Lopsided modified Euler-extrapolated Hermitian and skew-Hermitian splitting method for a class of complex symmetric linear systems

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Abstract

In this paper, a lopsided modified Euler-extrapolated Hermitian and skew-Hermitian splitting (LME-HS) iteration method is introduced for solving the complex symmetric linear systems. Under a loose restriction on parameter θ, we demonstrate that LME-HS iteration method is convergent. Moreover, we present the optimal parameter θ^* of the LME-HS method and discuss the spectral properties of corresponding preconditioned matrix. Finally, the numerical experiments are used to verify the effectiveness of the proposed method.

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1 Introduction

In this paper, we consider the system of linear equations of the form

$$Ax \equiv (W + iT)x = b, \quad \textit{and} \quad x, b \in \mathbb{C}^n, \tag{1.1}$$

where $A \in \mathbb{C}^{n\times n}$ is a complex symmetric matrix with $W \in \mathbb{R}^{n\times n}$ being symmetric positive definite and $T \in \mathbb{R}^{n\times n}$ being symmetric positive semi-definite. Here and in the sequel, we use $i = \sqrt{-1}$ to denote the imaginary unit. We suppose $W, T \neq 0$, which means that A is non-Hermitian.

Such linear systems can be widely used in scientific calculation and engineering applications, including diffuse optical tomography [1], quantum mechanics [2], molecular scattering [3], structural dynamics [4]. For more examples and additional references, we refer to [5, 6, 7].

In recent years, many effective iteration methods for solving this kind of complex symmetric linear systems have been presented. For example, Bai et al. proposed the Hermitian and skew-Hermitian splitting (HSS) iteration method [8]. See [9, 10, 11, 12, 13, 14, 15] for more details about the HSS iteration method and its variants. To avoid the complex arithmetic, the modified HSS (MHSS) iteration method [16] and the preconditioned MHSS (PMHSS) method [17] are introduced by Bai et al. Furthermore, Li et al. designed the lopsided PMHSS (LPMHSS) iteration method [18] based on LHSS method [19]. Besides, Pour et al. proposed the new HSS (NHSS) iteration method [20] for non-Hermitian positive-definite linear systems. However, notice that all the iteration schemes mentioned above are two-step methods. Recently, Gu et al. designed the Quasi-minimal residual variants of the COCG, COCR and SCBiCG methods [26, 27] for complex symmetric linear systems in electromagnetic simulations. Salkuyeh et al. defined the generalized successive overrelaxation (GSOR) method [25] to solve the equivalent real system. Li et al. proposed the

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single-step HSS (SHSS) iteration method [21] to solve the non-Hermitian positive definite linear systems. Based on the SHSS iteration method, Zeng et al. introduced a parameterized variant of the SHSS (PSHSS) iteration method [22]. Li et al. proposed an Euler-extrapolated Hermitian/skew-Hermitian splitting(E-HS) iteration method [23]. In this work, based on the ideas of [23], we establish a new approach named as the lopsided modified E-HS (LME-HS) iteration method to solve the complex symmetric linear system (1.1).

The rest of this paper is organized as follows. In Section 2, the LME-HS iteration method is described. In Section 3, we analyze the convergence properties, including the choices of iterative optimal parameter and the spectral properties of the corresponding preconditioned matrix. Numerical results are presented in Section 4 to demonstrate the effectiveness of this LME-HS iteration method.

2 The LME-HS iteration method

In order to improve the computational efficiency and make the eigenvalue of the preconditioned iterative matrix of the system (1.1) closer to the real number, firstly, we multiply the formula $-ie^{i\theta} = \sin(\theta) - i\cos(\theta)$ by the two sides of the system (1.1) yields

$$((\sin(\theta)W + \cos(\theta)T) + i(\sin(\theta)T - \cos(\theta)W))x = -ie^{i\theta}b, \tag{2.1}$$

where $\theta \in (0, \frac{\pi}{2})$. Thus, by (2.1), we present the lopsided modified Euler-extrapolated Hermitian and skew-Hermitian splitting (LME-HS) iteration method. The LME-HS iteration method is described as follows:

