Hyers-Ulam stability quintic functional equation in F-spaces: direct method

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Abstract

In this paper, we establish the Hyers-Ulam stability of quintic functional equation in F-spaces by using direct method.

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1 Introduction

Very often instead of a functional equation, we consider a functional inequality and one can ask the following question: When can one assert that the solutions of the inequality lie near to the solutions of the equation? A definition of stability in the case of homomorphisms between groups was suggested by a problem posed by S.M. Ulam [15] in 1940: Let $(G_1, *)$ be a group and let $(G_2, \diamond, d)$ be a metric group with the metric $d(.,.)$. Given $\varepsilon > 0$, does there exists a $\delta(\varepsilon) > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality $d(h(x * y), h(x) \diamond h(y)) < \delta$ for all $x, y \in G_1$, then there is a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \varepsilon$ for all $x \in G_1$? In other words, if a mapping is almost homomorphism then there is a true homomorphism near it with a small error as much as possible. If the answer is affirmative, we would call that the equation $H(x * y) = H(x) \diamond H(y)$ of homomorphism is stable. One year later, Hyers [6] provided a positive partial answer to Ulam's problem. In 1978, a generalized version of Hyer's result was proved by Rassias in [9]. Since then, the stability problems of several functional equations quadratic [5], cubic ([4], [7], [13]), quartic ([2], [14]), quintic ([17]), sextic ([17]), septic and octic ([18]), nonic ([3], [10]), decic ([1]), undecic ([11]) and quattuordecic ([12]) have been investigated by a number of authors with more general domains and co-domains.

In 2016, Xiuzhoung Yang [16] found the stability results for the quadratic functional equation and also orthogonal stability of pexiderixed quadratic functional equation in F-spaces.
Let us recall the basic definition of the F-Spaces.

Definition 1.1. [16] Let X be a linear space over $\mathbb{K}$ that denotes either complex or real numbers. A non-negative valued function $\|.\|$ defined on X is called F-norm if it satisfies the following conditions:
(1) $\|x\| = 0$ if and only if $x = 0$.
(2) $\|ax\| = \|x\|$ for all $a \in \mathbb{K}$, $|a| = 1$.
(3) $\|x + y\| \leq \|x\| + \|y\|$.
(4) $\|a_n x\| \to 0$ provided $a_n \to 0$.
(5) $\|ax_n\| \to 0$ provided $x_n \to 0$.
(6) $\|a_n x_n\| \to 0$ provided $a_n \to 0$, $x_n \to 0$.

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A linear space equipped with F-norm is called F^*-space which will be denoted by $(X, \|.\|)$ or X. A complete F^*-space is called F-space.

Definition 1.2. [16] Let X be a linear space over $\mathbb{K}$ that denotes either complex or real numbers and $0 < \beta \leq 1$. A non-negative valued function $\|.\|$ defined on X is called β-norm if it satisfies the following conditions:
(1) $\|x\| = 0$ if and only if $x = 0$.
(2) $\|ax\| = |a|^\beta \|x\|$ for all $a \in \mathbb{K}$.
(3) $\|x + y\| \leq \|x\| + \|y\|$.
A linear space equipped with β-norm is called β-normed space, which will be denoted by $(X, \|.\|)$ or briefly X.

2 Stability of quintic functional equation in F-spaces

In this section, we prove the Hyers-Ulam Stability of following Quintic Functional Equation

$$2f(2x + y) + 2f(2x - y) + f(x + 2y) + f(x - 2y) = 20f(x + y) + 20f(x - y) + 90f(x) \tag{2.1}$$

in F-Spaces by using Direct Method. The function $f(x) = x^5$ satisfies the functional equation (2.1), which is called a quintic functional equation and each solution of this equation is called a quintic mapping.

