On generalized Pentanacci quaternions

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Abstract

In this paper, we introduce the generalized Pentanacci quaternions and we present Binet's formulas, generating functions and the summation formulas for these quaternions and as special case for Pentanacci and Pentanacci-Lucas quaternions. Furthermore, we present matrices related to generalized Pentanacci quaternions.

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1 Introduction

In this paper, we define generalized Pentanacci quaternions and Pentanacci and Pentanacci-Lucas quaternions as special cases in the next section and give some properties of them. First, in this section, we present some background about quaternions and generalized Pentanacci numbers.

A quaternion is a hyper-complex number and is defined by

$$q = a_0 + ia_1 + ja_2 + ka_3 = (a_0, a_1, a_2, a_3)$$

where a_0, a_1, a_2 and a_3 are real numbers or scalers and $1, i, j, k$ are the standard orthonormal basis in $\mathbb{R}^4$, and

$$\begin{aligned} i^2 &= j^2 = k^2 = ijk = -1, \\ ij &= k = -ji,\ jk = i = -kj,\ ki = j = -ik. \end{aligned}$$

The set of all quaternions are denoted by $\mathbb{H}$. $\mathbb{H}$ is a 4 dimensional non-commutative, associative (Carley-Dickson) algebra over the reals.

A generalized Pentanacci sequence $\{V_n\}_{n\geq 0} = \{V_n(V_0, V_1, V_2, V_3, V_4)\}_{n\geq 0}$ is defined by the fifth-order recurrence relations

$$V_n = V_{n-1} + V_{n-2} + V_{n-3} + V_{n-4} + V_{n-5}, \tag{1.1}$$

with the initial values $V_0 = c_0, V_1 = c_1, V_2 = c_2, V_3 = c_3, V_4 = c_4$ not all being zero. The sequence $\{V_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$V_{-n} = -V_{-(n-1)} - V_{-(n-2)} - V_{-(n-3)} - V_{-(n-4)} + V_{-(n-5)}$$

for $n = 1, 2, 3, ...$ Therefore, recurrence (1.1) holds for all integer n.

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The first few generalized Pentanacci numbers with positive subscript and negative subscript are given in the following Table 1:

Table 1. A few generalized Pentanacci numbers

n	V_n	V_{-n}
0	c_0	c_0
1	c_1	$-c_0-c_1-c_2-c_3+c_4$
2	c_2	$2c_3-c_4$
3	c_3	$2c_2-c_3$
4	c_4	$2c_1-c_2$
5	$c_0+c_1+c_2+c_3+c_4$	$2c_0-c_1$
6	$c_0+2c_1+2c_2+2c_3+2c_4$	$-3c_0-2c_1-2c_2-2c_3+2c_4$
7	$2c_0+3c_1+4c_2+4c_3+4c_4$	$c_0+c_1+c_2+5c_3-3c_4$
8	$4c_0+6c_1+7c_2+8c_3+8c_4$	$4c_2-4c_3+c_4$
9	$8c_0+12c_1+14c_2+15c_3+16c_4$	$4c_1-4c_2+c_3$
10	$16c_0+24c_1+28c_2+30c_3+31c_4$	$4c_0-4c_1+c_2$

We consider two special cases of $\{V_n\}$. Pentanacci sequence $\{P_n\}_{n\geq 0}$ and Pentanacci-Lucas sequence $\{Q_n\}_{n\geq 0}$ are defined by the fifth-order recurrence relations

$$P_n = P_{n-1}+P_{n-2}+P_{n-3}+P_{n-4}+P_{n-5}, \qquad P_0=0, P_1=1, P_2=1, P_3=2, P_4=4 \tag{1.2}$$

and

$$Q_n = Q_{n-1}+Q_{n-2}+Q_{n-3}+Q_{n-4}+Q_{n-5}, \qquad Q_0=5, Q_1=1, Q_2=3, Q_3=7, Q_4=15 \tag{1.3}$$

respectively. Pentanacci sequence has been studied by many authors, see for example [9], [10], [12], [14].

In the following Table 2, we present the first few values of the Pentanacci and Pentanacci-Lucas numbers with positive and negative subscripts:

Table 2. A few Pentanacci and Pentanacci-Lucas Numbers

n	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10
P_n	-3	2	0	0	0	-1	1	0	0	0	0	1	1	2	4	8	16	31	61	120	236
Q_n	19	-1	-1	-1	-7	9	-1	-1	-1	-1	5	1	3	7	15	31	57	113	223	439	863

For all integers n, usual Pentanacci and Pentanacci-Lucas numbers can be expressed using Binet's formulas

$$\begin{aligned} P_n &= \frac{\alpha^{n+3}}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\lambda)}+\frac{\beta^{n+3}}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)(\beta-\lambda)} \\ &+\frac{\gamma^{n+3}}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)(\gamma-\lambda)}+\frac{\delta^{n+3}}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)(\delta-\lambda)}+\frac{\lambda^{n+3}}{(\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)(\lambda-\delta)} \end{aligned}$$

(see [14, Theorem 2.2]) or

$$P_n = \frac{\alpha-1}{6\alpha-10}\alpha^{n-1}+\frac{\beta-1}{6\beta-10}\beta^{n-1}+\frac{\gamma-1}{6\gamma-10}\gamma^{n-1}+\frac{\delta-1}{6\delta-10}\delta^{n-1}+\frac{\lambda-1}{6\lambda-10}\lambda^{n-1} \tag{1.4}$$

(see, for example, [5])

and

$$Q_n = \alpha^n + \beta^n + \gamma^n + \delta^n + \lambda^n$$

respectively, where $\alpha, \beta, \gamma, \delta$ and λ are the roots of the equation

$$x^5 - x^4 - x^3 - x^2 - x - 1 = 0. \tag{1.5}$$

Moreover, the approximate value of $\alpha, \beta, \gamma, \delta$ and λ are given by

$$\begin{aligned}
\alpha &= 1.965\,9 \\
\beta &= -0.678\,35 + 0.458\,54.i \\
\gamma &= -0.678\,35 - 0.458\,54.i \\
\delta &= 0.195\,38 + 0.848\,85.i \\
\lambda &= 0.195\,38 - 0.848\,85.i.
\end{aligned}$$

In fact, there are no solutions of the characteristic equation (1.5) in terms of radicals, see [19].

