Existence of solutions for infinite systems of regular fractional Sturm-Liouville problems in the spaces of tempered sequences

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Abstract

In this paper we investigate the existence of solutions for infinite systems of regular fractional Sturm-Liouville problems by an application of Meir-Keeler fixed point theorem in the tempered sequence spaces. We provide examples to check validity of our obtained results.

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1 Introduction

Fixed point theory is one of the most important area of research in Mathematics. In recent years, many results related to fixed point theorems in ordered metric spaces are established in [2, 5, 6, 36] and references therein. The results in this line was obtained by Ran and Reurings [40]. Subsequently, Nieto and Rodriguez-Lopez [35] extended the results by omitting the continuity hypothesis and applied their result to obtain a unique solution to a first order ordinary differential equation with periodic boundary conditions. Recently, Debnath et al.[13] introduced a new and proper extension of Kannans fixed point theorem to the case of multivalued maps using Wardowski's F-contraction and applied to a particular type of Volterra integral equation to find the existence of solutions.

In the recent years, the study of differential equations of fractional order has been in the limelight by many researchers in the areas of applied sciences, such as engineering, physics, biology and economics. This is basically because, it finds applications in several real world problems. For details on the theory and some applications of fractional differential equations, see the monographs of [22, 31, 37, 46]. In the qualitative theory of (classical and fractional) differential equations, various theorems have been extensively deployed by researchers in establishing the existence and uniqueness of solutions to both the initial and boundary value problems. For a detailed study on the existence and uniqueness of solutions of fractional equations, [1, 16, 18, 21, 29, 38, 39], and the references therein. In [41], Srivastava et al. studied the hybrid fixed point theorems of Krasnoselskii type, which involve product of two operators in partially ordered normed linear spaces and applied to fractional integral equations for establishing the existence of solutions under certain monotonicity conditions blending with the existence of the upper or lower solution.

In functional analysis the measure of noncompactness play important role which was introduced by Kuratowski [28]. Recently, Srivastava et al. [43] studied the solvability of non-linear functional integral equations of two variables by using the measure of noncompactness on $C([0,a]\times[0,a])$ and a fixed point theorem. Also, the idea of measure of noncompactness has been used by many

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researchers in obtaining the existence of solutions of infinite systems of integral and differential equations, see the monograph of [10]. In the theories of neural nets, branching processes and mechanics, infinite systems of ordinary differential equations describes many real life problems [11, 14]. In [32], Mursaleen and Muhiuddine established existence theorems for the infinite systems of differential equations in the space ℓ_p. Alotaibi et al. [4] existence theorems for the infinite systems of linear equations in ℓ_1 and ℓ_p was discussed. In [20], Hazarika et al. established the existence of solution for infinite system of nonlinear integral equations in the Banach spaces ℓ_p with the help of a technique associated with measure of noncompactness and generalized Meir-Keeler fixed point theorem. In [42] Srivastava et al. studied the existence of solutions of infinite systems of n^{th} order differential equations in the spaces c_0 and ℓ_1 via the measure of noncompactness. On the other hand, Mursaleen et al. [33] considered the following infinite systems of three-point fractional order boundary value problems in the spaces c_0 and ℓ_p,

$$\begin{cases} \mathfrak{D}^{\alpha}_{0^+}(\varpi_k(t)) + f_k(t, \varpi_k(t)), \quad t \in (0, \mathfrak{T}), \quad \alpha \in (1, 2), \\ \varpi_k(0) = 0, \ \varpi_k(\mathfrak{T}) = a\varpi_k(\xi), \quad a\xi^{\alpha-1} < \mathfrak{T}^{\alpha-1}, \quad k = 1, 2, 3, \cdots, \end{cases} \tag{1.1}$$

and derived necessary conditions for the existence of solutions for the infinite system (1.1). More recently, Das et al. [12] studied the infinite system of fractional order two-point boundary value problem

$$\begin{cases} \mathfrak{D}^{\alpha}_{0^+}(\varpi_k(t)) + f_k(t, \varpi_k(t)), \quad t \in (0, \mathfrak{T}), \quad \alpha \in (1, 2), \\ \varpi_k(0) = 0, \ \varpi_k(\mathfrak{T}) = 0, \quad k = 1, 2, 3, \cdots, \end{cases} \tag{1.2}$$

via Hausdorff measure of noncompactness in the tempered sequence spaces and established existence of solutions for the infinite system (1.2).

The Sturm-Liouville Problems(SLPs) and many of the associated theories were introduced over 170 years ago, and since then the resulting fields have played significant roles in many areas of science, engineering, and mathematics [45]. In [45], Zettl points out that there are thousands of papers have been written in this field since the early work of Sturm and Liouville on SLPs, yet this field remains very active, and dozens of papers are published on SLPs every year. In many engineering applications the Sturm-Liouville Problems (SLPs) arise as boundary value problems. In physics, solution of SLPs provide energy state and energy specturm of systems, for example hydrogen atoms. It is well-known that many different integral transforms can be deduced by carrying out a spectral expansion with respect to the eigenfunctions of the corresponding second-order Sturm-Liouville differential operators, see Srivastava et al.[44] and references therein. The Sturm-Liouville Theory (SLT) has been the basis for the development of fields such as spectral theory of differential operators, functional space and modern analysis,and theory of self-adjoint operators [7]. Many other differential equations such as Bessel, Hermite, Jacobi and Legendre equations can be transformed into Sturm-Liouville equations. Because of the importance of this field, several books have been written on the subject, and it is taught on a regular basis in engineering, physics, and mathematics [27, 17]. The recent works on fractional order Sturm-Liouville boundary value problems. see [19, 23, 24, 25, 26] and references therein.

Motivated by the aforementioned works in this paper, we considered an infinite system of regular fractional Sturm-Liouville two-point boundary value problems,

$$\begin{cases} {}^{\mathcal{C}}\mathfrak{D}^{\varrho}_{\mathfrak{T}^-}\left[p_k(t){}^{\mathcal{C}}\mathfrak{D}^{\varrho}_{0^+}(\varpi_k(t))\right] = \lambda_k f_k(t, \varpi(t)), \quad t \in (0, \mathfrak{T}), \quad \varrho \in (0, 1), \\ \varpi_k(0) = 0, \ {}^{\mathcal{C}}\mathfrak{D}^{\varrho}_{0^+}\varpi_k(\mathfrak{T}) = 0, \quad k = 1, 2, 3, \cdots, \end{cases} \tag{1.3}$$

where ${}^{\mathcal{C}}\mathfrak{D}^{\varrho}_{\mathfrak{T}-}$ and ${}^{\mathcal{C}}\mathfrak{D}^{\varrho}_{0+}$ denotes left and the right Liouville-Caputo fractional derivatives of order ϱ, respectively, $\varpi(t) = (\varpi_k(t))_{k=1}^{\infty} \in \mathbb{R}^{\infty}$, $f_k : [0, +\infty) \to [0, +\infty)$ are continuous, and p_k's are arbitrary positive functions in $C([0, \mathfrak{T}])$ and we establish necessary conditions for the existence of solutions for the system via the Hausdorff measure of noncompactness and MeirKeeler fixed point theorem in the tempered sequence spaces.

2 Kernel and its bounds

In this section, we present some definitions and preliminary lemmas that will be used in the proofs of the main results in the next section.

Definition 2.1 ([22]). Let $\mathfrak{g} : (0, \infty) \to \mathbb{R}$ be a function and $\sigma > 0$. Then the left Riemann-Liouville(RL) fractional integral of order σ of $\mathfrak{g}$ is defined by

$$\mathfrak{I}^{\sigma}_{0+}\mathfrak{g}(t) = \int_0^t \frac{(t-\tau)^{\sigma-1}}{\Gamma(\sigma)}\mathfrak{g}(\tau)d\tau, \quad t \in [0, \mathfrak{T}], \quad \mathfrak{T} > 0,$$

and the right RL fractional integral of order σ of $\mathfrak{g}$ is defined by

$$\mathfrak{I}^{\sigma}_{\mathfrak{T}-}\mathfrak{g}(t) = \int_t^{\mathfrak{T}} \frac{(\tau-t)^{\sigma-1}}{\Gamma(\sigma)}\mathfrak{g}(\tau)d\tau, \quad t \in [0, \mathfrak{T}], \quad \mathfrak{T} > 0.$$