Algorithm 2.1 (The LME-HS Method). Let $x^{(0)} \in \mathbb{C}^n$ be an arbitrary initial guess and $\theta \in (0, \frac{\pi}{2})$. For $k = 0, 1, 2, \cdots$ until the sequence of iterates $\{x^k\}_{k=0}^{\infty} \subset \mathbb{C}^n$ converges, compute the next iterate $x^{(k+1)}$ according to the following procedure:

$$\begin{cases} Wx^{(k+\frac{1}{2})} = -iTx^{(k)} + b, \\ (\sin(\theta)W + \cos(\theta)T)x^{(k+1)} = i(\cos(\theta)W - \sin(\theta)T)x^{(k+\frac{1}{2})} - ie^{i\theta}b. \end{cases} \tag{2.2}$$

As $W \in \mathbb{R}^{n\times n}$ is symmetric positive definite, $T \in \mathbb{R}^{n\times n}$ is symmetric positive semi-definite, and $\theta \in (0, \frac{\pi}{2})$, it is easy to observe that the matrix $\sin(\theta)W + \cos(\theta)T$ is symmetric positive definite. Therefore, the two linear sub-systems in (2.2) can also be effectively solved by either a sparse Cholesky factorization accurately or preconditioned conjugate gradient scheme inaccurately.

After simple derivation, we can restate the LME-HS iteration scheme into the standard form

$$x^{(k+1)} = Q(\theta)x^{(k)} + R(\theta)b, \quad k = 0, 1, 2, \cdots,$$

where

$$Q(\theta) = (\sin(\theta)W + \cos(\theta)T)^{-1}(\cos(\theta)W - \sin(\theta)T)W^{-1}T$$

and

$$R(\theta) = \sin(\theta)(\sin(\theta)W + \cos(\theta)T)^{-1}(I - iTW^{-1}).$$

Note that $Q(\theta)$ is the iteration matrix of the LME-HS method. Furthermore, if we introduce

matrices

$$B(\theta) = \frac{1}{\sin(\theta)}(I - iTW^{-1})^{-1}(\sin(\theta)W + \cos(\theta)T),$$
$$C(\theta) = \frac{1}{\sin(\theta)}(I - iTW^{-1})^{-1}(\cos(\theta)W - \sin(\theta)T)W^{-1}T,$$

then A holds that $A = B(\theta) - C(\theta)$, $Q(\theta)$ holds that $Q(\theta) = B(\theta)^{-1}C(\theta)$. Hence the LME-HS iteration scheme is derived by using the method of matrix splitting $A = B(\theta) - C(\theta)$. Then the matrix $B(\theta)$ can be regarded as a preconditioner to the complex coefficient matrix $A \in \mathbb{C}^{n\times n}$. We call it the LME-HS preconditioner.

3 Convergence analysis and preconditioning properties

Next, we discuss the convergence and preconditioning properties of the LME-HS iteration method. The spectral radius of the method is proposed and optimal parameters which minimize the spectral radius are also given.

Lemma 3.1 Let $W, T \in \mathbb{R}^{n\times n}$ be symmetric positive definite and symmetric positive semi-definite, respectively. Assume that $WT = TW$, then $W^{-1}T$ is also symmetric and its eigenvalues are non-negative.

Proof. Since W is symmetric positive definite, then W^{-1} is also symmetric positive definite. Notice that $WT = TW$, it is easy to know that $T = W^{-1}TW$. So we have $(W^{-1}T)^T = T^T(W^{-1})^T = TW^{-1} = (W^{-1}TW)W^{-1} = W^{-1}T$, i.e. $W^{-1}T$ is symmetric. Note that T is symmetric positive semi-definite, then $W^{-1}T$ is at least symmetric positive semi-definite. Thus, the eigenvalues of $W^{-1}T$ are non-negative. The proof is completed. Q.E.D.