Next we present the Binet's formula of the generalized Pentanacci sequence.

Lemma 1.1 ([14])**.** The Binet's formula of the generalized Pentanacci sequence $\{V_n\}$ is given as

$$V_n = P_{n-4}V_0 + (P_{n-4}+P_{n-5})V_1 + (P_{n-4}+P_{n-5}+P_{n-6})V_2 + (P_{n-4}+P_{n-5}+P_{n-6}+P_{n-7})V_3 + P_{n-3}V_4.$$

Proof. This Lemma is given in [14].

Corollary 1.2 ([14])**.** The Binet's formula of the generalized Pentanacci sequence $\{V_n\}$ is given as

$$V_n = A_1\alpha^{n-8} + A_2\beta^{n-8} + A_3\gamma^{n-8} + A_4\delta^{n-8} + A_5\lambda^{n-8}$$

where

$$\begin{aligned}
A_1 &= \frac{\alpha - 1}{6\alpha - 10}(V_4\alpha^4 + (V_0 + V_1 + V_2 + V_3)\alpha^3 + (V_1 + V_2 + V_3)\alpha^2 + (V_2 + V_3)\alpha + V_3), \\
A_2 &= \frac{\beta - 1}{6\beta - 10}(V_4\beta^4 + (V_0 + V_1 + V_2 + V_3)\beta^3 + (V_1 + V_2 + V_3)\beta^2 + (V_2 + V_3)\beta + V_3), \\
A_3 &= \frac{\gamma - 1}{6\gamma - 10}(V_4\gamma^4 + (V_0 + V_1 + V_2 + V_3)\gamma^3 + (V_1 + V_2 + V_3)\gamma^2 + (V_2 + V_3)\gamma + V_3), \\
A_4 &= \frac{\delta - 1}{6\delta - 10}(V_4\delta^4 + (V_0 + V_1 + V_2 + V_3)\delta^3 + (V_1 + V_2 + V_3)\delta^2 + (V_2 + V_3)\delta + V_3), \\
A_5 &= \frac{\lambda - 1}{6\lambda - 10}(V_4\lambda^4 + (V_0 + V_1 + V_2 + V_3)\lambda^3 + (V_1 + V_2 + V_3)\lambda^2 + (V_2 + V_3)\lambda + V_3).
\end{aligned}$$

Next, we give the ordinary generating function $\sum_{n=0}^{\infty} V_n x^n$ of the sequence $\{V_n\}$.

Lemma 1.3 ([14])**.** Suppose that $f_{V_n}(x) = \sum_{n=0}^{\infty} V_n x^n$ is the ordinary generating function of the generalized Pentanacci sequence $\{V_n\}_{n\geq 0}$.Then $f_{V_n}(x)$ is given by

$$f_{V_n}(x) = \frac{V_0 + (V_1 - V_0)x + (V_2 - V_1 - V_0)x^2 + (V_3 - V_2 - V_1 - V_0)x^3 + (V_4 - V_3 - V_2 - V_1 - V_0)x^4}{1 - x - x^2 - x^3 - x^4 - x^5}. \tag{1.6}$$

The previous Lemma gives the following results as particular examples: generating function of the Pentanacci sequence P_n is

$$f_{P_n}(x) = \sum_{n=0}^{\infty} P_n x^n = \frac{x}{1 - x - x^2 - x^3 - x^4 - x^5} \tag{1.7}$$

and generating function of the Pentanacci-Lucas sequence Q_n is

$$f_{Q_n}(x) = \sum_{n=0}^{\infty} Q_n x^n = \frac{5 - 4x - 3x^2 - 2x^3 - x^4}{1 - x - x^2 - x^3 - x^4 - x^5}.$$

2 Generalized Pentanacci quaternions and their generating functions and Binet's formulas

In this section, we define generalized Pentanacci quaternions and give generating functions and Binet formulas for them. First, we give some information about quaternion sequences from the literature.

There are various types of quaternion sequences which have been studied by many researchers. Horadam [8] introduced nth Fibonacci and nth Lucas quaternions as

$$Q_n = F_n + F_{n+1}e_1 + F_{n+2}e_2 + F_{n+3}e_3 = \sum_{s=0}^{3} F_{n+s}e_s$$

and

$$Q_n = L_n + L_{n+1}e_1 + L_{n+2}e_2 + L_{n+3}e_3 = \sum_{s=0}^{3} L_{n+s}e_s$$

respectively, where F_n and L_n are the nth Fibonacci and Lucas numbers respectively. He also defined generalized Fibonacci quaternion as

$$P_n = H_n + H_{n+1}e_1 + H_{n+2}e_2 + H_{n+3}e_3 = \sum_{s=0}^{3} H_{n+s}e_s$$

where H_n is the nth generalized Fibonacci number (which is now called Horadam number) by the recursive relation $H_1 = p$, $H_2 = p + q$, $H_n = H_{n-1} + H_{n-2}$ (p and q are arbitrary integers). Halici [6] gave the generating functions and Binet's formulas for the Fibonacci and Lucas quaternions. We can list a few references of quaternion sequences. We list the references of a few second order quaternion sequences as

[2],[7],[11],[15],[17]

and we list the references of a few third order quaternion sequences as

[1],[3],[4],[16],[18].

Soykan [13] introduced the Tetranacci and Tetranacci-Lucas quaternions as fourth order quaternion sequences.