Definition 2.2 ([22]). Let $\mathfrak{g} : (0, \infty) \to \mathbb{R}$ be a function and $0 < \sigma < 1$. The left and right Riemann-Liouville fractional derivatives of order σ of $\mathfrak{g}$ is defined by

$$\mathfrak{D}^{\sigma}_{0+}\mathfrak{g}(t) = \mathfrak{D}\mathfrak{I}^{1-\sigma}_{0+}\mathfrak{g}(t), \quad t \in (0, \mathfrak{T}], \quad \mathfrak{T} > 0,$$

$$\mathfrak{D}^{\sigma}_{\mathfrak{T}-}\mathfrak{g}(t) = -\mathfrak{D}\mathfrak{I}^{1-\sigma}_{\mathfrak{T}-}\mathfrak{g}(t), \quad t \in (0, \mathfrak{T}], \quad \mathfrak{T} > 0.$$

Definition 2.3 ([22]). Let $\mathfrak{g} \in AC([0, \mathfrak{T}])$ and $0 < \sigma < 1$. The left and right Liouville-Caputo fractional derivatives of order σ of $\mathfrak{g}$ is defined by

$${}^{\mathcal{C}}\mathfrak{D}^{\sigma}_{0+}\mathfrak{g}(t) = \mathfrak{D}^{\sigma}_{0+}[\mathfrak{g}(t) - \mathfrak{g}(0)], \quad t \in (0, \mathfrak{T}], \quad \mathfrak{T} > 0,$$

$${}^{\mathcal{C}}\mathfrak{D}^{\sigma}_{\mathfrak{T}-}\mathfrak{g}(t) = \mathfrak{D}^{\sigma}_{\mathfrak{T}-}[\mathfrak{g}(t) - \mathfrak{g}(\mathfrak{T})], \quad t \in [0, \mathfrak{T}), \quad \mathfrak{T} > 0.$$

Also satisfies the relations,

$${}^{\mathcal{C}}\mathfrak{D}^{\sigma}_{0+}\mathfrak{g}(t) = \mathfrak{I}^{1-\sigma}_{0+}\mathfrak{D}\mathfrak{g}(t), \quad t \in (0, \mathfrak{T}], \quad \mathfrak{T} > 0,$$

$${}^{\mathcal{C}}\mathfrak{D}^{\sigma}_{\mathfrak{T}-}\mathfrak{g}(t) = -\mathfrak{I}^{1-\sigma}_{\mathfrak{T}-}\mathfrak{D}\mathfrak{g}(t), \quad t \in (0, \mathfrak{T}], \quad \mathfrak{T} > 0.$$

Lemma 2.4. $\varpi(t)$ is a solution of (1.3) if and only if $\varpi(t)$ is a solution of the integral equation

$$\varpi_k(t) = \lambda_k \int_0^{\mathfrak{T}} \aleph_k(t, \tau) f_k(\tau, \varpi(\tau))d\tau, \tag{2.1}$$

where the kernel $\aleph(t, \tau)$ is given by

$$\aleph_k(t, \tau) = \frac{1}{\Gamma^2(\varrho)} \begin{cases} \aleph_{k_1}(t, \tau) := \int_0^{\tau} \frac{1}{p_k(\xi)}[(t-\xi)^{\varrho-1}(\tau-\xi)^{\varrho-1}]d\xi, & t \le \tau, \\ \aleph_{k_2}(t, \tau) := \int_0^{t} \frac{1}{p_k(\xi)}[(t-\xi)^{\varrho-1}(\tau-\xi)^{\varrho-1}]d\xi, & \tau < t. \end{cases}$$

Proof. Suppose ϖ_k is a solution of (1.3), then we have

$$\begin{aligned} p_k(t)^{\mathcal{C}}\mathfrak{D}^{\varrho}_{0^+}(\varpi_k(t)) &= \lambda_k \int_t^{\mathfrak{T}} \frac{(\tau-t)^{\varrho-1}}{\Gamma(\varrho)} f_k(\tau,\varpi(\tau))d\tau + p_k(\mathfrak{T})\mathfrak{D}^{\varrho}_{0^+}(\varpi_k(\mathfrak{T})) \\ &= \lambda_k \int_t^{\mathfrak{T}} \frac{(\tau-t)^{\varrho-1}}{\Gamma(\varrho)} f_k(\tau,\varpi(\tau))d\tau. \end{aligned}$$

Again integrating, we get

$$\begin{aligned} \varpi_k(t) &= \frac{\lambda_k}{\Gamma^2(\varrho)} \int_0^t \frac{(t-\tau)^{\varrho-1}}{p_k(\tau)} \left[\int_\tau^{\mathfrak{T}} (\xi-\tau)^{\varrho-1} f_k(\tau,\varpi(\xi))d\xi \right] d\tau + \varpi_k(0) \\ &= \frac{\lambda_k}{\Gamma^2(\varrho)} \int_0^t \frac{(t-\tau)^{\varrho-1}}{p_k(\tau)} \Big[\int_\tau^t (\xi-\tau)^{\varrho-1} f_k(\tau,\varpi(\xi))d\xi \\ &\qquad + \int_t^{\mathfrak{T}} (\xi-\tau)^{\varrho-1} f_k(\tau,\varpi(\xi))d\xi \Big] d\tau \\ &= \lambda_k \int_0^t \Big[\frac{1}{\Gamma^2(\varrho)} \int_0^\tau \frac{(t-\xi)^{\varrho-1}(\tau-\xi)^{\varrho-1}}{p_k(\xi)} d\xi \\ &\qquad + \int_t^{\mathfrak{T}} \frac{1}{\Gamma^2(\varrho)} \int_0^t \frac{(t-\xi)^{\varrho-1}(\tau-\xi)^{\varrho-1}}{p_k(\xi)} d\xi \Big] f_k(\tau,\varpi(\tau))d\tau \\ &= \lambda_k \int_0^{\mathfrak{T}} \aleph_k(t,\tau) f_k(\tau,\varpi(\tau))d\tau. \end{aligned}$$

The converse is evident and this completes the proof. Q.E.D.

Lemma 2.5. The kernel $\aleph_k(t,\tau)$ satisfies the following properties, for $k = 1, 2, 3, \cdots$,

(i) $\aleph_k(t,\tau) \in C([0,\mathfrak{T}] \times [0,\mathfrak{T}])$ and $\aleph_k(t,\tau) > 0$ for $(t,\tau) \in (0,\mathfrak{T}) \times (0,\mathfrak{T})$.

(ii) If $\varrho > 1/2$, then $\max\limits_{t\in[0,\mathfrak{T}]} \aleph_k(t,\tau) \le \dfrac{\mathfrak{T}^{2\varrho-1}}{(2\varrho-1)\Gamma^2(\varrho)}\, p_k^*$ for $\tau \in (0,\mathfrak{T})$, where

$$p_k^* = \max_{t\in[0,\mathfrak{T}]} \frac{1}{p_k(t)}.$$

Proof. From the definition of $\aleph_k(t,\tau)$, it is clear that $\aleph_k(t,\tau) \in C([0,\mathfrak{T}] \times [0,\mathfrak{T}])$ and $\aleph_k(t,\tau) > 0$ for $(t,\tau) \in (0,\mathfrak{T}) \times (0,\mathfrak{T})$. Now we prove (ii). Suppose $\varrho > 1/2$, then

$$\begin{aligned} \max_{t\in[0,\mathfrak{T}]} \aleph_{k_1}(t,\tau) &= \max_{t\in[0,\mathfrak{T}]} \int_0^\tau \frac{(t-\xi)^{\varrho-1}(\tau-\xi)^{\varrho-1}}{p_k(\xi)} d\xi \\ &\le p_k^* \int_0^{\mathfrak{T}} (\mathfrak{T}-\xi)^{2\varrho-2} d\xi \\ &\le \frac{\mathfrak{T}^{2\varrho-1}}{2\varrho-1}\, p_k^*, \end{aligned}$$

and

$$\begin{aligned}\max_{t\in[0,\mathfrak{T}]}\aleph_{k_2}(t,\tau) &= \max_{t\in[0,\mathfrak{T}]}\int_0^t \frac{(t-\xi)^{\varrho-1}(\tau-\xi)^{\varrho-1}}{p_k(\xi)}d\xi\\ &\leq p_k^*\int_0^{\mathfrak{T}}(\mathfrak{T}-\xi)^{2\varrho-2}d\xi\\ &\leq \frac{\mathfrak{T}^{2\varrho-1}}{2\varrho-1}\,p_k^*.\end{aligned}$$

Hence,

$$\max_{t\in[0,\mathfrak{T}]}\aleph_k(t,\tau)\leq \frac{\mathfrak{T}^{2\varrho-1}}{(2\varrho-1)\Gamma^2(\varrho)}\,p_k^*.$$

This completes the proof. Q.E.D.