Theorem 3.1. Let $A = W + iT \in \mathbb{C}^{n\times n}$ with $W, T \in \mathbb{R}^{n\times n}$ being symmetric positive definite and symmetric positive semi-definite, respectively. If $WT = TW$ and $\theta \in (0, \frac{\pi}{2})$. Then the spectral radius of the iteration matrix $Q(\theta)$ is

$$\rho(Q(\theta)) = \begin{cases} \rho_1, & \mu_0 \le \mu_{\min} \text{ or } \mu_0 \ge \mu_{\max}, \\ \mu_0^2, & \mu_{\min} \le \mu_0 \le \mu_{\max}, \end{cases} \tag{3.1}$$

where $\rho_1 = \max\left\{\left|\frac{\mu_{\min}-\tan(\theta)\mu_{\min}^2}{\tan(\theta)+\mu_{\min}}\right|, \left|\frac{\mu_{\max}-\tan(\theta)\mu_{\max}^2}{\tan(\theta)+\mu_{\max}}\right|\right\}$, $\mu_0 = \frac{1-\sin(\theta)}{\cos(\theta)}$. And $\mu_{\min}$ and $\mu_{\max}$ are the smallest and largest eigenvalues of the matrix $W^{-1}T$, respectively.

Proof. We first introduce the notation

$$\widetilde{T} = W^{-1}T.$$

It follows from Lemma 3.1 that $\widetilde{T}$ is a symmetric and positive semi-definite matrix. Then there exist an orthogonal matrix $U \in \mathbb{C}^{n\times n}$ and a non-negative diagonal matrix $D = diag(\mu_1, \mu_2, \ldots, \mu_n) \in \mathbb{R}^{n\times n}$ such that $\widetilde{T} = U^T DU$, where $\mu_j \ge 0$ $(j = 1, 2, \cdots, n)$ are the eigenvalues of the matrix $\widetilde{T}$.

Thus, we have

$$\begin{aligned}\rho(Q(\theta)) &= \rho((\sin(\theta)W+\cos(\theta)T)^{-1}(\cos(\theta)W-\sin(\theta)T)W^{-1}T)\\ &= \rho((\sin(\theta)I+\cos(\theta)\widetilde{T})^{-1}(\cos(\theta)I-\sin(\theta)\widetilde{T})\widetilde{T})\\ &= \max_{\mu_j\in\sigma(\widetilde{T})}\left|\frac{\cos(\theta)\mu_j-\sin(\theta)\mu_j^2}{\sin(\theta)+\cos(\theta)\mu_j}\right|\\ &= \max_{\mu_j\in\sigma(\widetilde{T})}\left|\frac{\mu_j-\tan(\theta)\mu_j^2}{\tan(\theta)+\mu_j}\right|,\end{aligned}$$

where $\sigma(\widetilde{T})$ denotes the spectrum of the matrix $\widetilde{T}$. Next, we use the function

$$f(\mu_j)=\frac{\mu_j-\tan(\theta)\mu_j^2}{\tan(\theta)+\mu_j}.$$

Then

$$f'(\mu_j)=\frac{-\tan(\theta)\mu_j^2-2\tan^2(\theta)\mu_j+\tan(\theta)}{(\tan(\theta)+\mu_j)^2}.$$

Let $f'(u_j)=0$, then we get $\mu_j=\frac{-\sin(\theta)\pm 1}{\cos(\theta)}$. And since $\mu_j\geq 0$, then we take $\mu_0=\frac{1-\sin(\theta)}{\cos(\theta)}$. Futhermore, let us do the following discussion:

case 1) if $0\leq\mu_{\min}\leq\mu_{\max}\leq\mu_0$, $f(\mu_j)$ is monotone increasing with regard to μ_j, then we have

$$f(\mu_j)_{\min}=\frac{\mu_{\min}-\tan(\theta)\mu_{\min}^2}{\tan(\theta)+\mu_{\min}},\quad f(\mu_j)_{\max}=\frac{\mu_{\max}-\tan(\theta)\mu_{\max}^2}{\tan(\theta)+\mu_{\max}},$$