We now define generalized Pentanacci quaternions over the quaternion algebra $\mathbb{H}$. The nth generalized Pentanacci quaternion is

$$\widehat{V}_n = V_n + iV_{n+1} + jV_{n+2} + kV_{n+3}. \tag{2.1}$$

As special cases, the nth Pentanacci quaternion and the nth Pentanacci-Lucas quaternion are given as

$$\widehat{P}_n = P_n + iP_{n+1} + jP_{n+2} + kP_{n+3}$$

and

$$\widehat{Q}_n = Q_n + iQ_{n+1} + jQ_{n+2} + kQ_{n+3}$$

respectively. It can be easily shown that

$$\widehat{V}_n = \widehat{V}_{n-1} + \widehat{V}_{n-2} + \widehat{V}_{n-3} + \widehat{V}_{n-4} + \widehat{V}_{n-5}. \tag{2.2}$$

The sequence $\{\widehat{V}_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\widehat{V}_{-n} = -\widehat{V}_{-(n-1)} - \widehat{V}_{-(n-2)} - \widehat{V}_{-(n-3)} - \widehat{V}_{-(n-4)} + \widehat{V}_{-(n-5)}.$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrence (2.2) holds for all integer n.

The first few generalized Pentanacci quaternions with positive subscript and negative subscript are given in the following Table 3:

Table 3. A few generalized Pentanacci quaternions

n	$\widehat{V}_n$	$\widehat{V}_{-n}$
0	$V_0 + iV_1 + jV_2 + kV_3$	$V_0 + iV_1 + jV_2 + kV_3$
1	$V_1 + iV_2 + jV_3 + kV_4$	$V_{-1} + iV_0 + jV_1 + kV_2$
2	$V_2 + iV_3 + jV_4 + kV_5$	$V_{-2} + iV_{-1} + jV_0 + kV_1$
3	$V_3 + iV_4 + jV_5 + kV_6$	$V_{-3} + iV_{-2} + jV_{-1} + kV_0$
4	$V_4 + iV_5 + jV_6 + kV_7$	$V_{-4} + iV_{-3} + jV_{-2} + kV_{-1}$
5	$V_5 + iV_6 + jV_7 + kV_8$	$V_{-5} + iV_{-4} + jV_{-3} + kV_{-2}$
6	$V_6 + iV_7 + jV_8 + kV_9$	$V_{-6} + iV_{-5} + jV_{-4} + kV_{-3}$
7	$V_7 + iV_8 + jV_9 + kV_{10}$	$V_{-7} + iV_{-6} + jV_{-5} + kV_{-4}$

The first few Pentanacci quaternions with positive subscript and negative subscript are given in the following Table 4:

Table 4. A few Pentanacci quaternions

n	$\widehat{P}_n$	$\widehat{P}_{-n}$
0	$i + j + 2k$	$i + j + 2k$
1	$1 + i + 2j + 4k$	$j + k$
2	$1 + 2i + 4j + 8k$	k
3	$2 + 4i + 8j + 16k$	0
4	$4 + 8i + 16j + 31k$	1
5	$8 + 16i + 31j + 61k$	$-1 + i$
6	$16 + 31i + 61j + 120k$	$-i + j$
7	$31 + 61i + 120j + 236k$	$-j + k$

The first few Pentanacci-Lucas quaternions with positive subscript and negative subscript are given in the following Table 5:

Table 5. A few Pentanacci-Lucas quaternions

n	$\widehat{Q}_n$	$\widehat{Q}_{-n}$
0	$5+i+3j+7k$	$5+i+3j+7k$
1	$1+3i+7j+15k$	$-1+5i+j+3k$
2	$3+7i+15j+31k$	$-1-i+5j+k$
3	$7+15i+31j+57k$	$-1-i-j+5k$
4	$15+31i+57j+113k$	$-1-i-j-k$
5	$31+57i+113+223k$	$9-i-j-k$
6	$57+113i+223j+439k$	$-7+9i-j-k$
7	$113+223i+439j+863k$	$-1-7i+9j-k$

Now, we will state Binet's formula for the generalized Pentanacci quaternions and in the rest of the paper, we fix the following notations:

$$\begin{aligned}
\widehat{\alpha} &= 1+i\alpha+j\alpha^2+k\alpha^3,\\
\widehat{\beta} &= 1+i\beta+j\beta^2+k\beta^3,\\
\widehat{\gamma} &= 1+i\gamma+j\gamma^2+k\gamma^3,\\
\widehat{\delta} &= 1+i\delta+j\delta^2+k\delta^3,\\
\widehat{\lambda} &= 1+i\lambda+j\lambda^2+k\lambda^3.
\end{aligned}$$

Theorem 2.1. (Binet's Formula) For any integer n, the nth generalized Pentanacci quaternion is

$$\widehat{V}_n = A_1\widehat{\alpha}\alpha^{n-8} + A_2\widehat{\beta}\beta^{n-8} + A_3\widehat{\gamma}\gamma^{n-8} + A_4\widehat{\delta}\delta^{n-8} + A_5\widehat{\lambda}\lambda^{n-8} \tag{2.3}$$

where A_1, A_2, A_3, A_4 and A_5 are as in Corollary 1.2.

Proof. Using Binet's formula of the generalized Pentanacci numbers, we obtain

$$\begin{aligned}
\widehat{V}_n &= V_n + iV_{n+1} + jV_{n+2} + kV_{n+3}\\
&= A_1\alpha^{n-8} + A_2\beta^{n-8} + A_3\gamma^{n-8} + A_4\delta^{n-8} + A_5\lambda^{n-8}\\
&\quad +i(A_1\alpha^{n-7} + A_2\beta^{n-7} + A_3\gamma^{n-7} + A_4\delta^{n-7} + A_5\lambda^{n-7})\\
&\quad +j(A_1\alpha^{n-6} + A_2\beta^{n-6} + A_3\gamma^{n-6} + A_4\delta^{n-6} + A_5\lambda^{n-6})\\
&\quad +k(A_1\alpha^{n-5} + A_2\beta^{n-5} + A_3\gamma^{n-5} + A_4\delta^{n-5} + A_5\lambda^{n-5})\\
&= A_1\widehat{\alpha}\alpha^{n-8} + A_2\widehat{\beta}\beta^{n-8} + A_3\widehat{\gamma}\gamma^{n-8} + A_4\widehat{\delta}\delta^{n-8} + A_5\widehat{\lambda}\lambda^{n-8}.
\end{aligned}$$

This proves (2.3).

