An extension of the operator Kantorovich inequality

Yaser Khatib, Mahmoud Hassani* and Maryam Amyari

Department of Mathematics, Mashhad Branch,Islamic Azad University, Mashhad, Iran
*Corresponding author

E-mail: yaserkhatibam@yahoo.com, mhassanimath@gmail.com, maryam_amyari@yahoo.com

Abstract

In this paper, we present an extension of Kantorovich inequality for two operators on a Hilbert space. Also, the multiple version and a related inequality for positive linear maps are obtained. Moreover, we introduce the concept of Specht's ratio and improve some inequalities related to Specht's ratio.

2000 Mathematics Subject Classification. **47A63**. L05, 47A60.
Keywords. Kantorovich inequality, positive operators, normalized positive linear map, geometric mean of operators.

1 Introduction and preliminaries

The Kantorovich inequality asserts that if A is a positive operator on a Hilbert space $\mathcal{H}$ satisfying $MI_{\mathcal{H}} \geq A \geq mI_{\mathcal{H}}$ for positive scalars M, m, then for every unit vector x in $\mathcal{H}$

$$\langle Ax, x\rangle\langle A^{-1}x, x\rangle \leq \frac{(M+m)^2}{4Mm}. \tag{1.1}$$

Also the Kantorovich type inequality for positive linear maps asserts that if A is a positive operator on a Hilbert space $\mathcal{H}$ satisfying $MI_{\mathcal{H}} \geq A \geq mI_{\mathcal{H}}$ for positive scalars M, m with $m < M$, and φ is a normalized positive linear map from $B(\mathcal{H})$ to $B(\mathcal{K})$, where $\mathcal{H}$, $\mathcal{K}$ are Hilbert spaces, then

$$\varphi(A^{-1}) \leq \frac{(M+m)^2}{4Mm}\varphi(A)^{-1}. \tag{1.2}$$

Several mathematicians have explored Kantorovich inequality for matrices and operators on a Hilbert space, see [1, 4, 6, 7, 9, 10]. This note intends to present an extension of inequalities (1.1) and (1.2). At first, let us recall some definitions and concepts from [8].

Suppose that $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ is a complex Hilbert space and $B(\mathcal{H})$ denotes the Banach algebra of all bounded linear operators on $\mathcal{H}$. An operator A in $B(\mathcal{H})$ is positive, denoted by $A \geq 0$, if A is self-adjoint and $\langle Ax, x\rangle \geq 0$ for every $x \in \mathcal{H}$; equivalently A is positive if and only if $A = B^*B$ for some operator $B \in B(\mathcal{H})$. In particular, for some scalars m and M, we write $m1_{\mathcal{H}} \leq A \leq M1_{\mathcal{H}}$ if $m \leq \langle Ax, x\rangle \leq M$ for every unit vector $x \in \mathcal{H}$. Note that for a self-adjoint operator A, $m1_{\mathcal{H}} \leq A \leq M1_{\mathcal{H}}$ if and only if $sp(A) \subset [m, M]$.

Tbilisi Mathematical Journal 13(4) (2020), pp. 183–191.
Tbilisi Centre for Mathematical Sciences.

Received by the editors: 09 December 2019 .
Accepted for publication: 17 October 2020.

A self-adjoint operator $A \in B(\mathcal{H})$ is strictly positive, denoted by $A > 0$, if there is a real number $m > 0$ such that $A \geq m1_{\mathcal{H}}$.
Let A be a strictly positive operator and B be a positive operator in $B(\mathcal{H})$. The operator ν-weighted geometric mean of A and B for $\nu \in [0, 1]$ is defined by

$$A\natural_\nu B \equiv A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\nu A^{\frac{1}{2}}.$$