therefore

$$\rho(Q(\theta))=\max|f(\mu_j)|=\max\left\{\left|\frac{\mu_{\min}-\tan(\theta)\mu_{\min}^2}{\tan(\theta)+\mu_{\min}}\right|,\left|\frac{\mu_{\max}-\tan(\theta)\mu_{\max}^2}{\tan(\theta)+\mu_{\max}}\right|\right\}.$$

case 2) if $0\leq\mu_{\min}\leq\mu_0\leq\mu_{\max}$, we have

$$\begin{aligned}f(\mu_j)_{\min}&=\min\left\{\frac{\mu_{\min}-\tan(\theta)\mu_{\min}^2}{\tan(\theta)+\mu_{\min}},\frac{\mu_{\max}-\tan(\theta)\mu_{\max}^2}{\tan(\theta)+\mu_{\max}}\right\},\\ f(\mu_j)_{\max}&=f(\mu_0)=\left(\frac{1-\sin(\theta)}{\cos(\theta)}\right)^2,\end{aligned}$$

therefore

$$\rho(Q(\theta))=\max\left\{\left|\frac{\mu_{\min}-\tan(\theta)\mu_{\min}^2}{\tan(\theta)+\mu_{\min}}\right|,\left|\frac{\mu_{\max}-\tan(\theta)\mu_{\max}^2}{\tan(\theta)+\mu_{\max}}\right|,\left(\frac{1-\sin(\theta)}{\cos(\theta)}\right)^2\right\}.$$

case 3) if $\mu_{\min}\geq\mu_0$, the situation is similar to *case* 1), we get

$$\rho(Q(\theta))=\max|f(\mu_j)|=\max\left\{\left|\frac{\mu_{\min}-\tan(\theta)\mu_{\min}^2}{\tan(\theta)+\mu_{\min}}\right|,\left|\frac{\mu_{\max}-\tan(\theta)\mu_{\max}^2}{\tan(\theta)+\mu_{\max}}\right|\right\}.$$

To sum up, the proof is completed. Q.E.D.

Theorem 3.2. Under the conditions of Theorem 3.1, the LME-HS method is convergent if the parameter θ satisfies

1) when $\mu_{\max} > 1$, $\cot(\theta) > \frac{\mu_{\max}^2-1}{2\mu_{\max}}$ with $\theta \in (0, \frac{\pi}{2})$;
2) when $\mu_{\max} \leq 1$, $\theta \in (0, \frac{\pi}{2})$.

Proof. The LME-HS method is convergent if and only if $\rho(Q(\theta)) < 1$, which is equal to

$$-1 < \frac{\mu_j - \tan(\theta)\mu_j^2}{\tan(\theta) + \mu_j} < 1$$

or equally $\tan(\theta)\mu_j^2 - 2\mu_j - \tan(\theta) < 0$, namely $\cot(\theta) > \frac{\mu_j^2-1}{2\mu_j}$ with $\mu_j > 0$ (if $\mu_j = 0$, note that $\theta \in (0, \frac{\pi}{2})$, obviously, $\cot(\theta) > 0$). Notice that $g(\mu_j) = \frac{\mu_j^2-1}{2\mu_j}$ is monotone increasing with regard to $\mu_j > 0$, then we get

1) if $\mu_{\max} > 1$, then $\cot(\theta) > \frac{\mu_{\max}^2-1}{2\mu_{\max}}$ with $\theta \in (0, \frac{\pi}{2})$;
2) if $\mu_{\max} \leq 1$, then $\cot(\theta) > 0$ with $\theta \in (0, \frac{\pi}{2})$.

Therefore, the proof is completed. Q.E.D.

