \leq 2\varepsilon + d\big(f(y), f(x)\big).$$

Consequently, we get for all $n \geq N_0$

$$\big|\Omega^c_{f_n}\big(x, U(x)\big) - \Omega^c_f(x, U(x))\big| \leq 2\varepsilon$$

and the assertion follows. Q.E.D.

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