Recall that a linear map $\varphi : B(\mathcal{H}) \to B(\mathcal{K})$ is positive, if preserves positivity. It is normalized if $\varphi(I_{\mathcal{H}}) = I_{\mathcal{K}}$ and is called multiplicative if $\varphi(AB) = \varphi(A)\varphi(B)$. The aim of this paper is to describe some new inequalities for operators in Hilbert space $\mathcal{H}$. We improve and generalize these inequalities by Specht's ratio concept.
Firstly, we state the Specht's ratio concept and some properties. The Specht's ratio [2, 11] was defined by

$$S(h) = \frac{h^{\frac{1}{h-1}}}{e \log h^{\frac{1}{h-1}}}, (h \neq 1 \;\; h > 0.) \tag{1.3}$$

The Specht's ratio has some properties in the following:

(i) $S(1) = 1$ and $S(h) = S(\frac{1}{h}) > 1$, for $h > 0$.

(ii) $S(h)$ is a monotone increasing function on $(1, \infty)$.

(iii) $S(h)$ is a monotone decreasing function on $(0, 1)$.

The classical Young inequality states that $(1-\nu)a + \nu b \geq a^{1-\nu}b^\nu$ for positive numbers a, b and $\nu \in [0, 1]$. The following lemma improves the classical Young inequality by Specht's ratio.

Lemma 1.1. [3, Theorem 1] For $a, b > 0$ and $\nu \in [0, 1]$,
$(1-\nu)a + \nu b \geq S((\frac{b}{a})^r)a^{1-\nu}b^\nu$,
where $r = \min\{\nu, 1-\nu\}$ and $S(.)$ is the Specht's ratio.

By applying the above lemma for positive operators, Furuichi [3] proved the following theorem.

Theorem 1.2. [3, Theorem 2] Let A and B be two positive operators and m, m', M, M' be positive real numbers satisfying one of the following conditions:

(i) $0 < m'I \leq A \leq mI < MI \leq B \leq M'I$.

(ii) $0 < m'I \leq B \leq mI < MI \leq A \leq M'I$,

with $h \equiv \frac{M}{m}$ and $h' \equiv \frac{M'}{m'}$, then we have

$$\begin{aligned}(1-\nu)A+\nu B &\geq S(h^r)A\natural_\nu B\\ &\geq A\natural_\nu B\\ &\geq S(h^r)(1-\nu)A^{-1}+\nu B^{-1}\\ &\geq (1-\nu)A^{-1}+\nu B^{-1}.\end{aligned}$$

where $\nu \in [0,1]$, $r = \min\{\nu, 1-\nu\}$, $S(.)$ is the Spech's ratio.

In the following we introduce an interesting corollary which leads to improving some inequalities that we get in this paper.

Corollary 1.3. Suppose that a, b and m, M, m', M' are positive real numbers satisfying one of the following conditions:

(i) $0 < m' \leq a \leq m < M \leq b \leq M'$,

(ii) $0 < m' \leq b \leq m < M \leq a \leq M'$.

If $h \equiv \frac{M}{m}$ and $h' \equiv \frac{M'}{m'}$,
then

$$S(\sqrt{h})\sqrt{ab} \leq \tfrac{a+b}{2}.$$

Proof. Put $A = a, B = b, \nu = \frac{1}{2}, r = \frac{1}{2}$, in Theorem 1.2. We get the desired result. Q.E.D.

2 A type of Kantorovich inequality

In this section, we present a type of Kantorovich inequality for two operators and improve them with Specht's ratio and Corollary 1.3.

In the following theorem we introduce an inequality by new conditions for two operators and we try improve it. Indeed, since in Corollary 1.3, we get an improvement for arithmetic-geometric mean by Specht's ratio, then we can use it for improvement inequalities in the following theorems.

Theorem 2.1. Let A and B be two positive operators on a Hilbert space $\mathcal{H}$, satisfying $A \leq aI_{\mathcal{H}}, B \leq bI_{\mathcal{H}}$, $AB = BA$ and $AB \leq I_{\mathcal{H}}$ for positive scalars a, b. If φ is a normalized positive linear map from $B(\mathcal{H})$ to $B(\mathcal{K})$, and m, m', M, M' are positive real numbers satisfying one of the following conditions:

(i) $0 < m'I \leq bA \leq mI < MI \leq aB \leq M'I$.