Definition 2.6 ([28]). Let $(\mathcal{X},d)$ be a metric space and $\mathcal{Y}\subset\mathcal{X}$. Then the kurtowski measure of noncompactness of $\mathcal{Y}$ is denoted by $\alpha(\mathcal{Y})$, if defined as

$$\alpha(\mathcal{Y})=\inf\Big\{\varepsilon>0:\mathcal{Y}\subset\bigcup_{k=1}^{n}\mathcal{E}_k,\ \mathcal{E}_k\subset\mathcal{X},\ diam(\mathcal{E}_k)<\varepsilon,\ n\in\mathbb{N}\Big\},$$

here α is called Kuratowski measure of noncompactness. It is evident that

$$\alpha(\mathcal{Y})\leq diam(\mathcal{Y})\ \ \forall\ \ \mathcal{Y}\subset\mathcal{X}.$$

Let $\mathcal{B}$ be a real Banach space with the norm $\|\cdot\|$ and $\mathcal{S}(a,a_0)$ be a closed ball in $\mathcal{B}$ centered at a and radius a_0. If $\mathcal{F}$ is a nonempty subset of $\mathcal{B}$ then $\overline{\mathcal{F}}$ and $conv(\mathcal{F})$ represents closure and convex closure of $\mathcal{F}$. Further, let $\mathfrak{M}_{\mathcal{B}}$ denote the family of all nonempty and bounded subsets of $\mathcal{B}$ and $\mathfrak{N}_{\mathcal{B}}$ its subfamily consisting of all relatively compact sets.

Definition 2.7 ([8]). A function $\mu:\mathfrak{M}_{\mathcal{B}}\to[0,1)$ is called a measure of noncompactness, iff

(C_1) the family $\mathrm{Ker}\mu=\{\mathcal{F}\in\mathfrak{M}_{\mathcal{B}}:\mu(X)=0\}$ is nonempty and $\mathrm{Ker}\mu\subset\mathfrak{N}_{\mathcal{B}}$.

(C_2) $\mathcal{F}\subset\mathcal{E}$ then $\mu(\mathcal{F})\leq\mu(\mathcal{E})$.

(C_3) $\mu(\overline{\mathcal{F}})=\mu(\mathcal{F})$.

(C_4) $\mu(conv\mathcal{F})=\mu(\mathcal{F})$.

(C_5) $\mu(\lambda\mathcal{E}+(1-\lambda)\mathcal{F})\leq\lambda\mu(\mathcal{E})+(1-\lambda)\mu(\mathcal{F})$ for $\lambda\in[0,1]$.

(C_6) if $\mathcal{E}_n\in\mathfrak{M}_{\mathcal{B}}$, $X_n=\overline{\mathcal{E}}_n$, $\mathcal{E}_{n+1}\subset\mathcal{E}_n$ for $n=1,2,3,\cdots$ and $\lim_{n\to\infty}\mu(\mathcal{E}_n)=0$ then $\cap_{n=1}^{\infty}\mathcal{E}_n\neq\varnothing$.

Definition 2.8 ([10]). Let $(\mathcal{X},d)$ be a metric space, $\mathcal{Y}$ be a bounded subset $\mathcal{X}$ and $\mathcal{S}(a,a_0)=\{b\in\mathcal{X}:d(a,b)<a_0\}$. Then the Hausdorff measure of noncompactness $\chi(\mathcal{Y})$ of $\mathcal{Y}$ is defined by

$$\chi(\mathcal{Y})=\inf\left\{\varepsilon>0:\mathcal{Y}\subset\bigcup_{k=1}^{n}\mathcal{S}(a_k,a_{0_k}),a_k\in\mathcal{X},a_{0_k}<\varepsilon,n\in\mathbb{N}\right\}.$$

Next, denote $\mathfrak{c}_0$, and $\mathfrak{c}$ be Banach spaces with sup norms, are defined as

$$\mathfrak{c}_0 = \left\{ a \in \omega : \lim_{k\to\infty} a_k = 0, \quad \|a\|_{\mathfrak{c}_0} = \sup_k |a_k| \right\},$$

$$\mathfrak{c} = \left\{ a \in \omega : \lim_{k\to\infty} a_k = l, \quad l \in \mathbb{C}, \quad \|a\|_{\mathfrak{c}} = \sup_k |a_k| \right\},$$

respectively.

Definition 2.9 ([10]). The Hausdorff measure of noncompactness χ on the Banach space $(\mathfrak{c}_0, \|\cdot\|_{\mathfrak{c}_0})$ is defined by

$$\chi(\mathcal{S}) = \lim_{k\to\infty} \left\{ \sup_{a(t)\in\mathcal{S}} \left[\max_{m\geq k} \left| a_m \right| \right] \right\}, \quad \mathcal{S} \in \mathfrak{M}_{\mathfrak{c}_0}.$$

Definition 2.10 ([34]). The (regular)measure of noncompactness μ on the Banach space $(\mathfrak{c}, \|\cdot\|_{\mathfrak{c}})$ is defined by

$$\mu(\mathcal{S}) = \lim_{k\to\infty} \left\{ \sup_{a(t)\in\mathcal{S}} \left[\sup_{m\geq k} \left| a_m - \lim_{n\to\infty} a_n \right| \right] \right\}, \quad \mathcal{S} \in \mathfrak{M}_{\mathfrak{c}}.$$

Definition 2.11 ([9]). Let $\delta = (\delta_k)$ be such that δ_k is positive and nonincreasing sequence, Then δ is called tempering sequence. Let $\mathcal{A}$ be set consisting of all real sequences $a = (a_k)_{k=1}^{\infty}$ such that $\delta_k a_k \to 0$ as $k \to \infty$. It is clear that $\mathcal{A}$ forms a linear space over the field of real numbers. We denote the space by $\mathfrak{c}_0^{\delta}$. Also, it can be see that $\mathfrak{c}_0^{\delta}$ is a Banach space with the norm

$$\|a\|_{\mathfrak{c}_0^{\delta}} = \sup \{\delta_k |a_k|\}.$$

Similarly, let $\mathcal{G}$ be set consisting of all real sequences $a = (a_k)_{k=1}^{\infty}$ such that $\delta_k a_k$ converges to finite limit. It is clear that $\mathcal{G}$ forms a linear space over the field of real numbers. We denote the space by $\mathfrak{c}^{\delta}$. Also, it can be see that $\mathfrak{c}^{\delta}$ is a Banach space with the norm

$$\|a\|_{\mathfrak{c}^{\delta}} = \sup \{\delta_k |a_k|\}.$$

Next, Consider the function spaces $\mathcal{C}(\mathfrak{J}, \mathfrak{c}_0^{\delta})$ and $\mathcal{C}(\mathfrak{J}, \mathfrak{c}^{\delta})$ where $\mathfrak{J} = [0, \mathfrak{T}]$, $\mathfrak{T} > 0$ the spaces of all continuous functions on $\mathfrak{J}$ with values in $\mathfrak{c}_0^{\delta}$ and the spaces of all continuous functions on $\mathfrak{J}$ with values in c^{β}respectively. Then $\mathcal{C}(\mathfrak{J}, \mathfrak{c}_0^{\delta})$ and $\mathcal{C}(\mathfrak{J}, \mathfrak{c}^{\delta})$ are Banach spaces with respect to the norms,

$$\|a\|_{\mathcal{C}(\mathfrak{J},\mathfrak{c}_0^{\delta})} = \max \left\{ \|a(t)\|_{\mathfrak{c}_0^{\delta}} : t \in \mathfrak{J} \right\}, \quad a \in \mathcal{C}(\mathfrak{J}, \mathfrak{c}_0^{\delta}),$$

$$\|a\|_{\mathcal{C}(\mathfrak{J},\mathfrak{c}^{\delta})} = \max \left\{ \|a(t)\|_{\mathfrak{c}^{\delta}} : t \in \mathfrak{J} \right\}, \quad a \in \mathcal{C}(\mathfrak{J}, \mathfrak{c}^{\delta}),$$

respectively(see [12]).