As special cases, for any integer n, the Binet's Formula of nth Pentanacci quaternion is

$$\widehat{P}_n = \frac{\alpha-1}{6\alpha-10}\widehat{\alpha}\alpha^{n-1} + \frac{\beta-1}{6\beta-10}\widehat{\beta}\beta^{n-1} + \frac{\gamma-1}{6\gamma-10}\widehat{\gamma}\gamma^{n-1} + \frac{\delta-1}{6\delta-10}\widehat{\delta}\delta^{n-1} + \frac{\lambda-1}{6\lambda-10}\widehat{\lambda}\lambda^{n-1}$$

and the Binet's Formula of nth Pentanacci-Lucas quaternion is

$$\widehat{Q}_n = \widehat{\alpha}\alpha^n + \widehat{\beta}\beta^n + \widehat{\gamma}\gamma^n + \widehat{\delta}\delta^n + \widehat{\lambda}\lambda^n.$$

Next, we present generating function.

Theorem 2.2. The generating function for the generalized Pentanacci quaternions is

$$\sum_{n=0}^{\infty} \widehat{V}_n x^n = \frac{\widehat{V}_0 + (\widehat{V}_1 - \widehat{V}_0)x + (\widehat{V}_2 - \widehat{V}_1 - \widehat{V}_0)x^2 + (\widehat{V}_3 - \widehat{V}_2 - \widehat{V}_1 - \widehat{V}_0)x^3 + \widehat{V}_{-1}x^4}{1 - x - x^2 - x^3 - x^4 - x^5} \tag{2.4}$$

Proof. Let

$$g(x) = \sum_{n=0}^{\infty} \widehat{V}_n x^n$$

be generating function of generalized Pentanacci quaternions. Then, using the definition of the generalized Pentanacci quaternions, and substracting $xg(x)$, $x^2g(x)$, $x^3g(x)$ and $x^4g(x)$ from $g(x)$, we obtain (note the shift in the index n in the third line)

$$\begin{aligned}
& (1 - x - x^2 - x^3 - x^4 - x^5)g(x) \\
= \; & \sum_{n=0}^{\infty} \widehat{V}_n x^n - x\sum_{n=0}^{\infty} \widehat{V}_n x^n - x^2\sum_{n=0}^{\infty} \widehat{V}_n x^n - x^3\sum_{n=0}^{\infty} \widehat{V}_n x^n - x^4\sum_{n=0}^{\infty} \widehat{V}_n x^n - x^5\sum_{n=0}^{\infty} \widehat{V}_n x^n \\
= \; & \sum_{n=0}^{\infty} \widehat{V}_n x^n - \sum_{n=0}^{\infty} \widehat{V}_n x^{n+1} - \sum_{n=0}^{\infty} \widehat{V}_n x^{n+2} - \sum_{n=0}^{\infty} \widehat{V}_n x^{n+3} - \sum_{n=0}^{\infty} \widehat{V}_n x^{n+4} - \sum_{n=0}^{\infty} \widehat{V}_n x^{n+5} \\
= \; & \sum_{n=0}^{\infty} \widehat{V}_n x^n - \sum_{n=1}^{\infty} \widehat{V}_{n-1} x^n - \sum_{n=2}^{\infty} \widehat{V}_{n-2} x^n - \sum_{n=3}^{\infty} \widehat{V}_{n-3} x^n - \sum_{n=4}^{\infty} \widehat{V}_{n-4} x^n - \sum_{n=5}^{\infty} \widehat{V}_{n-5} x^n \\
= \; & (\widehat{V}_0 + \widehat{V}_1 x + \widehat{V}_2 x^2 + \widehat{V}_3 x^3 + \widehat{V}_4 x^4) - (\widehat{V}_0 x + \widehat{V}_1 x^2 + \widehat{V}_2 x^3 + \widehat{V}_3 x^4) \\
& -(\widehat{V}_0 x^2 + \widehat{V}_1 x^3 + \widehat{V}_2 x^4) - (\widehat{V}_0 x^3 + \widehat{V}_1 x^4) - \widehat{V}_0 x^4 \\
& + \sum_{n=5}^{\infty} (\widehat{V}_n - \widehat{V}_{n-1} - \widehat{V}_{n-2} - \widehat{V}_{n-3} - \widehat{V}_{n-4} - \widehat{V}_{n-5}) x^n \\
= \; & \widehat{V}_0 + (\widehat{V}_1 - \widehat{V}_0)x + (\widehat{V}_2 - \widehat{V}_1 - \widehat{V}_0)x^2 + (\widehat{V}_3 - \widehat{V}_2 - \widehat{V}_1 - \widehat{V}_0)x^3 + (\widehat{V}_4 - \widehat{V}_3 - \widehat{V}_2 - \widehat{V}_1 - \widehat{V}_0)x^4.
\end{aligned}$$

Note that we used the recurrence relation $\widehat{V}_n = \widehat{V}_{n-1} + \widehat{V}_{n-2} + \widehat{V}_{n-3} + \widehat{V}_{n-4} + \widehat{V}_{n-5}$. Rearranging above equation, we get

$$g(x) = \frac{\widehat{V}_0 + (\widehat{V}_1 - \widehat{V}_0)x + (\widehat{V}_2 - \widehat{V}_1 - \widehat{V}_0)x^2 + (\widehat{V}_3 - \widehat{V}_2 - \widehat{V}_1 - \widehat{V}_0)x^3 + (\widehat{V}_4 - \widehat{V}_3 - \widehat{V}_2 - \widehat{V}_1 - \widehat{V}_0)x^4}{1 - x - x^2 - x^3 - x^4 - x^5},$$

or

$$g(x) = \frac{\widehat{V}_0 + (\widehat{V}_1 - \widehat{V}_0)x + (\widehat{V}_2 - \widehat{V}_1 - \widehat{V}_0)x^2 + (\widehat{V}_3 - \widehat{V}_2 - \widehat{V}_1 - \widehat{V}_0)x^3 + \widehat{V}_{-1}x^4}{1 - x - x^2 - x^3 - x^4 - x^5}$$

since $\widehat{V}_4 = \widehat{V}_3 + \widehat{V}_2 + \widehat{V}_1 + \widehat{V}_0 + \widehat{V}_{-1}$.