(ii) $0 < m'I \leq aB \leq mI < MI \leq bA \leq M'I$,

with $h \equiv \frac{M}{m}$ and $h' \equiv \frac{M'}{m'}$, then for every unit vector x in $\mathcal{H}$

$$\langle \varphi(A)x, x\rangle\langle \varphi(B)x, x\rangle \leq \frac{(ab+1)^2}{4S^2(\sqrt{h})ab}. \tag{2.1}$$

and

$$\langle Ax, x\rangle\langle Bx, x\rangle \leq \frac{(ab+1)^2}{4S^2(\sqrt{h})ab}. \tag{2.2}$$

Proof. We know $AB = BA$. Thus $(A - aI_{\mathcal{H}})(B - bI_{\mathcal{H}}) \geq 0$, or equivalently $AB + abI_{\mathcal{H}} \geq bA + aB$, and so $\varphi(AB) + abI_{\mathcal{H}} \geq b\varphi(A) + a\varphi(B)$, this implies that

$$1 + ab \geq \langle \varphi(AB)x, x\rangle + ab \geq b\langle \varphi(A)x, x\rangle + a\langle \varphi(B)x, x\rangle \tag{2.3}$$

for every unit vector x in $\mathcal{H}$. By using Corollary 1.3, we get the following inequlality.

$$b\langle \varphi(A)x, x\rangle + a\langle \varphi(B)x, x\rangle \geq 2S(\sqrt{h})(ab\langle \varphi(A)x, x\rangle\langle \varphi(B)x, x\rangle)^{\frac{1}{2}}. \tag{2.4}$$

Inequalities (2.3) and (2.4), imply that

$$\langle \varphi(A)x, x\rangle\langle \varphi(B)x, x\rangle \leq \tfrac{(ab+1)^2}{4S^2(\sqrt{h})ab}.$$

Taking $\varphi = I_{\mathcal{H}}$ in inequality (2.1) leads to inequality (2.2).

Put $B = A^{-1}$ in inequality (2.2), then we get an improvement for Kantorovich inequality. Q.E.D.

Example 2.2. Let

$$A = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{2}{3} \end{pmatrix} \text{ and } B = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{3}{4} \end{pmatrix}.$$

Therefore $A \leq 2I$, $B \leq 6I$, $AB = BA$ and $AB \leq I$. In this case the condition (ii) of Theorem 2.1 satisfy. Indeed we have

$$0 < m'I \leq 2B \leq 2I < 3I \leq 6A \leq M'I$$

$$\Rightarrow 0 < m'I \leq \begin{pmatrix} \frac{2}{3} & 0 \\ 0 & \frac{3}{2} \end{pmatrix} \leq 2I < 3I \leq \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix} \leq M'I$$

with $h \equiv \frac{M}{m} = \frac{3}{2}$. Then by equality 1.3, we have $S^2(\sqrt{h}) = 1/955967187$. Moreover, for unit vector $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, we have $\langle Ax, x\rangle = \frac{1}{2}|x_1|^2 + \frac{2}{3}|x_2|^2$.
Since $\|x\|^2 = \langle x, x\rangle = |x_1|^2 + |x_2|^2$, then

$$\tfrac{1}{2} = \min\{\tfrac{1}{2}, \tfrac{2}{3}\} \leq \langle Ax, x\rangle \leq \max\{\tfrac{1}{2}, \tfrac{2}{3}\} = \tfrac{2}{3}.$$

Similarly, we can write

$$\tfrac{1}{3} = \min\{\tfrac{1}{3}, \tfrac{3}{4}\} \leq \langle Bx, x\rangle \leq \max\{\tfrac{1}{3}, \tfrac{3}{4}\} = \tfrac{3}{4}.$$

Therefore $\langle Ax, x\rangle\langle Bx, x\rangle \leq \frac{1}{2}$. On the other hand, $\frac{(ab+1)^2}{4S^2(\sqrt{h})ab} = 1/8000472384$. Then we get

$$\langle Ax, x\rangle\langle Bx, x\rangle \leq \tfrac{(ab+1)^2}{4S^2(\sqrt{h})ab}.$$

Hence, we find two matrices that satisfy in conditions of inequality (2.2) in Theorem 2.1.