For any nonempty, closed, bounded and convex subset $\mathcal{X}$ of $\mathcal{C}(\mathfrak{J}, \mathfrak{c}_0^{\delta})$ or $\mathcal{C}(\mathfrak{J}, \mathfrak{c}^{\delta})$ and $t \in \mathfrak{J}$, we have

$$\chi_{\mathcal{C}(\mathfrak{J},\mathfrak{c}_0^{\delta})}(\mathcal{X}) = \sup \left\{ \chi_{\mathfrak{c}_0^{\delta}}(\mathcal{X}(t)) : t \in \mathfrak{J} \right\}$$

and

$$\mu_{\mathcal{C}(\mathfrak{J},\mathfrak{c}^{\delta})}(\mathcal{X}) = \sup \left\{ \mu_{\mathfrak{c}^{\delta}}(\mathcal{X}(t)) : t \in \mathfrak{J} \right\}.$$

Note that $\chi_{\mathcal{C}(\mathfrak{J},\mathfrak{c}_0^{\delta})}$ and $\mu_{\mathcal{C}(\mathfrak{J},\mathfrak{c}^{\delta})}$ satisfy all the axioms of measure of noncompactness on $\mathcal{C}(\mathfrak{J}, \mathfrak{c}_0^{\delta})$ and $\mathcal{C}(\mathfrak{J}, \mathfrak{c}^{\delta})$, respectively(see [12]).

Definition 2.12 ([30]). Let $(\mathcal{X}, d)$ be a metric space. Then a mapping $\mathcal{T}$ on $\mathcal{X}$ is said to be a Meir-Keeler contraction if for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\varepsilon \leq d(x, y) < \varepsilon + \delta \implies d(Tx, Ty) < \varepsilon, \quad \forall x, y \in \mathcal{X}.$$

Theorem 2.13 ([30]). Let $(\mathcal{X}, d)$ be a complete metric space. If $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ is a Meir-Keeler contraction, then $\mathcal{T}$ has a unique fixed point.

Definition 2.14 ([3]). Let $\mathcal{C}$ be a nonempty subset of a Banach space $\mathcal{E}$ and let μ be an arbitrary measure of noncompactness on $\mathcal{E}$. We say that an operator $\mathcal{T} : \mathcal{C} \to \mathcal{C}$ is a Meir-Keeler condensing operator if for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\varepsilon \leq \mu(\mathcal{X}) < \varepsilon + \delta \implies \mu(\mathcal{T}(\mathcal{X})) < \varepsilon$$

for any bounded subset $\mathcal{X}$ of $\mathcal{C}$.

Theorem 2.15 ([3]). Let $\mathcal{C}$ be a nonempty, bounded, closed and convex subset of a Banach space $\mathcal{E}$ and let μ be an arbitrary measure of noncompactness on $\mathcal{E}$. If $\mathcal{T} : \mathcal{C} \to \mathcal{C}$ is a continuous and Meir-Keeler condensing operator, then $\mathcal{T}$ has at least one fixed point and the set of all fixed points of $\mathcal{T}$ in $\mathcal{C}$ is compact.

3 Existence of solutions for (1.3) in $\mathcal{C}(\mathfrak{J}, \mathfrak{c}_0^\delta)$

In this section, we investigate the existence of solutions for infinite systems of regular fractional Sturm-Liouville problems (1.3) in the sequences spaces $\mathcal{C}(\mathfrak{J}, \mathfrak{c}_0^\delta)$.

Suppose

$(\mathfrak{M}_1)$ $f_k : \mathfrak{J} \to \mathbb{R}^\infty$ and define an operator $f : \mathfrak{J} \times \mathfrak{c}_0^\delta \to \mathfrak{c}_0^\delta$ as

$$(t, \varpi(t)) \to (f\varpi)(t) = \big(f_k(t, \varpi(t))\big)_{k=1}^\infty$$

is the class of all functions $\big((f\varpi)(t)\big)_{t\in\mathfrak{J}}$ and equicontinuous on $\mathfrak{c}_0^\delta$.

$(\mathfrak{M}_2)$ $\mathfrak{p}_k(t), \mathfrak{q}_k(t) : \mathfrak{J} \to \mathbb{R}$ are continuous functions such that the sequence $\delta_k \mathfrak{p}_k(t)$ converges uniformly to zero on $\mathfrak{J}$ and the sequence $(\mathfrak{q}_k(t))$ is equibounded on $\mathfrak{J}$, so we take $\mathfrak{q}(t) = \sup\{\mathfrak{q}_k(t) : k \in \mathbb{N}\}$, $\mathfrak{q}^* = \sup\{\mathfrak{q}(t) : t \in \mathfrak{J}\}$, $\delta^* = \sup\{\delta_k \mathfrak{p}_k(t) : k \in \mathbb{N}, t \in \mathfrak{J}\}$ and

$$\big|f_k(t, \varpi(t))\big| \leq \mathfrak{p}_k(t) + \mathfrak{q}_k(t)\big|\varpi_k(t)\big|, \quad \varpi_k \in \mathfrak{c}_0^\delta, \ t \in \mathfrak{J}, k = 1, 2, 3, \cdots.$$

Theorem 3.1. Suppose $\varrho > 1/2$, $\frac{\mathfrak{q}^* \mathfrak{T}^{2\varrho}}{(2\varrho-1)\Gamma^2(\varrho)} < 1$ and $(\mathfrak{M}_1) - (\mathfrak{M}_2)$ hold, then the infinite systems of fractional order boundary value problems (1.3) has at least one solution $\varpi(t) = \big(\varpi_k(t)\big)$ in $\mathcal{C}(\mathfrak{J}, \mathfrak{c}_0^\delta)$ provide

$$0 < \lambda_k < \frac{1}{p_k^*}, \quad k \in \mathbb{N}.$$

Proof. Since $\sup\{\delta_k|\varpi_k(t)|\} < +\infty$ for all $\varpi(t) = \left(\varpi_k(t)\right)_{k=1}^{\infty} \in \mathcal{C}(\mathfrak{J}, \mathfrak{c}_0^{\delta})$ and $t \in \mathfrak{J}$, there exists $\mathfrak{L} > 0$ such that $\sup\{\delta_k|\varpi_k(t)|\} < \mathfrak{L}$. From $(\mathfrak{M}_2)$ and (2.1), we get

$$\begin{aligned}
\|\varpi(t)\|_{\mathfrak{c}_0^{\delta}} &= \sup_{k\in\mathbb{N}}\left\{\delta_k\left|\lambda_k\int_0^{\mathfrak{T}}\aleph_k(t,\tau)f_k(\tau,\varpi(\tau))d\tau\right|\right\}\\
&\leq \sup_{k\in\mathbb{N}}\left\{\delta_k\lambda_k\int_0^{\mathfrak{T}}|\aleph_k(t,\tau)|\,|f_k(\tau,\varpi(\tau))d\tau|\right\}\\
&\leq \frac{\mathfrak{T}^{2\varrho-1}}{(2\varrho-1)\Gamma^2(\varrho)}\sup_{k\in\mathbb{N}}\left\{\delta_k\lambda_k p_k^*\int_0^{\mathfrak{T}}\left[\mathfrak{p}_k(\tau)+\mathfrak{q}_k(\tau)\left|\varpi_k(\tau)\right|\right]d\tau\right\}\\
&\leq \frac{\mathfrak{T}^{2\varrho-1}}{(2\varrho-1)\Gamma^2(\varrho)}\sup_{k\in\mathbb{N}}\left\{\int_0^{\mathfrak{T}}\left[\delta^*+\mathfrak{q}^*\mathfrak{L}\right]d\tau\right\}\\
&\leq \frac{(\delta^*+\mathfrak{q}^*\mathfrak{L})\mathfrak{T}^{2\varrho}}{(2\varrho-1)\Gamma^2(\varrho)}.
\end{aligned}$$

Therefore,

$$\max_{t\in\mathfrak{J}}\|\varpi(t)\|_{\mathfrak{c}_0^{\delta}} \leq r, \quad i.e., \ \|\varpi(t)\|_{\mathcal{C}(\mathfrak{J},\mathfrak{c}_0^{\delta})} \leq r,$$

where

$$r = \frac{(\delta^*+\mathfrak{q}^*\mathfrak{L})\mathfrak{T}^{2\varrho}}{(2\varrho-1)\Gamma^2(\varrho)}.$$

Now, let the closed ball $\mathcal{S} = \mathcal{S}(\varpi^0(t), r)$ centered at $\varpi^0(t) = \left(\varpi^0(t)\right)_{k=1}^{\infty}$, $\varpi^0(t) = 0\,\forall\, t \in \mathfrak{J}$, $k \in \mathbb{N}$ and radius r. So, $\mathcal{S}$ is a nonempty bounded closed convex subset of $\mathcal{C}(\mathfrak{J}, \mathfrak{c}_0^{\delta})$.