Corollary 3.1. Let the conditions of Theorem 3.1 be satisfied. Then, the optimal parameter θ^* minimizing the spectral radius $\rho(Q(\theta))$ is

$$\theta^* = \left\{ \arctan\left(\frac{-(\mu_{\min}^2\mu_{\max} - \mu_{\min} - \mu_{\max} + \mu_{\min}\mu_{\max}^2) + \sqrt{d}}{2(\mu_{\min}^2 + \mu_{\max}^2)} \right) \in (0, \frac{\pi}{2}) \right\} \tag{3.2}$$

where $d = (\mu_{\min}^2\mu_{\max} - \mu_{\min} - \mu_{\max} + \mu_{\min}\mu_{\max}^2)^2 + 8\mu_{\min}\mu_{\max}(\mu_{\min}^2 + \mu_{\max}^2)$, and the corresponding optimal method of the LME-HS method is given by

$$\frac{\mu_{\min} - \tan(\theta^*)\mu_{\min}^2}{\tan(\theta^*) + \mu_{\min}} = \frac{\tan(\theta^*)\mu_{\max}^2 - \mu_{\max}}{\tan(\theta^*) + \mu_{\max}}. \tag{3.3}$$

Proof. To minimize the spectral radius $\rho(Q(\theta))$, we discuss the following situation.
1) When $\mu_{\min} \leq \mu_0 \leq \mu_{\max}$, the optimal paremeter θ^* must satisfy either of the following three conditions:

(a) $\left|\frac{\mu_{\min}-\tan(\theta^*)\mu_{\min}^2}{\tan(\theta^*)+\mu_{\min}}\right| = \left|\frac{\mu_{\max}-\tan(\theta^*)\mu_{\max}^2}{\tan(\theta^*)+\mu_{\max}}\right| \geq (\frac{1-\sin(\theta^*)}{\cos(\theta^*)})^2$,

(b) $\left|\frac{\mu_{\max}-\tan(\theta^*)\mu_{\max}^2}{\tan(\theta^*)+\mu_{\max}}\right| = (\frac{1-\sin(\theta^*)}{\cos(\theta^*)})^2 \geq \left|\frac{\mu_{\min}-\tan(\theta^*)\mu_{\min}^2}{\tan(\theta^*)+\mu_{\min}}\right|$,

(c) $(\frac{1-\sin(\theta^*)}{\cos(\theta^*)})^2 = \left|\frac{\mu_{\min}-\tan(\theta^*)\mu_{\min}^2}{\tan(\theta^*)+\mu_{\min}}\right| \geq \left|\frac{\mu_{\max}-\tan(\theta^*)\mu_{\max}^2}{\tan(\theta^*)+\mu_{\max}}\right|$.

After a simple analysis, we can see that

$$\frac{\mu_{\min} - \tan(\theta^*)\mu_{\min}^2}{\tan(\theta^*) + \mu_{\min}} = \frac{\tan(\theta^*)\mu_{\max}^2 - \mu_{\max}}{\tan(\theta^*) + \mu_{\max}}.$$

2) When $\mu_0 \leq \mu_{\min}$ or $\mu_0 \geq \mu_{\max}$, it's easy for us to know $f(\mu_j) = \frac{\mu_j - \tan(\theta)\mu_j^2}{\tan(\theta)+\mu_j}$ is monotone. Thus,

$$\rho(Q(\theta)) = \max|f(\mu_j)| = \max\left\{ \left|\frac{\mu_{\min} - \tan(\theta)\mu_{\min}^2}{\tan(\theta) + \mu_{\min}}\right|, \left|\frac{\mu_{\max} - \tan(\theta)\mu_{\max}^2}{\tan(\theta) + \mu_{\max}}\right| \right\}.$$

Note that $f(\theta) = \frac{\mu_j - \tan(\theta)\mu_j^2}{\tan(\theta)+\mu_j}$ is monotone, hence the optimal parameter θ^* can also satisfy equality (3.3). After a simple calculation, we get the result of Corollary 3.1. Q.E.D.