The following theorem is a multiple version of Theorem 2.1.

Theorem 2.3. Let A_i and B_i be positive operators on a Hilbert space $\mathcal{H}$, for $i = 1, 2, ..., n$, satisfying $A_i \leq aI_{\mathcal{H}}, B_i \leq bI_{\mathcal{H}}$, $A_iB_i = B_iA_i$ and $A_iB_i \leq I_{\mathcal{H}}$, for positive scalars a, b and for $i = 1, 2, ..., n$. If φ is a normalized positive linear map from $B(\mathcal{H})$ to $B(\mathcal{K})$, and m, m', M, M' are positive real numbers satisfying one of the following conditions:

(i) $0 < m'I \leq bA_i \leq mI < MI \leq aB_i \leq M'I$.

(ii) $0 < m'I \leq aB_i \leq mI < MI \leq bA_i \leq M'I$

with $h \equiv \frac{M}{m}$ and $h' \equiv \frac{M'}{m'}$, then for every unit vector x in $\mathcal{H}$

$$(\sum_{i=1}^{n} \langle\varphi(A_i)x_i, x_i\rangle)(\sum_{i=1}^{n} \langle\varphi(B_i)x_i, x_i\rangle) \leq \frac{(ab+1)^2}{4S^2(\sqrt{h})ab}, \tag{2.5}$$

and

$$(\sum_{i=1}^{n} \langle A_ix_i, x_i\rangle)(\sum_{i=1}^{n} \langle B_ix_i, x_i\rangle) \leq \frac{(ab+1)^2}{4S^2(\sqrt{h})ab}. \tag{2.6}$$

Proof. Put $\tilde{A} = \begin{pmatrix} A_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & A_n \end{pmatrix}, \tilde{B} = \begin{pmatrix} B_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & B_n \end{pmatrix}$ and

$$\tilde{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \tilde{I_{\mathcal{H}}} = \begin{pmatrix} I_{\mathcal{H}} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & I_{\mathcal{H}} \end{pmatrix}.$$

By assumption, we have $\tilde{A} \leq a\tilde{I_{\mathcal{H}}}$ and $\tilde{B} \leq b\tilde{I_{\mathcal{H}}}$. Also $\tilde{A}\tilde{B} \leq \tilde{I_{\mathcal{H}}}$, since

$$\tilde{A}\tilde{B} = \begin{pmatrix} A_1B_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & A_nB_n \end{pmatrix} \leq \begin{pmatrix} I_{\mathcal{H}} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & I_{\mathcal{H}} \end{pmatrix} = \tilde{I_{\mathcal{H}}}.$$

Now, properties of Theorem 2.1 hold and we can write

$$\textstyle\sum_{i=1}^{n} \langle \varphi(A_i)x_i, x_i\rangle = \langle \varphi(\tilde{A})\tilde{x}, \tilde{x}\rangle, \sum_{i=1}^{n} \langle \varphi(B_i)x_i, x_i\rangle = \langle \varphi(\tilde{B})\tilde{x}, \tilde{x}\rangle,$$

for each $i \in \mathbb{N}$. Note that, similarly $\tilde{A}$, $\tilde{B}$ satisfy in conditions (i) or (ii) as follows:

(i) $0 < m'\tilde{I} \leq b\tilde{A} \leq m\tilde{I} < M\tilde{I} \leq a\tilde{B} \leq M'\tilde{I}$.