Theorem 2.1. Let $\mathcal{X}$ be a real vector space and $\mathcal{Y}$ be F-space in which there exist $\frac{1}{2} \leq c < 1$ such that $\left\|\frac{y}{2}\right\| \leq c\|y\|$ for all $x \in \mathcal{Y}$, and let $\varphi : \mathcal{X} \times \mathcal{X} \to \mathbb{R}^+$ be a function such that

$$\sum_{i=0}^{\infty} c^{5i}\varphi(2^i x, 0) \tag{2.2}$$

converges and

$$\lim_{n\to\infty} c^{5n}\varphi(2^n x, 2^n y) = 0 \tag{2.3}$$

for all $x, y \in \mathcal{X}$. Suppose that f satisfies
$\|2f(2x + y) + 2f(2x - y) + f(x + 2y) + f(x - 2y) - 20f(x + y)$

$$-20f(x - y) - 90f(x)\| \leq \varphi(x, y) \tag{2.4}$$

for all $x, y \in \mathcal{X}$. Then there exist a unique quintic function $g : \mathcal{X} \to \mathcal{Y}$ which satisfies the equation (2.1) and the inequality

$$\|f(x) - g(x)\| \leq c^7 \sum_{i=0}^{\infty} c^{5i}\varphi(2^i x, 0), \quad for \quad all \quad x \in \mathcal{X}. \tag{2.5}$$

Function g is given by

$$g(x) = \lim_{n\to\infty} 2^{-5n} f(2^n x), \quad for \quad all \quad x \in \mathcal{X}. \tag{2.6}$$

Proof. Replace $y = 0$ in inequality (2.4), we get

$$\|4f(2x) - 128f(x)\| \leq \varphi(x, 0) \tag{2.7}$$

for all $x \in \mathcal{X}$. Divide by 128, we obtain

$$\left\|f(x) - 32^{-1}f(x)\right\| \leq c^7\varphi(x, 0) \tag{2.8}$$

for all $x \in \mathcal{X}$. Change x by $2x$ in inequality (2.8), we have

$$\left\|f(2x) - 32^{-1}f(2^2x)\right\| \leq c^7\varphi(2x, 0) \tag{2.9}$$

for all $x \in \mathcal{X}$. Multiply 32^{-1} on both sides of the above inequality, we get

$$\left\|32^{-1}f(2x) - 32^{-2}f(2^2x)\right\| \leq c^7c^4\varphi(2x, 0) \tag{2.10}$$

for all $x \in \mathcal{X}$. Now adding the inequality (2.8) and the inequality (2.10), we obtain
$\left\|f(x) - 32^{-1}f(2x)\right\| + \left\|32^{-1}f(2x) - 32^{-2}f(2^2x)\right\|$

$$\leq c^7\varphi(x, 0) + c^7c^5\varphi(2x, 0)$$

$$\Rightarrow \left\|f(x) - 32^{-2}f(2^2x)\right\| \leq c^7[\varphi(x, 0) + c^5\varphi(2x, 0)] \tag{2.11}$$

for all $x \in \mathcal{X}$. Change x by $2x$ in the inequality (2.10), we have

$$\left\|32^{-1}f(2^2x) - 32^{-2}f(2^3x)\right\| \leq c^7c^5\varphi(2^2x, 0)$$

for all $x \in \mathcal{X}$. Now multiply 32^{-1} on both sides of the above equation we get

$$\left\|32^{-2}f(2^2x) - 32^{-3}f(2^3x)\right\| \leq c^7c^{10}\varphi(2^2x, 0) \tag{2.12}$$

for all $x \in \mathcal{X}$. Now adding the inequality (2.8), (2.10), (2.12) we have
$\left\|f(x) - 32^{-1}f(2x)\right\| + \left\|32^{-1}f(2x) - 32^{-2}f(2^2x)\right\| + \left\|32^{-2}f(2^2x) - 32^{-3}f(2^3x)\right\|$

$$\leq c^7\varphi(x, 0) + c^7c^5\varphi(2x, 0) + c^7c^{10}\varphi(2^2x, 0)$$

$$\left\|f(x) - 32^{-3}f(2^3x)\right\| \leq c^7[\varphi(x, 0) + c^5\varphi(2x, 0) + c^{10}\varphi(2^2x, 0)] \tag{2.13}$$

for all $x \in \mathcal{X}$. By induction hypothesis on positive integer n, we obtain

$$\left\|f(x) - 32^{-n}f(2^nx)\right\| \leq c^7\sum_{i=0}^{n-1} c^{5i}\varphi(2^ix, 0) \tag{2.14}$$