As special cases, the generating functions for the Pentanacci and Pentanacci-Lucas quaternions are

$$\sum_{n=0}^{\infty} \widehat{P}_n x^n = \frac{(i + j + 2k) + (1 + j + 2k)x + (j + 2k)x^2 + (j + 2k)x^3 + (j + k)x^4}{1 - x - x^2 - x^3 - x^4 - x^5}$$

and

$$\sum_{n=0}^{\infty}\widehat{Q}_n x^n=\frac{\begin{array}{c}(5+i+3j+7k)+(-4+2i+4j+8k)x+(-3+3i+5j+9k)x^2\\+(-2+4i+6j+4k)x^3+(-1+5i+j+3k)x^4\end{array}}{1-x-x^2-x^3-x^4-x^5}$$

respectively.

The following Theorem present some linear sum identities of generalized Pentanacci numbers.

Theorem 2.3. For $n\geq 0$, we have the following formulas:

(a) $\sum_{p=0}^{n}V_p=\frac{1}{4}(V_{n+4}-V_{n+2}-2V_{n+1}+V_n-V_4+V_2+2V_1+3V_0)$,

(b) $\sum_{p=0}^{n}V_{2p+1}=\frac{1}{8}(3V_{2n+2}+4V_{2n+1}+V_{2n}+2V_{2n-1}-V_{2n-2}-3V_4+4V_3-V_2+6V_1+V_0)$

(c) $\sum_{p=0}^{n}V_{2p}=\frac{1}{8}(-V_{2n+2}+4V_{2n+1}+5V_{2n}+2V_{2n-1}+3V_{2n-2}+V_4-4V_3+3V_2-2V_1+5V_0)$.

Proof. This is a Theorem in Soykan [14, Theorem 2.9].

Next, we give some linear sum identities of generalized Pentanacci quaternions.

Theorem 2.4. For $n\geq 0$, we have the following formulas:

(a) $\sum_{p=0}^{n}\widehat{V}_p=\frac{1}{4}(\widehat{V}_{n+4}-\widehat{V}_{n+2}-2\widehat{V}_{n+1}+\widehat{V}_n+c)$,

(b) $\sum_{p=0}^{n}\widehat{V}_{2p+1}=\frac{1}{8}(3\widehat{V}_{2n+2}+4\widehat{V}_{2n+1}+\widehat{V}_{2n}+2\widehat{V}_{2n-1}-\widehat{V}_{2n-2}+d)$,

(c) $\sum_{p=0}^{n}\widehat{V}_{2p}=\frac{1}{8}(-\widehat{V}_{2n+2}+4\widehat{V}_{2n+1}+5\widehat{V}_{2n}+2\widehat{V}_{2n-1}+3\widehat{V}_{2n-2}+e)$,

where

$$\begin{aligned}
c &= -V_4+V_2+2V_1+3V_0+i(-V_4+V_2+2V_1-V_0)\\
&\quad +j(-V_4+V_2-2V_1-V_0)+k(-V_4-3V_2-2V_1-V_0),\\
d &= (-3V_4+4V_3-V_2+6V_1+V_0)+i(V_4-4V_3+3V_2-2V_1-3V_0)\\
&\quad +j(-3V_4+4V_3-V_2-2V_1+V_0)+k(V_4-4V_3-5V_2-2V_1-3V_0),\\
e &= (V_4-4V_3+3V_2-2V_1+5V_0)+i(-3V_4+4V_3-V_2+6V_1+V_0)\\
&\quad +j(V_4-4V_3+3V_2-2V_1-3V_0)\\
&\quad +k(-3V_4+4V_3-V_2-2V_1+V_0).
\end{aligned}$$

Proof. We prove (a). (b) and (c) can be proved similarly. Using definition of generalized Pentanacci quaternions, we obtain

$$\begin{aligned}
\sum_{p=0}^{n}\widehat{V}_p &= \sum_{p=0}^{n}V_p+i\sum_{p=0}^{n}V_{p+1}+j\sum_{p=0}^{n}V_{p+2}+k\sum_{p=0}^{n}V_{p+3}\\
&= (V_0+...+V_n)+i(V_1+...+V_{n+1})\\
&\quad +j(V_2+...+V_{n+2})+k(V_3+...+V_{n+3}).
\end{aligned}$$

and so, from Theorem 2.3 (a), we have

$$\begin{aligned} 4\sum_{p=0}^{n}\widehat{V}_p &= (V_{n+4}-V_{n+2}-2V_{n+1}+V_n-V_4+V_2+2V_1+3V_0) \\ &\quad +i(V_{n+4}-V_{n+2}-2V_{n+1}+V_n-V_4+V_2+2V_1+3V_0-4V_0) \\ &\quad +j(V_{n+5}-V_{n+3}-2V_{n+2}+V_{n+1}-V_4+V_2+2V_1+3V_0-4(V_0+V_1)) \\ &\quad +k(V_{n+6}-V_{n+4}-2V_{n+3}+V_{n+1}-V_4+V_2+2V_1+3V_0-4(V_0+V_1+V_2)) \\ &= \widehat{V}_{n+4}-\widehat{V}_{n+2}-2\widehat{V}_{n+1}+\widehat{V}_n+c \end{aligned}$$

where

$$\begin{aligned} c &= -V_4+V_2+2V_1+3V_0+i(-V_4+V_2+2V_1+3V_0-4V_0) \\ &\quad +j(-V_4+V_2+2V_1+3V_0-4(V_0+V_1))+k(-V_4+V_2+2V_1+3V_0-4(V_0+V_1+V_2)) \\ &= -V_4+V_2+2V_1+3V_0+i(-V_4+V_2+2V_1-V_0) \\ &\quad +j(-V_4+V_2-2V_1-V_0)+k(-V_4-3V_2-2V_1-V_0). \end{aligned}$$

Hence

$$\sum_{p=0}^{n}\widehat{V}_p=\frac{1}{4}(\widehat{V}_{n+4}-\widehat{V}_{n+2}-2\widehat{V}_{n+1}+\widehat{V}_n+c).$$

This proves (a).