(ii) $0 < m'\tilde{I} \leq b\tilde{A} \leq m\tilde{I} < M\tilde{I} \leq a\tilde{B} \leq M'\tilde{I}$.

Hence, by using Corollary 1.3 and the proof of Theorem 2.1, we have the desired inequality as follows:

$$\Big(\sum_{i=1}^{n} \langle \varphi(A_i)x_i, x_i\rangle\Big)\Big(\sum_{i=1}^{n} \langle \varphi(B_i)x_i, x_i\rangle\Big) \leq \frac{(ab+1)^2}{4S^2(\sqrt{h})ab}. \tag{2.7}$$

We can reach inequality (2.6), by taking $\varphi = I_{\mathcal{H}}$ in inequality (2.7). Q.E.D.

Next, we present a kind of Kantorovich inequality for positive linear maps.

Theorem 2.4. Let A and B be two positive operators on $\mathcal{H}$ satisfying $0 < A \leq aI_{\mathcal{H}},\ 0 < B \leq bI_{\mathcal{H}}$, $AB = BA$ and $AB \leq I_{\mathcal{H}}$ for positive scalars a, b. If φ is a normalized positive linear map from $B(\mathcal{H})$ to $B(\mathcal{K})$, and m, m', M, M' are positive real numbers satisfying one of the following conditions:

(i) $0 < m'I \leq A \leq mI < MI \leq B \leq M'I$.

(ii) $0 < m'I \leq B \leq mI < MI \leq A \leq M'I$,

with $h \equiv \frac{M}{m}$ and $h' \equiv \frac{M'}{m'}$, then

$$\varphi(A) \natural \varphi(B) \leq \tfrac{ab+1}{2S(\sqrt{h})(ab)^{\frac{1}{2}}}.$$

Proof. By assumption, we conclude that $AB + abI_H \geq bA + aB$, applying φ on the both sides, we get $\varphi(AB) + ab\varphi(I_{\mathcal{H}}) \geq b\varphi(A) + a\varphi(B)$. Since φ is normalized, then $\varphi(AB) + abI_{\mathcal{K}} \geq b\varphi(A) + a\varphi(B)$. Moreover, by using Theorem 1.2 we get

$$1 + ab \geq 2S(\sqrt{h})(ab)^{\frac{1}{2}}\ \varphi(A) \natural \varphi(B)$$

and so $\varphi(A) \natural \varphi(B) \leq \frac{ab+1}{2S(\sqrt{h})(ab)^{\frac{1}{2}}}$. Q.E.D.

Corollary 2.5. Suppose that the conditions of Theorem 2.4 hold and U_i, $1 \leq i \leq n$ is a contraction on $\mathcal{H}$, with $\sum_{i=1}^{n} U_i{}^* U_i = I_{\mathcal{H}}$. Then

$$\left(\sum_{i=1}^{n} U_i{}^* \varphi(A) U_i\right) \natural \left(\sum_{i=1}^{n} U_i{}^* \varphi(B) U_i\right) \leq \frac{ab+1}{2S(\sqrt{h})(ab)^{\frac{1}{2}}}, \tag{2.8}$$

and

$$\left(\sum_{i=1}^{n} U_i{}^* A U_i\right) \natural \left(\sum_{i=1}^{n} U_i{}^* B U_i\right) \leq \frac{ab+1}{2S(\sqrt{h})(ab)^{\frac{1}{2}}}. \tag{2.9}$$

Proof. Take $\varphi'(X) = \sum_{i=1}^{n} U_i{}^* \varphi(X) U_i$, for normalized positive linear map φ, so Theorem 2.4 implies inequality (2.8). Putting $\varphi = I_{\mathcal{H}}$ in inequality (2.8) leads to inequality (2.9). Q.E.D.