For fixed $t \in \mathfrak{J}$, define an operator $\Omega = (\Omega_k)_{k=1}^{\infty} : \mathcal{C}(\mathfrak{J}, \mathfrak{c}_0^{\delta}) \to \mathcal{C}(\mathfrak{J}, \mathfrak{c}_0^{\delta})$ as

$$(\Omega\varpi)(t) = \{(\Omega_k\varpi)(t)\}_{k=1}^{\infty} = \left\{\lambda_k\int_0^{\mathfrak{T}}\aleph_k(t,\tau)f_k(\tau,\varpi(\tau))d\tau\right\}_{k=1}^{\infty}.$$

Since $\left(f_k(t,\varpi(t))\right)_{k=1}^{\infty} \in \mathfrak{c}_0^{\delta}$, for $t \in \mathfrak{J}$, it follows that

$$\begin{aligned}
\lim_{k\to\infty}\{\delta_k(\Omega_k\varpi)(t)\} &= \lim_{k\to\infty}\left\{\lambda_k\int_0^{\mathfrak{T}}\aleph_k(t,\tau)f_k(\tau,\varpi(\tau))d\tau\right\}\\
&\leq \frac{\mathfrak{T}^{2\varrho-1}}{(2\varrho-1)\Gamma^2(\varrho)}\int_0^{\mathfrak{T}}\lim_{k\to\infty}[\lambda_k\mathfrak{p}_k^*\delta_k f_k(\tau,\varpi(\tau))]d\tau\\
&\leq \frac{\mathfrak{T}^{2\varrho-1}}{(2\varrho-1)\Gamma^2(\varrho)}\int_0^{\mathfrak{T}}\lim_{k\to\infty}[\delta_k f_k(\tau,\varpi(\tau))]d\tau\\
&= 0.
\end{aligned}$$

Thus, $(\Omega\varpi)(t) \in \mathcal{C}(\mathfrak{J}, \mathfrak{c}_0^{\delta})$. Also, it can be seen that $(\Omega_k\varpi)(t)$ satisfies boundary conditions, i.e., $(\Omega_k\varpi)(0) = 0$ and ${}^{\mathcal{C}}\mathfrak{D}_{0^+}^{\varrho}(\Omega_k\varpi)(\mathfrak{T}) = 0$.

For fixed $t \in \mathfrak{T}$ and $\varpi(t) \in \mathcal{S}$, we get

$$\begin{aligned}\|(\Omega\varpi)(t) - \varpi^0(t)\|_{\mathfrak{c}_0^\delta} \leq r \implies & \max_{t\in\mathfrak{J}} \|(\Omega\varpi)(t) - \varpi^0(t)\|_{\mathfrak{c}_0^\delta} \leq r \\ \implies & \|(\Omega\varpi)(t) - \varpi^0(t)\|_{\mathcal{C}(\mathfrak{J},\mathfrak{c}_0^\delta)} \leq r,\end{aligned}$$

which proves that Ω is self mapping on $\mathcal{S}$. From $(\mathfrak{M}_1)$, for any $\vartheta(t) = \big(\vartheta_k(t)\big)_{k=1}^{\infty} \in \mathcal{S}$ and for any $\varepsilon > 0$ there exists $\eta > 0$ such that $\|(f\varpi)(t) - (f\vartheta)(t)\|_{\mathfrak{c}_0^\delta} < \frac{\varepsilon(2\varrho-1)\Gamma^2(\varrho)}{\mathfrak{T}^{2\varrho}}$ for each $\varpi(t) \in \mathcal{S}$, whenever $|\varpi(t) - \vartheta(t)| \leq \eta$, where $t \in \mathfrak{J}$. So, for $t \in \mathfrak{J}$, we have

$$\begin{aligned}\|(\Omega\varpi)(t) - (\Omega\vartheta)(t)\|_{\mathfrak{c}_0^\delta} =& \sup_{k\in\mathbb{N}} \left\{ \delta_k\lambda_k \int_0^{\mathfrak{T}} |\aleph_k(t,\tau)||f_k(\tau,\varpi(\tau)) - f_k(\tau,\vartheta(\tau))|d\tau \right\} \\ \leq& \frac{\mathfrak{T}^{2\varrho-1}}{(2\varrho-1)\Gamma^2(\varrho)} \sup_{k\in\mathbb{N}} \left\{ \delta_k\lambda_k\mathfrak{p}_k^* \int_0^{\mathfrak{T}} |f_k(\tau,\varpi(\tau)) - f_k(\tau,\vartheta(\tau))|d\tau \right\} \\ \leq& \frac{\mathfrak{T}^{2\varrho-1}}{(2\varrho-1)\Gamma^2(\varrho)} \frac{\varepsilon(2\varrho-1)\Gamma^2(\varrho)}{\mathfrak{T}^{2\varrho}}\mathfrak{T} < \varepsilon.\end{aligned}$$

Since t is arbitrary, it follows that Ω is continuous on $\mathcal{S}$ for every $t \in \mathfrak{J}$.

Now, we have

$$\begin{aligned}\chi(\Omega\mathcal{S}) =& \lim_{k\to\infty} \left\{ \sup_{\varpi(t)\in\mathcal{S}} \sup_{m\geq k} \left[\delta_m \Big| \lambda_m \int_0^{\mathfrak{T}} \aleph_m(t,\tau) f_m(\tau,\varpi(\tau))d\tau \Big| \right] \right\} \\ \leq& \frac{\mathfrak{T}^{2\varrho-1}}{(2\varrho-1)\Gamma^2(\varrho)} \lim_{k\to\infty} \left\{ \sup_{\varpi(t)\in\mathcal{S}} \sup_{m\geq k} \left[\int_0^{\mathfrak{T}} \big(\delta_m\mathfrak{p}_m(\tau) + \delta_m\mathfrak{q}_m(\tau)\big|\varpi_m(\tau)\big| \big) \right] \right\} \\ \leq& \frac{\mathfrak{q}^*\mathfrak{T}^{2\varrho}}{(2\varrho-1)\Gamma^2(\varrho)} \chi(\mathcal{S}).\end{aligned}$$

Thus,

$$\sup_{t\in\mathfrak{J}} \chi(\Omega\mathcal{S}) \leq \frac{\mathfrak{q}^*\mathfrak{T}^{2\varrho}}{(2\varrho-1)\Gamma^2(\varrho)} \sup_{t\in\mathfrak{J}} \chi(\mathcal{S}).$$

It follows that,

$$\chi_{\mathcal{C}(\mathfrak{J},\mathfrak{c}_0^\delta)}(\Omega\mathcal{S}) \leq \frac{\mathfrak{q}^*\mathfrak{T}^{2\varrho}}{(2\varrho-1)\Gamma^2(\varrho)} \chi_{\mathcal{C}(\mathfrak{J},\mathfrak{c}_0^\delta)}(\mathcal{S}) < \varepsilon \implies \chi_{\mathcal{C}(\mathfrak{J},\mathfrak{c}_0^\delta)}(\mathcal{S}) < \frac{\varepsilon(2\varrho-1)\Gamma^2(\varrho)}{\mathfrak{q}^*\mathfrak{T}^{2\varrho}}.$$

Letting $\eta = \dfrac{\varepsilon}{\mathfrak{q}^*\mathfrak{T}^{2\varrho}} \left[(2\varrho-1)\Gamma^2(\varrho) - \mathfrak{q}^*\mathfrak{T}^{2\varrho}\right]$, we obtain $\varepsilon \leq \chi_{\mathcal{C}(\mathfrak{J},\mathfrak{c}_0^\delta)}(\mathcal{S}) < \varepsilon + \eta$.

Therefore, Ω is a Meir-Keeler condensing operator on $\mathcal{S}$. Since t is arbitrary, so for every $t \in \mathfrak{J}$, Ω satisfies all the conditions of Theorem 2.15, i.e., Ω has a fixed point in $\mathcal{S}$. Hence, the infinite systems (1.3) has a solution in $\mathcal{C}(\mathfrak{J},\mathfrak{c}_0^\delta)$. Q.E.D.

4 Existence of solutions for (1.3) in $\mathcal{C}(\mathfrak{J}, \mathfrak{c}^\delta)$

In this section, we investigate the existence of solutions for infinite systems of regular fractional Sturm-Liouville problems (1.3) in the sequences spaces $\mathcal{C}(\mathfrak{J}, \mathfrak{c}^\delta)$.

Here, we assume that $p_k(t) = p(t)$, a nonnegative real-valued continuous function defined on $\mathfrak{J}$ for all $k \in \mathbb{N}$ and

$(\mathfrak{N}_1)$ $f_k : \mathfrak{J} \to \mathbb{R}^\infty$ and define an operator $f : \mathfrak{J} \times \mathfrak{c}^\delta \to \mathfrak{c}^\delta$ as

$$(t, \varpi(t)) \to (f\varpi)(t) = \big(f_k(t, \varpi(t))\big)_{k=1}^{\infty}$$

is the class of all functions $\big((f\varpi)(t)\big)_{t\in\mathfrak{J}}$ and equicontinuous on $\mathfrak{c}^\delta$.