As special cases of the above Theorem, we have the following two Corollaries. First one present some summing formulas of Pentanacci quaternions.

Corollary 2.5. For $n\geq 0$, we have the following formulas:

(a) $\sum_{p=0}^{n}\widehat{P}_p=\frac{1}{4}(\widehat{P}_{n+4}-\widehat{P}_{n+2}-2\widehat{P}_{n+1}+\widehat{P}_n-1-i-5j-9k)$,

(b) $\sum_{p=0}^{n}\widehat{P}_{2p+1}=\frac{1}{8}(3\widehat{P}_{2n+2}+4\widehat{P}_{2n+1}+\widehat{P}_{2n}+2\widehat{P}_{2n-1}-\widehat{P}_{2n-2}+1-3i-7j-11k)$,

(c) $\sum_{p=0}^{n}\widehat{P}_{2p}=\frac{1}{8}(-\widehat{P}_{2n+2}+4\widehat{P}_{2n+1}+5\widehat{P}_{2n}+2\widehat{P}_{2n-1}+3\widehat{P}_{2n-2}-3+i-3j-7k)$.

Second Corollary gives some summing formulas of Pentanacci-Lucas quaternions.

Corollary 2.6. For $n\geq 0$, we have the following formulas:

(a) $\sum_{p=0}^{n}\widehat{Q}_p=\frac{1}{4}(\widehat{Q}_{n+4}-\widehat{Q}_{n+2}-2\widehat{Q}_{n+1}+\widehat{Q}_n+5-15i-19j-31k)$,

(b) $\sum_{p=0}^{n}\widehat{Q}_{2p+1}=\frac{1}{8}(3\widehat{Q}_{2n+2}+4\widehat{Q}_{2n+1}+\widehat{Q}_{2n}+2\widehat{Q}_{2n-1}-\widehat{Q}_{2n-2}-9-21i-17j-45k)$,

(c) $\sum_{p=0}^{n}\widehat{Q}_{2p}=\frac{1}{8}(-\widehat{Q}_{2n+2}+4\widehat{Q}_{2n+1}+5\widehat{Q}_{2n}+2\widehat{Q}_{2n-1}+3\widehat{Q}_{2n-2}+19-9i-21j-17k)$.

3 Matrices related to generalized Pentanacci quaternions

We define the square matrix B of order 5 as:

$$B = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

such that $\det B = 1$. Induction proof may be used to establish

$$B^n = \begin{pmatrix} P_{n+1} & P_n + P_{n-1} + P_{n-2} + P_{n-3} & P_n + P_{n-1} + P_{n-2} & P_n + P_{n-1} & P_n \\ P_n & P_{n-1} + P_{n-2} + P_{n-3} + P_{n-4} & P_{n-1} + P_{n-2} + P_{n-3} & P_{n-1} + P_{n-2} & P_{n-1} \\ P_{n-1} & P_{n-2} + P_{n-3} + P_{n-4} + P_{n-5} & P_{n-2} + P_{n-3} + P_{n-4} & P_{n-2} + P_{n-3} & P_{n-2} \\ P_{n-2} & P_{n-3} + P_{n-4} + P_{n-5} + P_{n-6} & P_{n-3} + P_{n-4} + P_{n-5} & P_{n-3} + P_{n-4} & P_{n-3} \\ P_{n-3} & P_{n-4} + P_{n-5} + P_{n-6} + P_{n-7} & P_{n-4} + P_{n-5} + P_{n-6} & P_{n-4} + P_{n-5} & P_{n-4} \end{pmatrix} \tag{3.1}$$

and (the matrix formulation of V_n)

$$\begin{pmatrix} V_{n+4} \\ V_{n+3} \\ V_{n+2} \\ V_{n+1} \\ V_n \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} V_4 \\ V_3 \\ V_2 \\ V_1 \\ V_0 \end{pmatrix}. \tag{3.2}$$

Now, we define the matrices B_V as

$$B_V = \begin{pmatrix} \widehat{V}_6 & \widehat{V}_5 + \widehat{V}_4 + \widehat{V}_3 + \widehat{V}_2 & \widehat{V}_5 + \widehat{V}_4 + \widehat{V}_3 & \widehat{V}_5 + \widehat{V}_4 & \widehat{V}_5 \\ \widehat{V}_5 & \widehat{V}_4 + \widehat{V}_3 + \widehat{V}_2 + \widehat{V}_1 & \widehat{V}_4 + \widehat{V}_3 + \widehat{V}_2 & \widehat{V}_4 + \widehat{V}_3 & \widehat{V}_4 \\ \widehat{V}_4 & \widehat{V}_3 + \widehat{V}_2 + \widehat{V}_1 + \widehat{V}_0 & \widehat{V}_3 + \widehat{V}_2 + \widehat{V}_1 & \widehat{V}_3 + \widehat{V}_2 & \widehat{V}_3 \\ \widehat{V}_3 & \widehat{V}_2 + \widehat{V}_1 + \widehat{V}_0 + \widehat{V}_{-1} & \widehat{V}_2 + \widehat{V}_1 + \widehat{V}_0 & \widehat{V}_2 + \widehat{V}_1 & \widehat{V}_2 \\ \widehat{V}_2 & \widehat{V}_1 + \widehat{V}_0 + \widehat{V}_{-1} + \widehat{V}_{-2} & \widehat{V}_1 + \widehat{V}_0 + \widehat{V}_{-1} & \widehat{V}_1 + \widehat{V}_0 & \widehat{V}_1 \end{pmatrix}.$$