Theorem 3.3. Under the assumptions of Theorem 3.1 and Corollary 3.1, the eigenvalues of the matrix $B(\theta)^{-1}A$ are

$$\beta_j = 1 + \frac{\tan(\theta)\mu_j^2 - \mu_j}{\tan(\theta) + \mu_j}, \quad j = 1, 2, \cdots, n. \tag{3.4}$$

where μ_j is an arbitrary eigenvalue of the matrix $W^{-1}T$.

Proof. According to Theorem 3.1, it is easy to get the corresponding results. Q.E.D.

Remark 3.1 For the optimal parameter θ^* of the LME-HS preconditioner, the eigenvalues (3.4) of the preconditioned matrix $B(\theta)^{-1}A$ are real numbers, which is different from the similar results of the previous classical algorithms [16, 18, 23].

4 Numerical experiments

Next, we experiment the efficiency of LME-HS iteration method as a solver for solving complex symmetric linear system (1.1) by comparing the number of iterations (denoted as IT) and in terms of the CPU times (in seconds, denoted as CPU) of LME-HS method with those of PMHSS, GSOR and E-HS methods. In our implementations, we choose $x^{(0)} = \mathbf{0}$ as an initial guess and the iteration is terminated once the current iterate $x^{(k)}$ satisfies $\frac{||b-Ax^{(k)}||_2}{||b||_2} \leq 10^{-6}$. The iteration parameters used in PMHSS, GSOR, E-HS and LME-HS methods are the experimental optimal ones α^* and θ^*, respectively. These parameters minimize the numbers of iteration steps and iteration time. Moreover, the sparse Cholesky factorization is used to solve the linear subsystems involved in each step of the PMHSS, GSOR, E-HS and LME-HS iteration methods. Furthermore, all codes and experiments were run in MATLAB 2016b.

Example 1 (See [7],[24]). Consider the following complex Helmholtz equation describing the propagation of harmonics with damping,

$$\Delta u + \sigma_1 u + i\sigma_2 u = f,$$

where σ_1 and σ_2 are real coefficient functions, which describes the propagation of damped time-harmonic waves. u satisfies Dirichlet boundary conditions in $D = [0, 1] \times [0, 1]$. We take K the five-point centered difference matrix approximating the negative Laplacian operator on an uniform mesh $h = \frac{1}{m+1}$, where m is the number of discretization points in each direction. The matrix $K \in \mathbb{R}^{n\times n}$ possesses the tensor-product form $K = I \otimes D_m + D_m \otimes I$, with $D_m = h^{-2}tridiag(-1, 2, -1) \in \mathbb{R}^{m\times m}$. Therefore, K is an $n \times n$ block-tridiagonal matrix with $n = m^2$. This results in a complex symmetric linear system (1.1) in the form $[(K + \sigma_1 I) + i\sigma_2 I]x = b$. Here, $A = [(K + \sigma_1 I) + i\sigma_2 I]$. Furthermore, we take the right-hand side vector b to be $b = (1 + i)A\mathbf{1}$, with $\mathbf{1}$ being the vector of all entries equal to 1. Moreover, we also standardize the system by multiplying both parties by h^2.

For Example 1, when $\sigma_1 = 1$, $\sigma_2 = 10$, it can be seen from the results of Table 1 that the LME-HS iteration method has been better than the PMHSS and GSOR iteration methods in terms of iteration numbers and CPU times. For small scale problems, the CPU times of the LME-HS

Table 1
Numerical results of PMHSS, GSOR, E-HS and LME-HS methods for Example 1 when $\sigma_1 = 1$, $\sigma_2 = 10$.

Methods	$m \times m$	8×8	16×16	32×32	64×64	128×128
PMHSS	α^*	0.860	0.960	0.960	0.960	0.960
	IT	35	38	40	40	40
	CPU	0.0069	0.0272	0.1048	0.4865	3.0578
GSOR	α^*	0.947	0.948	0.948	0.948	0.948
	IT	6	6	5	5	5
	CPU	0.0042	0.0046	0.0127	0.0741	0.3566
E-HS	θ^*	0.235	0.227	0.210	0.190	0.150
	IT	9	9	9	9	8
	CPU	0.0058	0.0050	0.0131	0.0602	0.3427
LME-HS	θ^*	1.150	1.125	1.122	1.122	1.122
	IT	4	4	4	4	3
	CPU	0.0018	0.0032	0.0101	0.0535	0.2497

Table 2
Numerical results of PMHSS, GSOR, E-HS and LME-HS methods for Example 1 when $\sigma_1 = \sigma_2 = 100$.