for all $x \in \mathcal{X}$. We are going to prove that the sequence $\{32^{-n}f(2^nx)\}$ is convergence. Here we change x by 2^mx in the inequality (2.14), we have

$$\left\|f(2^mx) - 32^{-n}f(2^n2^mx)\right\| \leq c^7\sum_{i=0}^{n-1} c^{5i}\varphi(2^i2^mx, 0) \tag{2.15}$$

for all $m, n > 0$. Now multiply 32^{-m} on both sides of the above inequality (2.15), we get

$$\left\|32^{-m}f(2^m x) - 32^{-n}32^{-m}f(2^n 2^m x)\right\| \leq c^7 \sum_{i=0}^{n-1} c^{5i}c^{5m}\varphi(2^i 2^m x, 0)$$

$$\left\|32^{-(m+n)}f(2^{m+n}x) - 32^{-m}f(2^m x)\right\| \leq c^7 \sum_{i=0}^{n-1} c^{5(m+i)}\varphi(2^{m+i}x, 0)$$

$$\left\|32^{-(m+n)}f(2^{m+n}x) - 32^{-m}f(2^m x)\right\| \leq c^7 \sum_{i=m}^{n} c^{5i}\varphi(2^i x, 0) \tag{2.16}$$

The right hand side tends to 0 as n tends to infinity.
The sequence $\{32^{-n}f(2^n x)\}$ is a Cauchy sequence.
$\sum_{i=0}^{\infty} c^{5i}\varphi(2^i x, 0)$ is converges.
Given

$$\lim_{n\to\infty} c^{5n}\varphi(2^n x, 2^n y) = 0$$

for all $x \in \mathcal{X}$. Now we define the function

$$g(x) = \lim_{n\to\infty} 2^{-5n}f(2^n x)$$

for all $x \in \mathcal{X}$. Now taking the limit on both sides of the inequality (2.14), we have

$$\lim_{n\to\infty} \left\|f(x) - 32^{-n}f(2^n x)\right\| \leq \lim_{n\to\infty} c^7 \sum_{i=0}^{n-1} c^{5i}\varphi(2^i x, 0)$$

$$\Rightarrow \|f(x) - g(x)\| \leq c^7 \sum_{i=0}^{\infty} c^{5i}\varphi(2^i x, 0)$$

for all $x \in \mathcal{X}$. Now we are going to prove T satisfies the equation (2.1).
We change x by $2^n x$ and y by $2^n y$ in the inequality (2.4), we obtain
$\left\|2f(2^{n+1}x + 2^n y) + 2f(2^{n+1}x - 2^n y) + f(2^n x + 2^{n+1}y) + f(2^n x - 2^{n+1}y)\right.$

$$\left. -20f(2^n x + 2^n y) - 20f(2^n x - 2^n y) - 90f(2^n x)\right\| \leq \varphi(2^n x, 2^n y) \tag{2.17}$$

for all $x \in \mathcal{X}$. Multiply 32^{-n} on both sides of the inequality (2.17), we have
$\|32^{-n}[2f(2^n(2x + y)) + 2f(2^n(2x - y)) - f(2^n(x + 2y)) + f(2^n(x - 2y)$

$$-20f(2^n(x + y)) - 20f(2^n(x - y)) - 90f(2^n x)]\| \leq c^{5n}\varphi(2^n x, 2^n y) \tag{2.18}$$

Here taking limit as $n \to \infty$ on the above inequality, we find that g satisfies the equation (2.1) for all $x, y \in \mathcal{X}$.
$\lim_{n\to\infty} \|32^{-n}[2f(2^n(2x + y)) + 2f(2^n(2x - y)) - f(2^n(x + 2y)) + f(2^n(x - 2y)$

$$-20f(2^n(x + y)) - 20f(2^n(x - y)) - 90f(2^n x)]\| \leq \lim_{n\to\infty} c^{5n}\varphi(2^n x, 2^n y)$$

$$\|2g(2x+y)+2g(2x-y)+g(x+2y)+g(x-2y)-20g(x+y)$$
$$-20g(x-y)-90g(x)\| \leq \lim_{n\to\infty} c^{5n}\varphi(2^n x, 2^n y)$$

$$2g(2x+y)+2g(2x-y)+g(x+2y)+g(x-2y)-20g(x+y)-20g(x-y)-90g(x)=0$$

for all $x, y \in \mathcal{X}$. Now we are going to prove the uniqueness of a quintic function f subject to the inequality (2.5), let us assume that there exist a quintic function $S : \mathcal{X} \to \mathcal{Y}$ which satisfies the equation (2.1) and the inequality (2.5).