Theorem 2.6. Let A and B be two positive operators on a Hilbert space $\mathcal{H}$, satisfying $A \leq aI_{\mathcal{H}}$, $B \leq bI_{\mathcal{H}}$, $AB = BA$ and $AB \leq I_{\mathcal{H}}$ for positive scalars a, b. If φ is a normalized multiplicative positive linear map from $B(\mathcal{H})$ to $B(\mathcal{K})$, and m, m', M, M' are positive real numbers satisfying one of the following conditions:

(i) $0 < m'I \leq A \leq mI < MI \leq B \leq M'I$.

(ii) $0 < m'I \leq B \leq mI < MI \leq A \leq M'I$,

with $h \equiv \frac{M}{m}$ and $h' \equiv \frac{M'}{m'}$, then

$$\langle \varphi(A^2)x, x\rangle \langle \varphi(A)\varphi(B)x, x\rangle \leq \frac{(ab+1)^2}{4S^2(\sqrt{h})ab} \langle \varphi(A)x, x\rangle^2, \tag{2.10}$$

and

$$\langle A^2 x, x\rangle \langle ABx, x\rangle \leq \frac{(ab+1)^2}{4S^2(\sqrt{h})ab} \langle Ax, x\rangle^2. \tag{2.11}$$

Proof. Note that $\varphi(B)$ and $(\varphi(A))^{\frac{1}{2}}$ commute. Taking $y = \frac{(\varphi(A))^{\frac{1}{2}} x}{\|(\varphi(A))^{\frac{1}{2}} x\|}$ for a unit vector x in $\mathcal{H}$ in inequality (2.1), we get

$$\frac{\langle \varphi(A)(\varphi(A))^{\frac{1}{2}} x, (\varphi(A))^{\frac{1}{2}} x\rangle}{\|(\varphi(A))^{\frac{1}{2}} x\|^2} \frac{\langle \varphi(B)(\varphi(A))^{\frac{1}{2}} x, (\varphi(A))^{\frac{1}{2}} x\rangle}{\|(\varphi(A))^{\frac{1}{2}} x\|^2} \leq \frac{(ab+1)^2}{4S^2(\sqrt{h})ab}$$

and so

$$\langle \varphi(A^2)x, x\rangle \langle \varphi(A)\varphi(B)x, x\rangle \leq \frac{(ab+1)^2}{4S^2(\sqrt{h})ab} \langle \varphi(A)x, x\rangle^2. \tag{2.12}$$

Also, by putting $\varphi = I_{\mathcal{H}}$ in inequality (2.12), we reach inequality (2.11). Q.E.D.

Theorem 2.7. Let A and B be two positive operators on a Hilbert space $\mathcal{H}$, satisfying $A \leq aI_{\mathcal{H}}$, $B \leq bI_{\mathcal{H}}$, $AB = BA$ and $AB \leq I_{\mathcal{H}}$ for positive scalars a, b and A is invertible. If φ is a normalized positive linear map from $B(\mathcal{H})$ to $B(\mathcal{K})$, then for every unit vector x in $\mathcal{H}$

$$\varphi(B) - \varphi(A)^{-1} \leq (\sqrt{b} - \tfrac{1}{\sqrt{a}})^2 I_{\mathcal{K}}.$$

Proof. From Theorem 2.1, we have

$$1 + ab \geq \langle \varphi(AB)x, x\rangle + ab \geq b\langle \varphi(A)x, x\rangle + a\langle \varphi(B)x, x\rangle.$$

Thus

$$\langle \varphi(B)x, x\rangle \leq \frac{1+ab}{a} - \frac{b}{a}\langle \varphi(A)x, x\rangle$$

for every unit vector x in $\mathcal{H}$.