$(\mathfrak{N}_2)$ $\mathfrak{d}_k(t), \mathfrak{e}_k(t) : \mathfrak{J} \to \mathbb{R}$ are continuous functions such that the sequence $\delta_k \mathfrak{d}_k(t)$ converges uniformly to zero on $\mathfrak{J}$ and the sequence $(\mathfrak{e}_k(t))$ is convergence on $\mathfrak{J}$, so we take $\mathfrak{e}(t) = \sup\{\mathfrak{e}_k(t) : k \in \mathbb{N}\}$, $\mathfrak{e}^* = \sup\{\mathfrak{e}(t) : t \in \mathfrak{J}\}$, $\delta^* = \sup\{\delta_k \mathfrak{d}_k(t) : k \in \mathbb{N}, t \in \mathfrak{J}\}$ and

$$f_k(t, \varpi(t)) \le \mathfrak{d}_k(t) + \mathfrak{e}_k(t)\varpi_k(t), \quad \varpi_k \in \mathfrak{c}^\delta, \ t \in \mathfrak{J}, k = 1, 2, 3, \cdots.$$

Theorem 4.1. Suppose $\varrho > 1/2$, $\frac{\mathfrak{e}^* \mathfrak{T}^{2\varrho}}{(2\varrho-1)\Gamma^2(\varrho)} < 1$ and $(\mathfrak{N}_1) - (\mathfrak{N}_2)$ hold, then the infinite systems of fractional order boundary value problems (1.3) has at least one solution $\varpi(t) = \big(\varpi_k(t)\big)$ in $\mathcal{C}(\mathfrak{J}, \mathfrak{c}^\delta)$ provide

$$0 < \lambda_k < \frac{1}{p^*}, \quad k \in \mathbb{N},$$

where $p^* = \sup\{p(t) : t \in \mathfrak{J}\}$.

Proof. Since $\sup\{\delta_k |\varpi_k(t)|\} < +\infty$ for all $\varpi(t) = \big(\varpi_k(t)\big)_{k=1}^{\infty} \in \mathcal{C}(\mathfrak{J}, \mathfrak{c}^\delta)$ and $t \in \mathfrak{J}$, there exists $\mathfrak{Z} > 0$ such that $\sup\{\delta_k |\varpi_k(t)|\} < \mathfrak{Z}$. From $(\mathfrak{N}_2)$ and (2.1), we get

$$\begin{aligned}
\|\varpi(t)\|_{\mathfrak{c}^\delta} &= \sup_{k\in\mathbb{N}} \left\{ \delta_k \left| \lambda_k \int_0^{\mathfrak{T}} \aleph_k(t, \tau) f_k(\tau, \varpi(\tau)) d\tau \right| \right\} \\
&\le \sup_{k\in\mathbb{N}} \left\{ \delta_k \lambda_k \int_0^{\mathfrak{T}} |\aleph_k(t, \tau)| \, |f_k(\tau, \varpi(\tau)) d\tau| \right\} \\
&\le \frac{\mathfrak{T}^{2\varrho-1}}{(2\varrho-1)\Gamma^2(\varrho)} \sup_{k\in\mathbb{N}} \left\{ \delta_k \lambda_k p^* \int_0^{\mathfrak{T}} \left[\mathfrak{d}_k(\tau) + \mathfrak{e}_k(\tau) \big| \varpi_k(\tau) \big| \right] d\tau \right\} \\
&\le \frac{\mathfrak{T}^{2\varrho-1}}{(2\varrho-1)\Gamma^2(\varrho)} \sup_{k\in\mathbb{N}} \left\{ \int_0^{\mathfrak{T}} \left[\delta^* + \mathfrak{e}^* \mathfrak{Z} \right] d\tau \right\} \\
&\le \frac{(\delta^* + \mathfrak{e}^* \mathfrak{Z}) \mathfrak{T}^{2\varrho}}{(2\varrho-1)\Gamma^2(\varrho)}.
\end{aligned}$$

Therefore,

$$\max_{t\in\mathfrak{J}} \|\varpi(t)\|_{\mathfrak{c}^\delta} \le r_1, \quad i.e., \ \|\varpi(t)\|_{\mathcal{C}(\mathfrak{J}, \mathfrak{c}^\delta)} \le r_1,$$

where

$$r_1 = \frac{(\delta^* + \mathfrak{e}^*\mathfrak{Z})\mathfrak{T}^{2\varrho}}{(2\varrho-1)\Gamma^2(\varrho)}.$$

Now, let the closed ball $\mathcal{B} = \mathcal{B}(\varpi^0(t), r_1)$ centered at $\varpi^0(t) = \big(\varpi^0(t)\big)_{k=1}^{\infty}$, $\varpi^0(t) = 0\,\forall\, t \in \mathfrak{J}$, $k \in \mathbb{N}$ and radius r_1. So, $\mathcal{B}$ is a nonempty bounded closed convex subset of $\mathcal{C}(\mathfrak{J}, \mathfrak{c}^\delta)$.

For fixed $t \in \mathfrak{J}$, define an operator $\Omega = (\Omega_k)_{k=1}^{\infty} : \mathcal{C}(\mathfrak{J}, \mathfrak{c}_0^\delta) \to \mathcal{C}(\mathfrak{J}, \mathfrak{c}_0^\delta)$ as

$$(\Omega\varpi)(t) = \{(\Omega_k\varpi)(t)\}_{k=1}^{\infty} = \left\{\lambda_k \int_0^{\mathfrak{T}} \aleph_k(t,\tau) f_k(\tau, \varpi(\tau)) d\tau\right\}_{k=1}^{\infty}.$$

Since $\big(f_k(t, \varpi(t))\big)_{k=1}^{\infty} \in \mathfrak{c}^\delta$, and $p_k(t) = p(t)$ for $t \in \mathfrak{J}$, it follows that $\aleph_k(t,\tau) = \aleph_l(t,\tau)$ for any $k, l \in \mathbb{N}$. Therefore, we have

$$\begin{aligned}
&|\delta_k(\Omega_k\varpi)(t) - \delta_l(\Omega_l\varpi)(t)| \\
&= \left|\delta_k\lambda_k \int_0^{\mathfrak{T}} \aleph_k(t,\tau) f_k(\tau,\varpi(\tau)) d\tau - \delta_l\lambda_l \int_0^{\mathfrak{T}} \aleph_l(t,\tau) f_l(\tau,\varpi(\tau)) d\tau\right| \\
&\le \left| \int_0^{\mathfrak{T}} \aleph_l(t,\tau)[\delta_k\lambda_k\mathfrak{d}_k(\tau) + \delta_k\lambda_k\mathfrak{e}_k(\tau)\varpi_k(\tau)] - \int_0^{\mathfrak{T}} \aleph_l(t,\tau)[\delta_l\lambda_l\mathfrak{d}_l(\tau) + \delta_l\lambda_l\mathfrak{e}_l(\tau)\varpi_l(t)]\right| \\
&\le \int_0^{\mathfrak{T}} |\aleph_l(t,\tau)||\delta_k\lambda_k\mathfrak{d}_k(\tau) - \delta_l\lambda_l\mathfrak{d}_l(\tau)| d\tau \\
&\qquad\qquad + \int_0^{\mathfrak{T}} |\aleph_l(t,\tau)||\delta_k\lambda_k\mathfrak{e}_k(\tau)\varpi_k(\tau) - \delta_l\lambda_l\mathfrak{e}_l(\tau)\varpi_l(\tau)| d\tau \\
&\le \int_0^{\mathfrak{T}} |\aleph_l(t,\tau)||\delta_k\lambda_k\mathfrak{d}_k(\tau) - \delta_l\lambda_l\mathfrak{d}_l(\tau)| d\tau \\
&\qquad + \int_0^{\mathfrak{T}} |\aleph_l(t,\tau)|\Big[\delta_k\lambda_k|\varpi_k(\tau)||\mathfrak{e}_k(\tau) - \mathfrak{e}_l(\tau)| + |\mathfrak{e}_l(\tau)||\delta_k\lambda_k\varpi_k(\tau) - \delta_l\lambda_l\varpi_l(\tau)|\Big]
\end{aligned}$$

Since (λ_k), $(\mathfrak{e}_k)$, $(\delta_k\mathfrak{d}_k)$ are convergent on $\mathfrak{J}$ and $\lambda_k\varpi_k \in \mathfrak{c}^\delta$, it follows that

$$|\delta_k(\Omega_k\varpi)(t) - \delta_l(\Omega_l\varpi)(t)| \to 0 \qquad \text{as} \qquad k, l \to \infty.$$

Hence, $(\Omega\varpi)(t) \in \mathcal{C}(\mathfrak{J}, \mathfrak{c}^\delta)$. Also, it can be seen that $(\Omega_k\varpi)(t)$ satisfies boundary conditions, i.e., $(\Omega_k\varpi)(0) = 0$ and ${}^{\mathcal{C}}\mathfrak{D}_{0^+}^{\varrho}(\Omega_k\varpi)(\mathfrak{T}) = 0$.