$$S(2^n x) = 32^n S(x) \quad and \quad T(2^n x) = 32^n T(x)$$

$$\begin{aligned}\|S(x)-T(x)\| &= \left\|32^{-n}S(2^n x) - 32^{-n}T(2^n x)\right\| \\ &= \left\|32^{-n}[S(2^n x)-f(2^n x)] + 32^{-n}[f(2^n x)-T(2^n x)]\right\| \\ &\leq c^7 \sum_{i=0}^{\infty} c^{5(n+i)}\varphi(2^{n+i}x, 0)\end{aligned}$$

for all $x \in \mathcal{X}$ and $n \in N$. Now taking $n \to \infty$ in the proceeding inequality, we obtain the uniqueness of g. Q.E.D.

Let us consider $\mathcal{X}$ be a real vector space and $\mathcal{Y}$ be a complete β- normed space in the following corollary.

Corollary 2.2. Let $\mathcal{X}$ be a real vector space and $\mathcal{Y}$ be a complete β-normed space $(0 < \beta \leq 1)$ and let $\varphi : \mathcal{X} \times \mathcal{X} \to \mathbb{R}^+$ be a function such that

$$\sum_{i=0}^{\infty} 32^{-\beta i}\varphi(2^i x, 0) \tag{2.19}$$

converges and

$$\lim_{n\to\infty} 32^{-\beta n}\varphi(2^n x, 2^n y) = 0 \tag{2.20}$$

for all $x, y \in \mathcal{X}$. Suppose that h satisfies
$\|2f(2x+y)+2f(2x-y)+f(x+2y)+f(x-2y)-20f(x+y)$

$$-20f(x-y)-90f(x)\| \leq \varphi(x, y) \tag{2.21}$$

for all $x, y \in \mathcal{X}$. Then there exist a unique quintic function $g : \mathcal{X} \to \mathcal{Y}$ which satisfies the equation (2.1) and the inequality

$$\|f(x)-g(x)\| \leq 128^{-\beta} \sum_{i=0}^{\infty} 32^{-\beta n}\varphi(2^i x, 0), \quad for \quad all \quad x \in \mathcal{X}. \tag{2.22}$$

Function g is given by

$$g(x) = \lim_{n\to\infty} 2^{-5n} f(2^n x), \quad for \quad all \quad x \in \mathcal{X}. \tag{2.23}$$

Proof. In this Corollary, we are replacing constant c by $2^{-\beta}$ and obtain the proof of the above. The proof of this Corollary can be followed by replacing constant c^7 by $128^{-\beta}$ and c^5 by $32^{-\beta}$ in the Theorem (2.1).
Replacing $y = 0$ in the inequality (2.21), we get
$\|2f(2x+0) + 2f(2x-0) + f(x+20) + f(x-20) - 20f(x+0) - 20f(x-0)$

$$-90f(x)\| \leq \varphi(x,0)$$

$$\Rightarrow \|4f(2x) - 128f(x)\| \leq \varphi(x,0) \tag{2.24}$$

for all $x \in \mathcal{X}$. Dividing by 128, we obtain

$$\left\|f(x) - 32^{-1}f(2x)\right\| \leq 128^{-\beta}\varphi(x,0) \tag{2.25}$$

for all $x \in \mathcal{X}$. Changing x by $2x$ in the inequality (2.25), we have

$$\left\|f(2x) - 32^{-1}f(2^2x)\right\| \leq 128^{-\beta}\varphi(2x,0) \tag{2.26}$$

for all $x \in \mathcal{X}$. Multiply 32^{-1} on both sides of the equation (2.26), we get

$$\left\|32^{-1}f(2x) - 32^{-2}f(2^2x)\right\| \leq 128^{-\beta}32^{-\beta}\varphi(2x,0) \tag{2.27}$$