Methods	$m \times m$	8×8	16×16	32×32	64×64	128×128
PMHSS	α^*	0.650	0.650	0.900	0.900	0.900
	IT	22	30	36	39	40
	CPU	0.0048	0.0206	0.919	0.4779	3.0373
GSOR	α^*	0.855	0.855	0.868	0.868	0.868
	IT	8	8	8	8	8
	CPU	0.0023	0.0056	0.0197	0.1158	0.6235
E-HS	θ^*	0.455	0.380	0.354	0.330	0.310
	IT	10	12	13	13	13
	CPU	0.0020	0.0053	0.0179	0.0889	0.5434
LME-HS	θ^*	0.959	0.910	0.886	0.975	0.975
	IT	6	6	6	5	5
	CPU	0.0019	0.0046	0.0149	0.0692	0.4037

Table 3
Numerical results of PMHSS, GSOR, E-HS and LME-HS methods for Example 2.

Methods	$m \times m$	8×8	16×16	32×32	64×64	128×128
PMHSS	α^*	0.800	0.800	0.800	0.800	0.800
	IT	28	34	37	38	38
	CPU	0.0110	0.0208	0.1099	0.4834	3.0661
GSOR	α^*	0.450	0.453	0.456	0.454	0.455
	IT	24	24	23	23	23
	CPU	0.0106	0.0230	0.0628	0.2719	1.4845
E-HS	θ^*	0.690	0.653	0.635	0.625	0.615
	IT	33	37	38	38	36
	CPU	0.0099	0.0202	0.0582	0.2508	1.4277
LME-HS	θ^*	0.400	0.400	0.400	0.400	0.415
	IT	15	14	13	12	10
	CPU	0.0094	0.0160	0.0395	0.1534	0.7575

method is very close to that of the E-HS method. Moreover, when the problem becomes larger scale ($m = 128$), the LME-HS method needs the least CPU times.

For Example 1, when $\sigma_1 = \sigma_2 = 100$, we can get that LME-HS method outperforms the PMHSS, GSOR and E-HS iteration methods from Table 2, because the LME-HS method needs less CPU