For fixed $t \in \mathfrak{T}$ and $\varpi(t) \in \mathcal{B}$, we get

$$\begin{aligned}
\|(\Omega\varpi)(t) - \varpi^0(t)\|_{\mathfrak{c}^\delta} \le r_1 &\implies \max_{t\in\mathfrak{J}} \|(\Omega\varpi)(t) - \varpi^0(t)\|_{\mathfrak{c}^\delta} \le r_1 \\
&\implies \|(\Omega\varpi)(t) - \varpi^0(t)\|_{\mathcal{C}(\mathfrak{J},\mathfrak{c}^\delta)} \le r_1,
\end{aligned}$$

which proves that Ω is self mapping on $\mathcal{B}$. From $(\mathfrak{N}_1)$, for any $\vartheta(t) = \big(\vartheta_k(t)\big)_{k=1}^{\infty} \in \mathcal{B}$ and for any $\varepsilon > 0$ there exists $\eta > 0$ such that $\|(f\varpi)(t) - (f\vartheta)(t)\|_{\mathfrak{c}^\delta} < \frac{\varepsilon(2\varrho-1)\Gamma^2(\varrho)}{\mathfrak{T}^{2\varrho}}$ for each $\varpi(t) \in \mathcal{B}$, whenever

$|\varpi(t)-\vartheta(t)|\leq\eta$, where $t\in\mathfrak{J}$. So, for $t\in\mathfrak{J}$, we have

$$\begin{aligned}\|(\Omega\varpi)(t)-(\Omega\vartheta)(t)\|_{\mathfrak{c}^\delta}&=\sup_{k\in\mathbb{N}}\left\{\delta_k\lambda_k\int_0^{\mathfrak{T}}|\aleph_k(t,\tau)||f_k(\tau,\varpi(\tau))-f_k(\tau,\vartheta(\tau))|d\tau\right\}\\&\leq\frac{\mathfrak{T}^{2\varrho-1}}{(2\varrho-1)\Gamma^2(\varrho)}\sup_{k\in\mathbb{N}}\left\{\delta_k\lambda_k p^*\int_0^{\mathfrak{T}}|f_k(\tau,\varpi(\tau))-f_k(\tau,\vartheta(\tau))|d\tau\right\}\\&\leq\frac{\mathfrak{T}^{2\varrho-1}}{(2\varrho-1)\Gamma^2(\varrho)}\frac{\varepsilon(2\varrho-1)\Gamma^2(\varrho)}{\mathfrak{T}^{2\varrho}}\mathfrak{T}<\varepsilon.\end{aligned}$$

Since t is arbitrary, it follows that Ω is continuous on $\mathcal{B}$ for every $t\in\mathfrak{J}$.

Now, we have

$$\begin{aligned}\mu(\Omega\mathcal{B})&=\lim_{k\to\infty}\left\{\sup_{\varpi(t)\in\mathcal{B}}\sup_{m\geq k}\left[\left|\delta_m\lambda_m\int_0^{\mathfrak{T}}\aleph_m(t,\tau)f_m(\tau,\varpi(\tau))d\tau\right.\right.\right.\\&\qquad\left.\left.\left.-\lim_{n\to\infty}\left(\delta_n\lambda_n\int_0^{\mathfrak{T}}\aleph_n(t,\tau)f_n(\tau,\varpi(\tau))d\tau\right)\right|\right]\right\}\\&=\lim_{k\to\infty}\left\{\sup_{\varpi(t)\in\mathcal{B}}\sup_{m\geq k}\left[\int_0^{\mathfrak{T}}|\aleph_m(t,\tau)|\Big[|\delta_m\lambda_m f_m(\tau,\varpi(\tau))\right.\right.\\&\qquad\left.\left.-\lim_{n\to\infty}\delta_n\lambda_n f_n(\tau,\varpi(\tau))|\Big]d\tau\right]\right\}\\&\leq\frac{\mathfrak{T}^{2\varrho-1}}{(2\varrho-1)\Gamma^2(\varrho)}\lim_{k\to\infty}\left\{\sup_{\varpi(t)\in\mathcal{B}}\sup_{m\geq k}\left[\int_0^{\mathfrak{T}}\Big|\delta_m\lambda_m\mathfrak{e}_m(\tau)\varpi_m(\tau)\right.\right.\\&\qquad\left.\left.-\lim_{n\to\infty}\delta_n\lambda_n\mathfrak{e}_n(\tau)\varpi_n(\tau)\Big|d\tau\right]\right\}\\&\leq\frac{\mathfrak{T}^{2\varrho-1}}{(2\varrho-1)\Gamma^2(\varrho)}\lim_{k\to\infty}\left\{\sup_{\varpi(t)\in\mathcal{B}}\sup_{m\geq k}\left[\int_0^{\mathfrak{T}}\Big(\lambda_m|\mathfrak{e}_m(\tau)|\Big|\delta_m\varpi_m(\tau)-\lim_{n\to\infty}\delta_n\varpi_n(\tau)\Big|\right.\right.\\&\qquad\left.\left.-\Big|\lim_{n\to\infty}\delta_n\varpi_n(\tau)\big(\lambda_m\mathfrak{e}_m(\tau)-\lambda_n\mathfrak{e}_n(\tau)\big)\Big|\Big)d\tau\right]\right\}\\&\leq\frac{\mathfrak{e}^*\mathfrak{T}^{2\varrho}}{(2\varrho-1)\Gamma^2(\varrho)}\,\mu(\mathcal{B}).\end{aligned}$$

Thus,

$$\sup_{t\in\mathfrak{J}}\mu(\Omega\mathcal{B})\leq\frac{\mathfrak{e}^*\mathfrak{T}^{2\varrho}}{(2\varrho-1)\Gamma^2(\varrho)}\sup_{t\in\mathfrak{J}}\mu(\mathcal{B}).$$

It follows that,

$$\mu_{\mathcal{C}(\mathfrak{J},\mathfrak{c}^\delta)}(\Omega\mathcal{B})\leq\frac{\mathfrak{e}^*\mathfrak{T}^{2\varrho}}{(2\varrho-1)\Gamma^2(\varrho)}\,\mu_{\mathcal{C}(\mathfrak{J},\mathfrak{c}^\delta)}(\mathcal{B})<\varepsilon\implies\mu_{\mathcal{C}(\mathfrak{J},\mathfrak{c}^\delta)}(\mathcal{B})<\frac{\varepsilon(2\varrho-1)\Gamma^2(\varrho)}{\mathfrak{e}^*\mathfrak{T}^{2\varrho}}.$$

Letting $\eta = \frac{\varepsilon}{\mathfrak{e}^*\mathfrak{T}^{2\varrho}}\left[(2\varrho-1)\Gamma^2(\varrho) - \mathfrak{e}^*\mathfrak{T}^{2\varrho}\right]$, we obtain $\varepsilon \leq \mu_{\mathcal{C}(\mathfrak{J},\mathfrak{c}^\delta)}(\mathcal{B}) < \varepsilon + \eta$.
Therefore, Ω is a Meir-Keeler condensing operator on $\mathcal{S}$. Since t is arbitrary, so for every $t \in \mathfrak{J}$, Ω satisfies all the conditions of Theorem 2.15, i.e., Ω has a fixed point in $\mathcal{B}$. Hence, the infinite systems (1.3) has a solution in $\mathcal{C}(\mathfrak{J},\mathfrak{c}^\delta)$. Q.E.D.

5 Applications

In this section, we present two examples to illustrate our main results.