for all $x \in \mathcal{X}$. Now adding the inequality (2.25) and the inequality (2.27), we obtain
$\left\|f(x) - 32^{-1}f(2x)\right\| + \left\|32^{-1}f(2x) - 32^{-2}f(2^2x)\right\|$

$$\leq 128^{-\beta}\varphi(x,0) + 128^{-\beta}32^{-\beta}\varphi(2x,0)$$

$$\left\|f(x) - 32^{-2}f(2^2x)\right\| \leq 128^{-\beta}[\varphi(x,0) + 32^{-\beta}\varphi(2x,0)] \tag{2.28}$$

for all $x \in \mathcal{X}$. Changing x by $2x$ in the inequality (2.27), we have

$$\left\|32^{-1}f(2^2x) - 32^{-2}f(2^3x)\right\| \leq 128^{-\beta}32^{-\beta}\varphi(2^2x,0)$$

for all $x \in \mathcal{X}$. Now multiply 32^{-1} on both sides of the above equation we get

$$\left\|32^{-2}f(2^2x) - 32^{-3}f(2^3x)\right\| \leq 128^{-\beta}32^{-2\beta}\varphi(2^2x,0) \tag{2.29}$$

for all $x \in \mathcal{X}$. Now adding the inequalities (2.25), (2.27), (2.29) we have
$\left\|f(x) - 32^{-1}f(2x)\right\| + \left\|32^{-1}f(2x) - 32^{-2}f(2^2x)\right\| + \left\|32^{-2}f(2^2x) - 32^{-3}f(2^3x)\right\|$

$$\leq 128^{-\beta}\varphi(x,0) + 128^{-\beta}32^{-\beta}\varphi(2x,0) + 128^{-\beta}32^{-2\beta}\varphi(2^2x,0)$$

$$\left\|f(x) - 32^{-3}f(2^3x)\right\| \leq 128^{-\beta}[\varphi(x,0) + 32^{-\beta}\varphi(2x,0) + 32^{-2\beta}\varphi(2^2x,0)] \tag{2.30}$$

for all $x \in \mathcal{X}$. By induction hypothesis on positive integer n, we obtain

$$\left\|f(x) - 32^{-n}f(2^nx)\right\| \leq 128^{-\beta}\sum_{i=0}^{n-1} 32^{-\beta i}\varphi(2^ix,0) \tag{2.31}$$

for all $x \in \mathcal{X}$. We are going to prove that the sequence $\{32^{-n} f(2^n x)\}$ is convergent.
We change u by $2^m x$ in the inequality (2.31), we have

$$\left\| f(2^m x) - 32^{-n} f(2^n 2^m x) \right\| \leq 128^{-\beta} \sum_{i=0}^{n-1} 32^{-\beta i} \varphi(2^i 2^m x, 0) \tag{2.32}$$

for all $m, n > 0$. Now multiply 32^{-m} on both sides of the above inequality (2.32), we get

$$\left\| 32^{-m} f(2^m x) - 32^{-n} 32^{-m} f(2^n 2^m x) \right\| \leq 128^{-\beta} \sum_{i=0}^{n-1} 32^{-\beta i} 32^{-\beta m} \varphi(2^i 2^m x, 0)$$

$$\left\| 32^{-(n+m)} f(2^{m+n} x) - 32^{-m} f(2^m x) \right\| \leq 128^{-\beta} \sum_{i=0}^{n-1} 32^{-\beta(m+i)} \varphi(2^{m+i} x, 0)$$

$$\left\| 32^{-(n+m)} f(2^{m+n} x) - 32^{-m} f(2^m x) \right\| \leq 128^{-\beta} \sum_{i=m}^{n} 32^{-\beta i} \varphi(2^i x, 0) \tag{2.33}$$