This matrice B_V is called generalized Pentanacci quaternion matrix. As special cases, Pentanacci quaternion matrix and Pentanacci-Lucas quaternion matrix are

$$B_P = \begin{pmatrix} \widehat{P}_6 & \widehat{P}_5 + \widehat{P}_4 + \widehat{P}_3 + \widehat{P}_2 & \widehat{P}_5 + \widehat{P}_4 + \widehat{P}_3 & \widehat{P}_5 + \widehat{P}_4 & \widehat{P}_5 \\ \widehat{P}_5 & \widehat{P}_4 + \widehat{P}_3 + \widehat{P}_2 + \widehat{P}_1 & \widehat{P}_4 + \widehat{P}_3 + \widehat{P}_2 & \widehat{P}_4 + \widehat{P}_3 & \widehat{P}_4 \\ \widehat{P}_4 & \widehat{P}_3 + \widehat{P}_2 + \widehat{P}_1 + \widehat{P}_0 & \widehat{P}_3 + \widehat{P}_2 + \widehat{P}_1 & \widehat{P}_3 + \widehat{P}_2 & \widehat{P}_3 \\ \widehat{P}_3 & \widehat{P}_2 + \widehat{P}_1 + \widehat{P}_0 + \widehat{P}_{-1} & \widehat{P}_2 + \widehat{P}_1 + \widehat{P}_0 & \widehat{P}_2 + \widehat{P}_1 & \widehat{P}_2 \\ \widehat{P}_2 & \widehat{P}_1 + \widehat{P}_0 + \widehat{P}_{-1} + \widehat{P}_{-2} & \widehat{P}_1 + \widehat{P}_0 + \widehat{P}_{-1} & \widehat{P}_1 + \widehat{P}_0 & \widehat{P}_1 \end{pmatrix} \text{ and}$$

$$B_Q = \begin{pmatrix} \widehat{Q}_6 & \widehat{Q}_5 + \widehat{Q}_4 + \widehat{Q}_3 + \widehat{Q}_2 & \widehat{Q}_5 + \widehat{Q}_4 + \widehat{Q}_3 & \widehat{Q}_5 + \widehat{Q}_4 & \widehat{Q}_5 \\ \widehat{Q}_5 & \widehat{Q}_4 + \widehat{Q}_3 + \widehat{Q}_2 + \widehat{Q}_1 & \widehat{Q}_4 + \widehat{Q}_3 + \widehat{Q}_2 & \widehat{Q}_4 + \widehat{Q}_3 & \widehat{Q}_4 \\ \widehat{Q}_4 & \widehat{Q}_3 + \widehat{Q}_2 + \widehat{Q}_1 + \widehat{Q}_0 & \widehat{Q}_3 + \widehat{Q}_2 + \widehat{Q}_1 & \widehat{Q}_3 + \widehat{Q}_2 & \widehat{Q}_3 \\ \widehat{Q}_3 & \widehat{Q}_2 + \widehat{Q}_1 + \widehat{Q}_0 + \widehat{Q}_{-1} & \widehat{Q}_2 + \widehat{Q}_1 + \widehat{Q}_0 & \widehat{Q}_2 + \widehat{Q}_1 & \widehat{Q}_2 \\ \widehat{Q}_2 & \widehat{Q}_1 + \widehat{Q}_0 + \widehat{Q}_{-1} + \widehat{Q}_{-2} & \widehat{Q}_1 + \widehat{Q}_0 + \widehat{Q}_{-1} & \widehat{Q}_1 + \widehat{Q}_0 & \widehat{Q}_1 \end{pmatrix},$$

respectively.

Theorem 3.1. For $n \geq 0$, the following is valid:

$$B_V \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}^n \tag{3.3}$$

$$= \begin{pmatrix} \widehat{V}_{n+6} & \widehat{V}_{n+5}+\widehat{V}_{n+4}+\widehat{V}_{n+3}+\widehat{V}_{n+2} & \widehat{V}_{n+5}+\widehat{V}_{n+4}+\widehat{V}_{n+3} & \widehat{V}_{n+5}+\widehat{V}_{n+4} & \widehat{V}_{n+5} \\ \widehat{V}_{n+5} & \widehat{V}_{n+4}+\widehat{V}_{n+3}+\widehat{V}_{n+2}+\widehat{V}_{n+1} & \widehat{V}_{n+4}+\widehat{V}_{n+3}+\widehat{V}_{n+2} & \widehat{V}_{n+4}+\widehat{V}_{n+3} & \widehat{V}_{n+4} \\ \widehat{V}_{n+4} & \widehat{V}_{n+3}+\widehat{V}_{n+2}+\widehat{V}_{n+1}+\widehat{V}_{n+0} & \widehat{V}_{n+3}+\widehat{V}_{n+2}+\widehat{V}_{n+1} & \widehat{V}_{n+3}+\widehat{V}_{n+2} & \widehat{V}_{n+3} \\ \widehat{V}_{n+3} & \widehat{V}_{n+2}+\widehat{V}_{n+1}+\widehat{V}_n+\widehat{V}_{n-1} & \widehat{V}_{n+2}+\widehat{V}_{n+1}+\widehat{V}_n & \widehat{V}_{n+2}+\widehat{V}_{n+1} & \widehat{V}_{n+2} \\ \widehat{V}_{n+2} & \widehat{V}_{n+1}+\widehat{V}_n+\widehat{V}_{n-1}+\widehat{V}_{n-2} & \widehat{V}_{n+1}+\widehat{V}_n+\widehat{V}_{n-1} & \widehat{V}_{n+1}+\widehat{V}_n & \widehat{V}_{n+1} \end{pmatrix}.$$

Proof. We prove by mathematical induction on n. If $n = 0$, then the result is clear. Now, we assume it is true for $n = k$, that is