$$\langle \varphi(B)x, x\rangle - \langle \varphi(A)x, x\rangle^{-1} \leq \tfrac{1+ab}{a} - \tfrac{b}{a}\langle \varphi(A)x, x\rangle - \langle \varphi(A)x, x\rangle^{-1} = (\sqrt{b} - \tfrac{1}{\sqrt{a}})^2 - (\tfrac{\sqrt{b}}{\sqrt{a}}\langle \varphi(A)x, x\rangle^{\frac{1}{2}} - \langle \varphi(A)x, x\rangle^{-\frac{1}{2}})^2 \leq (\sqrt{b} - \tfrac{1}{\sqrt{a}})^2$$

and so, we have

$$\langle \varphi(B)x, x\rangle - \langle \varphi(A)x, x\rangle^{-1} \leq (\sqrt{b} - \frac{1}{\sqrt{a}})^2. \tag{2.13}$$

Hence by using Holder–McCarthy inequality [8, Theorem 1.4], we obtain

$$\langle \varphi(A)^{-1}x, x\rangle \geq \langle \varphi(A)x, x\rangle^{-1} \text{ then } \langle \varphi(B)x, x\rangle - \langle \varphi(A)^{-1}x, x\rangle \leq \langle \varphi(B)x, x\rangle - \langle \varphi(A)x, x\rangle^{-1}$$

then, by inequality (2.13)

$$\langle \varphi(B)x, x\rangle - \langle \varphi(A)^{-1}x, x\rangle \leq (\sqrt{b} - \tfrac{1}{\sqrt{a}})^2,$$

and so,

$$\varphi(B) - \varphi(A)^{-1} \leq (\sqrt{b} - \tfrac{1}{\sqrt{a}})^2 I_{\mathcal{K}}.$$

Q.E.D.

Corollary 2.8. Let A and B be two positive operators on a Hilbert space $\mathcal{H}$, satisfying $A \leq aI_{\mathcal{H}}$, $B \leq bI_{\mathcal{H}}$, $AB = BA$ and $AB \leq I_{\mathcal{H}}$ for positive scalars a, b and A is invertible. Then for every unit vector x in $\mathcal{H}$

$$\langle Bx, x\rangle - \langle Ax, x\rangle^{-1} \leq (\sqrt{b} - \frac{1}{\sqrt{a}})^2.$$

Proof. Take $\varphi = I_{\mathcal{H}}$ in inequality (2.13). We deduce the desired inequality. Q.E.D.

Acknowledgements
The authors would like to thank the referees for their useful comments.

References

[1] J.K. Baksalary and S. Puntanem, *Generalized matrix versions of the Cauchy Schwartz and Kantorovich inequalities*, Aequationes Math. **41** (1991), 103–110.

[2] J.I. Fujii, S. Izumino and Y. Seo, *Determinant for positive operators and Specht's theorem*, Sci. Math. **1** (1998), no. 3, 307–310.

[3] S. Furuichi, *Refined Young inequalities with Specht's ratio*, J. Egyptian Math. Soc. **20** (2012), no. 1, 46–49.

[4] W. Greub and W. Rheinboldt, *On a generalistion of an inequality of L.V. Kantorovich*, Proc. Amer. Math. Soc **10** (1959), 407–415.

[5] F. Kubo and T. Ando, *Means of positive operators*, Math. Ann. **264** (1980) 205–224.

[6] A.W. Marshall and I. Olkin, *Matrix versions of Cauchy and Kantorovich inequalities*, Aequationes Math. **40** (1990), 89–93.

[7] M. Newman, *Kantorovich's inequality*, J. Res. Nat. Bur. Standards Sect. B, **64B** (1960), 33–34.

[8] J. Peč arić, T. Furuta, J. Mić ić Hot and Y. Seo, *Mond-Peč arić Method in Operator Inequalities*, Element, Zagreb, (2005).

[9] W.G. Strang, *On the Kantorovich inequality*, Proc. Amer. Math. Soc **11** (1960), no 3, 468.

[10] P.G. Spain, *Operator versions of the Kantorovich inequality*, Proc. Amer. Math. Soc. **124** (1996), 2813–2819.

[11] W. Specht, *Zur Theorie der elementaren Mittel.* (German) Math. Z **74** (1960), 91–98.