The right hand side $\to 0$ as $n \to \infty$.
The sequence $\{32^{-n} f(2^n x)\}$ is a Cauchy sequence.
$\sum_{i=0}^{\infty} 32^{-\beta i} \varphi(2^i x, 0)$ is convergent.
Given

$$\lim_{n \to \infty} 32^{-\beta i} \varphi(2^n x, 2^n y) = 0$$

for all $x, y \in \mathcal{X}$. Now we define the function

$$g(x) = \lim_{n \to \infty} 2^{-5n} f(2^n x)$$

for all $x \in \mathcal{X}$. Now taking limit on both sides of the inequality (2.31), we have

$$\lim_{n \to \infty} \left\| f(x) - 32^{-n} f(2^n x) \right\| \leq \lim_{n \to \infty} 128^{-\beta} \sum_{i=0}^{n-1} 32^{-\beta i} \varphi(2^i x, 0)$$

$$\| f(x) - g(x) \| \leq 128^{-\beta} \sum_{i=0}^{\infty} 32^{-\beta i} \varphi(2^i x, 0)$$

for all $x \in \mathcal{X}$. Now we are going to prove T satisfies the equation (2.1).
We change x by $2^n x$ and y by $2^n y$ in the inequality (2.21), we obtain
$\left\| 2f(2^{n+1}x + 2^n y) + 2f(2^{n+1}x - 2^n y) + f(2^n x + 2^{n+1} y) + f(2^n x - 2^{n+1} y) \right.$

$$\left. -20 f(2^n x + 2^n y) - 20 f(2^n x - 2^n y) - 90 f(2^n x) \right\| \leq \varphi(2^n x, 2^n y) \tag{2.34}$$

Multiply 32^{-n} on both sides of the inequality (2.17), we have
$\| 32^{-n} [2f(2^n(2x + y)) + 2f(2^n(2x - y)) - f(2^n(x + 2y)) + f(2^n(x - 2y)$

$$-20 f(2^n(x + y)) - 20 f(2^n(x - y)) - 90 f(2^n x)] \| \leq 32^{-\beta n} \varphi(2^n x, 2^n y) \tag{2.35}$$

Here taking limit as $n \to \infty$ on the above inequality, we find that g satisfies the equation (2.1) for all $x, y \in \mathcal{X}$.

$$\lim_{n\to\infty} \|32^{-n}[2f(2^n(2x+y)) + 2f(2^n(2x-y)) - f(2^n(x+2y)) + f(2^n(x-2y)$$

$$-20f(2^n(x+y)) - 20f(2^n(x-y)) - 90f(2^n x)]\| \leq \lim_{n\to\infty} 32^{-\beta n}\varphi(2^n x, 2^n y)$$

$$\|2g(2x+y) + 2g(2x-y) + g(x+2y) + g(x-2y) - 20g(x+y) - 20g(x-y) - 90g(x)\|$$

$$\leq \lim_{n\to\infty} 32^{-\beta n}\varphi(2^n x, 2^n y)$$

$$\|2g(2x+y) + 2g(2x-y) + g(x+2y) + g(x-2y) - 20g(x+y) - 20g(x-y) - 90g(x)\| = 0$$

for all $x, y \in \mathcal{X}$.

$$\Rightarrow 2g(2x+y) + 2g(2x-y) + g(x+2y) + g(x-2y) - 20g(x+y) - 20g(x-y) - 90g(x) = 0$$

for all $x, y \in \mathcal{X}$. Now we are going to prove the uniqueness of a quintic function T subject to the inequality (2.22), let us assume that there exist a quintic function $S : \mathcal{X} \to \mathcal{Y}$ which satisfies the equation (2.1) and the inequality (2.22).

$$\Rightarrow S(2^n x) = 32^n S(x) \quad and \quad T(2^n x) = 32^n T(x) \quad for \quad all \quad x \in \mathcal{X} \quad and \quad n \in N.$$

$$\begin{aligned}
\Rightarrow \|S(x) - T(x)\| &= \left\|32^{-n}S(2^n x) - 32^{-n}T(2^n x)\right\| \\
&= \left\|32^{-n}[S(2^n x) - f(2^n x)] + 32^{-n}[f(2^n x) - T(2^n x)]\right\| \\
&\leq 128^{-\beta}\sum_{i=0}^{\infty} 32^{-\beta(n+i)}\varphi(2^{n+i}x, 0) \quad for \quad all \quad x \in \mathcal{X}.
\end{aligned}$$

Now taking $n \to \infty$ in the proceeding inequality, we obtain the uniqueness of g. Hence the proof is complete. Q.E.D.

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