$$B_V B^k = \begin{pmatrix} \widehat{V}_{k+6} & \widehat{V}_{k+5}+\widehat{V}_{k+4}+\widehat{V}_{k+3}+\widehat{V}_{k+2} & \widehat{V}_{k+5}+\widehat{V}_{k+4}+\widehat{V}_{k+3} & \widehat{V}_{k+5}+\widehat{V}_{k+4} & \widehat{V}_{k+5} \\ \widehat{V}_{k+5} & \widehat{V}_{k+4}+\widehat{V}_{k+3}+\widehat{V}_{k+2}+\widehat{V}_{k+1} & \widehat{V}_{k+4}+\widehat{V}_{k+3}+\widehat{V}_{k+2} & \widehat{V}_{k+4}+\widehat{V}_{k+3} & \widehat{V}_{k+4} \\ \widehat{V}_{k+4} & \widehat{V}_{k+3}+\widehat{V}_{k+2}+\widehat{V}_{k+1}+\widehat{V}_{k+0} & \widehat{V}_{k+3}+\widehat{V}_{k+2}+\widehat{V}_{k+1} & \widehat{V}_{k+3}+\widehat{V}_{k+2} & \widehat{V}_{k+3} \\ \widehat{V}_{k+3} & \widehat{V}_{k+2}+\widehat{V}_{k+1}+\widehat{V}_k+\widehat{V}_{k-1} & \widehat{V}_{k+2}+\widehat{V}_{k+1}+\widehat{V}_k & \widehat{V}_{k+2}+\widehat{V}_{k+1} & \widehat{V}_{k+2} \\ \widehat{V}_{k+2} & \widehat{V}_{k+1}+\widehat{V}_k+\widehat{V}_{k-1}+\widehat{V}_{k-2} & \widehat{V}_{k+1}+\widehat{V}_k+\widehat{V}_{k-1} & \widehat{V}_{k+1}+\widehat{V}_k & \widehat{V}_{k+1} \end{pmatrix}.$$

If we use (2.1), then we have $\widehat{V}_{k+5} = \widehat{V}_{k+4}+\widehat{V}_{k+3}+\widehat{V}_{k+2}+\widehat{V}_{k+1}+\widehat{V}_k$. Then, by induction hypothesis, we obtain

$$\begin{aligned} & B_V B^{k+1} \\ = \; & (B_V B^k) B \\ = \; & \begin{pmatrix} \widehat{V}_{k+6} & \widehat{V}_{k+5}+\widehat{V}_{k+4}+\widehat{V}_{k+3}+\widehat{V}_{k+2} & \widehat{V}_{k+5}+\widehat{V}_{k+4}+\widehat{V}_{k+3} & \widehat{V}_{k+5}+\widehat{V}_{k+4} & \widehat{V}_{k+5} \\ \widehat{V}_{k+5} & \widehat{V}_{k+4}+\widehat{V}_{k+3}+\widehat{V}_{k+2}+\widehat{V}_{k+1} & \widehat{V}_{k+4}+\widehat{V}_{k+3}+\widehat{V}_{k+2} & \widehat{V}_{k+4}+\widehat{V}_{k+3} & \widehat{V}_{k+4} \\ \widehat{V}_{k+4} & \widehat{V}_{k+3}+\widehat{V}_{k+2}+\widehat{V}_{k+1}+\widehat{V}_{k+0} & \widehat{V}_{k+3}+\widehat{V}_{k+2}+\widehat{V}_{k+1} & \widehat{V}_{k+3}+\widehat{V}_{k+2} & \widehat{V}_{k+3} \\ \widehat{V}_{k+3} & \widehat{V}_{k+2}+\widehat{V}_{k+1}+\widehat{V}_k+\widehat{V}_{k-1} & \widehat{V}_{k+2}+\widehat{V}_{k+1}+\widehat{V}_k & \widehat{V}_{k+2}+\widehat{V}_{k+1} & \widehat{V}_{k+2} \\ \widehat{V}_{k+2} & \widehat{V}_{k+1}+\widehat{V}_k+\widehat{V}_{k-1}+\widehat{V}_{k-2} & \widehat{V}_{k+1}+\widehat{V}_k+\widehat{V}_{k-1} & \widehat{V}_{k+1}+\widehat{V}_k & \widehat{V}_{k+1} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \\ = \; & \begin{pmatrix} \widehat{V}_{k+7} & \widehat{V}_{k+6}+\widehat{V}_{k+5}+\widehat{V}_{k+4}+\widehat{V}_{k+3} & \widehat{V}_{k+6}+\widehat{V}_{k+5}+\widehat{V}_{k+4} & \widehat{V}_{k+6}+\widehat{V}_{k+5} & \widehat{V}_{k+6} \\ \widehat{V}_{k+6} & \widehat{V}_{k+5}+\widehat{V}_{k+4}+\widehat{V}_{k+3}+\widehat{V}_{k+2} & \widehat{V}_{k+5}+\widehat{V}_{k+4}+\widehat{V}_{k+3} & \widehat{V}_{k+5}+\widehat{V}_{k+4} & \widehat{V}_{k+5} \\ \widehat{V}_{k+5} & \widehat{V}_{k+4}+\widehat{V}_{k+3}+\widehat{V}_{k+2}+\widehat{V}_{k+1} & \widehat{V}_{k+4}+\widehat{V}_{k+3}+\widehat{V}_{k+2} & \widehat{V}_{k+4}+\widehat{V}_{k+3} & \widehat{V}_{k+4} \\ \widehat{V}_{k+4} & \widehat{V}_{k+3}+\widehat{V}_{k+2}+\widehat{V}_{k+1}+\widehat{V}_k & \widehat{V}_{k+3}+\widehat{V}_{k+2}+\widehat{V}_{k+1} & \widehat{V}_{k+3}+\widehat{V}_{k+2} & \widehat{V}_{k+3} \\ \widehat{V}_{k+3} & \widehat{V}_{k+2}+\widehat{V}_{k+1}+\widehat{V}_k+\widehat{V}_{k-1} & \widehat{V}_{k+2}+\widehat{V}_{k+1}+\widehat{V}_k & \widehat{V}_{k+2}+\widehat{V}_{k+1} & \widehat{V}_{k+2} \end{pmatrix}. \end{aligned}$$

Thus, (3.3) holds for all non-negative integers n.

Corollary 3.2. For $n \geq 0$, the following holds:

$$\widehat{V}_{n+2} = \widehat{V}_2 P_{n+1} + \left(\widehat{V}_1+\widehat{V}_0+\widehat{V}_{-1}+\widehat{V}_{-2}\right) P_n + \left(\widehat{V}_1+\widehat{V}_0+\widehat{V}_{-1}\right) P_{n-1} + \left(\widehat{V}_1+\widehat{V}_0\right) P_{n-2} + \widehat{V}_1 P_{n-3}.$$

Proof. The proof can be seen by the coefficient of the matrix B_V and (3.1).

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