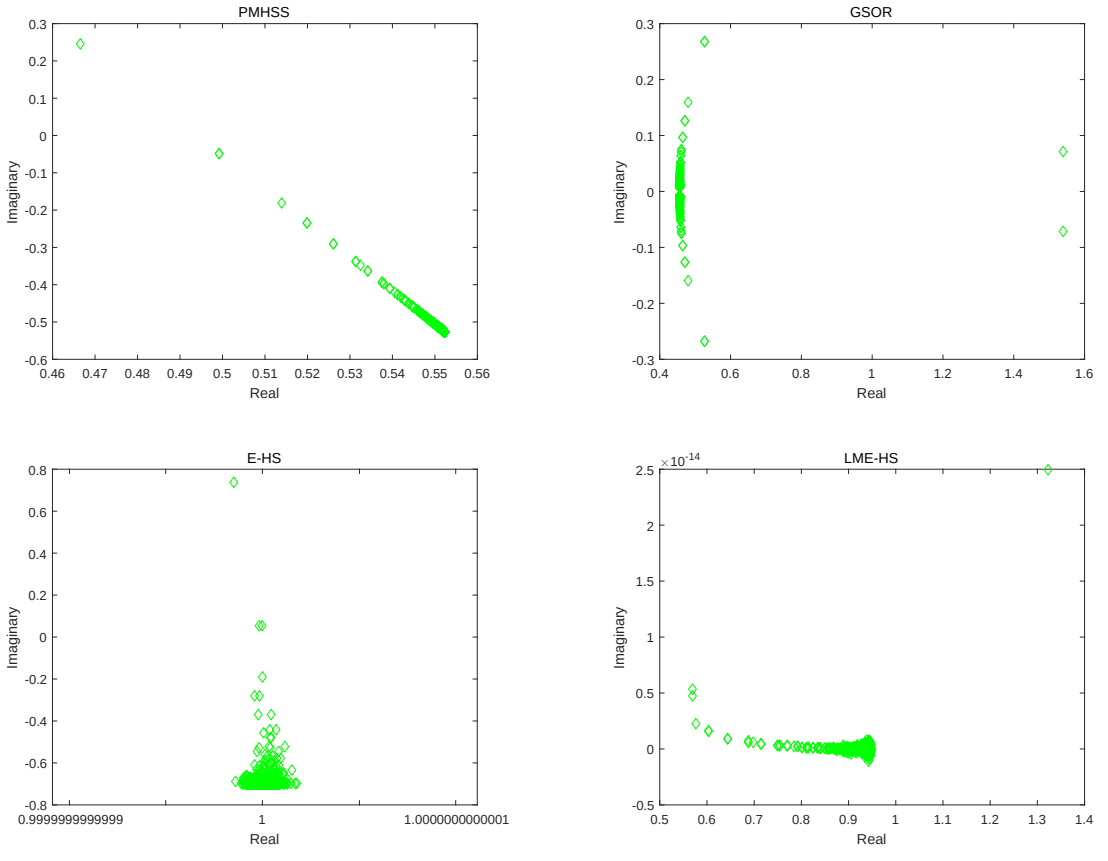


Fig.1. Eigenvalues distribution of the four preconditioned matrices for Example 2 with $m = 32$.

times. And the bigger the m, the better the effect.

Example 2 (See[5, 16]). The system of linear equations (1.1) is of the form

$$[(-\omega^2 M + K) + i(\omega C_V + C_H)]x = b,$$

where M and K are the inertia and the stiffness matrices, C_V and C_H are the viscous and the hysteretic damping matrices, respectively, and ω is the driving circular frequency. We take $C_V = 10I, M = I, C_H = \mu K$ with μ a damping coefficient, and $K \in \mathbb{R}^{n\times n}$ possesses the tensor-product form $K = I \otimes D_m + D_m \otimes I$, with $D_m = h^{-2} tridiag(-1, 2, -1) \in \mathbb{R}^{m\times m}$. Then $A = [(-\omega^2 M + K) + i(\omega C_V + C_H)]$. In addition, we set $n = m^2$, $\omega = \pi, \mu = 0.02$ and the right-hand side vector b to be $b = (1 + i)A\mathbf{1}$, with $\mathbf{1}$ being the vector of all entries equal to 1.

For Example 2, it is easy to see from Table 3 that the LME-HS iteration method is more efficient than PMHSS, GSOR and E-HS iteration methods in terms of the iteration steps and CPU times.

As m increases, this advantage becomes more obvious.

Fig.1 shows eigenvalues distribution of the PMHSS preconditioned matrix, the GSOR preconditioned matrix, the E-HS preconditioned matrix and the LME-HS preconditioned matrix when $m = 32$. From Fig.1, we can also see that the eigenvalues of the LME-HS preconditioned matrix $B(\alpha, \theta)^{-1}A$ are close to real numbers, although there may be some small error accumulation effects here. Thus, the LME-HS method may be more efficient than the PMHSS, E-HS and GSOR methods in terms of the iteration steps and CPU times.

5 Conclusions

In this paper, the LME-HS method for solving a class of complex symmetric linear systems was introduced. The theoretical analysis shows that for any initial guess and a loosen restriction of parameters θ, the LME-HS method converges to the unique solution of the complex linear system (1.1). The convergence of the LME-HS method and the optimal parameters are analyzed in detail. In addition, numerical examples show that CPU times of LME-HS method are less than the results of the PMHSS, GSOR and E-HS methods.

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