Example 5.1. Consider the infinite systems of regular fractional Sturm-Liouville boundary value problems.

$$\begin{cases} {}^{\mathcal{C}}\mathfrak{D}^{4/5}_{1^-}\Big[(t^2+1)\big({}^{\mathcal{C}}\mathfrak{D}^{4/5}_{0^+}(\varpi_k(t))\big)\Big] = \lambda\left[\dfrac{e^{-kt^2}\sin(kt)}{k} + \displaystyle\sum_{n=k}^{\infty}\frac{\varpi_k(t)}{6n^2}\right] \\ \varpi_k(0) = 0,\ {}^{\mathcal{C}}\mathfrak{D}^{4/5}_{0^+}\varpi_k(1) = 0,\ k \in \mathbb{N}. \end{cases} \tag{5.1}$$

So, $\varrho = \frac{4}{5}$, $p_k(t) = t^2+1$, $\mathfrak{T} = 1$, $\mathfrak{J} = [0,1]$, and $f_k(t,\varpi(t)) = \frac{e^{-kt^2}\sin(kt)}{k} + \sum_{n=k}^{\infty}\frac{\varpi_k(t)}{6n^2}$.
Let $\delta_k = \frac{1}{k^2}$ for all $k \in \mathbb{N}$. Now, for $\varpi(t) \in \mathcal{C}(\mathfrak{J},\mathfrak{c}_0^\delta)$, we have

$$\lim_{k\to\infty}\delta_k f_k(t,\varpi(t)) = \lim_{k\to\infty}\left[\frac{e^{-kt^2}\sin(kt)}{k^3} + \frac{1}{k^2}\sum_{n=k}^{\infty}\frac{\varpi_k(t)}{6n^2}\right] = 0.$$

Next, let for any $\vartheta(t) = \big(\vartheta_k(t)\big)_{k=1}^{\infty} \in \mathcal{C}(\mathfrak{J},\mathfrak{c}_0^\delta)$. Let $\varepsilon > 0$ be given and $\eta = \dfrac{36\varepsilon}{\pi^2}$ such that $\|\varpi(t) - \vartheta(t)\|_{\mathcal{C}(\mathfrak{J},\mathfrak{c}_0^\delta)} < \eta$. Then

$$\begin{aligned} \|(f\varpi)(t) - (f\vartheta)(t)\|_{\mathfrak{c}_0^\delta} &= \sup_{k\in\mathbb{N}}\Big\{\delta_k\big|f_k(t,\varpi(t)) - f_k(t,\vartheta(t))\big|\Big\} \\ &= \sup_{k\in\mathbb{N}}\left\{\frac{1}{k^2}\sum_{n=k}^{\infty}\frac{1}{6n^2}\big|\varpi_k(t) - \vartheta_k(t)\big|\right\} \\ &\leq \frac{\pi^2}{36}\|\varpi_k(t) - \vartheta_k(t)\|_{\mathfrak{c}_0^\delta} < \varepsilon. \end{aligned}$$

Thus, $((f\varpi)(t))_{t\in\mathfrak{J}}$ is equicontinuous on $\mathfrak{c}_0^\delta$. Also, for $t \in \mathfrak{J}$ and $k \in \mathbb{N}$, we have

$$\begin{aligned} |f_k(t,\varpi(t))| &\leq \frac{e^{-kt^2}|\sin(kt)|}{k} + \sum_{n=k}^{\infty}\frac{1}{n^2}|\varpi_k(t)| \\ &\leq \frac{e^{-kt^2}}{k} + \frac{\pi^2}{36}|\varpi_k(t)|, \end{aligned}$$

where $\mathfrak{p}_k(t) = \dfrac{e^{-kt^2}}{k}$ and $\mathfrak{q}_k(t) = \dfrac{\pi^2}{36}$. So, $\mathfrak{q}^* = \dfrac{\pi^2}{36}$. We note that $(\delta\mathfrak{p}_k(t)) = \left(e^{-kt^2}/k^3\right)$ converges uniformly to zero on $\mathfrak{J}$ and the sequence $\mathfrak{q}_k(t)$ is equibounded on $\mathfrak{J}$. Also,

$$\frac{\mathfrak{q}^*\mathfrak{T}^{2\varrho}}{(2\varrho-1)\Gamma^2(\varrho)} = \frac{5\pi^2}{108(\Gamma(4/5))^2} < 1.$$

Hence, by Theorem 3.1 the infinite systems (5.1) has a solution in $\mathcal{C}(\mathfrak{J}, \mathfrak{c}_0^{\delta})$ provided

$$\lambda < \max_{t\in[0,1]} \frac{1}{t^2+1} = 1.$$

Example 5.2. Consider the infinite systems of regular fractional Sturm-Liouville boundary value problems.

$$\begin{cases} {}^{\mathcal{C}}\mathfrak{D}^{3/5}_{1/2^-}\Big[(\sqrt{t^2+2})\big({}^{\mathcal{C}}\mathfrak{D}^{3/5}_{0^+}(\varpi_k(t))\big)\Big] = \lambda\left[\dfrac{\cos(kt)}{k} + \displaystyle\sum_{n=k}^{\infty}\frac{\varpi_k(t)}{6n^2}\right] \\ \varpi_k(0) = 0,\ {}^{\mathcal{C}}\mathfrak{D}^{3/5}_{0^+}\varpi_k(1) = 0,\ k\in\mathbb{N}. \end{cases} \tag{5.2}$$

So, $\varrho = \frac{3}{5}$, $p_k(t) = \sqrt{t^2+2}$, $\mathfrak{T} = \frac{1}{2}$, $\mathfrak{J} = [0, \frac{1}{2}]$, and $f_k(t, \varpi(t)) = \frac{\cos(kt)}{k} + \sum_{n=k}^{\infty}\frac{\varpi_k(t)}{2n^2}$.
Let $\delta_k = \frac{1}{k^2}$ for all $k\in\mathbb{N}$. Now, for $\varpi(t)\in\mathcal{C}(\mathfrak{J}, \mathfrak{c}^{\delta})$, we have

$$\lim_{k\to\infty}\delta_k f_k(t,\varpi(t)) = \lim_{k\to\infty}\left[\frac{\cos(kt)}{k^3} + \frac{1}{k^2}\sum_{n=k}^{\infty}\frac{\varpi_k(t)}{2n^2}\right] = 0 \text{ (finite limit)}.$$

This shows that $\big(f_k(t,\varpi(t))\big)\in\mathfrak{c}^{\delta}$. Next, let for any $\vartheta(t) = \big(\vartheta_k(t)\big)_{k=1}^{\infty}\in\mathcal{C}(\mathfrak{J},\mathfrak{c}^{\delta})$. Let $\varepsilon>0$ be given and $\eta = \dfrac{12\varepsilon}{\pi^2}$ such that $\|\varpi(t)-\vartheta(t)\|_{\mathcal{C}(\mathfrak{J},\mathfrak{c}^{\delta})} < \eta$. Then

$$\begin{aligned} \|(f\varpi)(t) - (f\vartheta)(t)\|_{\mathfrak{c}^{\delta}} &= \sup_{k\in\mathbb{N}}\Big\{\delta_k\big|f_k(t,\varpi(t)) - f_k(t,\vartheta(t))\big|\Big\} \\ &= \sup_{k\in\mathbb{N}}\left\{\frac{1}{k^2}\sum_{n=k}^{\infty}\frac{1}{2n^2}\big|\varpi_k(t)-\vartheta_k(t)\big|\right\} \\ &\le \frac{\pi^2}{12}\|\varpi_k(t)-\vartheta_k(t)\|_{\mathfrak{c}_0^{\delta}} < \varepsilon. \end{aligned}$$

Thus, $((f\varpi)(t))_{t\in\mathfrak{J}}$ is equicontinuous on $\mathfrak{c}^{\delta}$. Also, for $t\in\mathfrak{J}$ and $k\in\mathbb{N}$, we have $\mathfrak{p}_k(t) = \dfrac{\cos kt}{k}$ and $\mathfrak{q}_k(t) = \dfrac{\pi^2}{12}$. So, $\mathfrak{q}^* = \dfrac{\pi^2}{12}$. We note that $(\delta\mathfrak{p}_k(t)) = \big(\cos(kt)/k^3\big)$ converges uniformly to zero on $\mathfrak{J}$ and the sequence $\mathfrak{q}_k(t)$ is convergent on $\mathfrak{J}$. Also,

$$\frac{\mathfrak{q}^*\mathfrak{T}^{2\varrho}}{(2\varrho-1)\Gamma^2(\varrho)} = \frac{5\pi^2}{12\times(64)^{1/5}(\Gamma(3/5))^2} < 1.$$

Hence, by Theorem 4.1 the infinite systems (5.2) has a solution in $\mathcal{C}(\mathfrak{J},\mathfrak{c}^{\delta})$ provided

$$\lambda < \max_{t\in[0,1/2]}\frac{1}{\sqrt{t^2+2}} = \frac{1}{\sqrt{2}}.$$

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