Some estimation about Tayler-Maclaurin coefficients of generalized subclasses of bi-univalent functions

S.A. Saleh[1], Alaa H. El-Qadeem[2], Mohamed A. Mamon[3]

[1,3]Department of Mathematics, Faculty of science, Tanta University, Tanta 31527, Egypt,
[2]Department of Mathematics, Faculty of science, Zagazig University, Zagazig 44519, Egypt.
E-mail: Dr.sasaleh@hotmail.com[1], ahhassan@science.zu.edu.eg[2], mohamed.saleem@science.tanta.edu.eg[3]

Abstract

In this study, two bi-univalent function subclasses are defined. The functions are defined in the open unit disc U. The main purpose is to determine some estimations on the initial Tayler-Maclaurin coefficients for functions belong to these subclasses. Finally, some earlier results are pointed out and improved.

2000 Mathematics Subject Classification. **30C45**. 30C50.
Keywords. analytic functions, univalent and bi-univalent functions, Maclaurin series, coefficient bounds.

1 Introduction, definitions and notations

Let $\mathcal{A}$ denote the class of all analytic functions f defined in the open unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ and normalized by the condition $f(0) = f'(0) - 1 = 0$. Thus each $f \in \mathcal{A}$ has a Taylor-Maclaurin series expansion of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (z \in \mathbb{U}). \tag{1.1}$$

Further, let $\mathcal{S}$ denote the class of all functions $f \in \mathcal{A}$ which are univalent in $\mathbb{U}$ (for details, see [8]; see also some of the recent investigations [1, 2]). Two of the important and well-investigated subclasses of the analytic and univalent function class $\mathcal{S}$ are the class $\mathcal{S}^*(\alpha)$ of starlike functions of order α in $\mathbb{U}$ and the class $\mathcal{K}(\alpha)$ of convex functions of order α in $\mathbb{U}$. By definition, we have

$$\mathcal{S}^*(\alpha) := \left\{ f : \ f \in \mathcal{S} \ \text{ and } \ \mathrm{Re}\left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha, \quad (z \in \mathbb{U}; 0 \le \alpha < 1) \right\}, \tag{1.2}$$

and

$$\mathcal{K}(\alpha) := \left\{ f : \ f \in \mathcal{S} \ \text{ and } \ \mathrm{Re}\left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \alpha, \quad (z \in \mathbb{U}; 0 \le \alpha < 1) \right\}. \tag{1.3}$$

It is clear from the definitions (1.2) and (1.3) that $\mathcal{K}(\alpha) \subset \mathcal{S}^*(\alpha)$, and also

$$f \in \mathcal{K}(\alpha) \quad \Leftrightarrow \quad zf' \in \mathcal{S}^*(\alpha),$$

and

$$f(z) \in \mathcal{S}^*(\alpha) \quad \Leftrightarrow \quad \int_0^z \frac{f(t)}{t} dt = F(z) \in \mathcal{K}(\alpha).$$

It is well-known [8] that every function $f \in \mathcal{S}$ has an inverse map f^{-1} that satisfies the following conditions:

Tbilisi Mathematical Journal 13(4) (2020), pp. 23–32.
Tbilisi Centre for Mathematical Sciences.
Received by the editors: 09 July 2019.
Accepted for publication: 17 September 2020.

$$f^{-1}(f(z)) = z \quad (z \in \mathbb{U}),$$

and

$$f\left(f^{-1}(w)\right) = w \qquad \left(|w| < r_0(f); r_0(f) \leq \tfrac{1}{4}\right).$$

In fact, the inverse function is given by

$$f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots . \tag{1.4}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both $f(z)$ and $f^{-1}(z)$ are univalent in $\mathbb{U}$. Let Σ denote the class of bi-univalent functions in $\mathbb{U}$ given by (1.1). For a brief history and some interesting examples of functions and characterization of the class Σ, see Srivastava et al. [18]. Examples of functions in the class Σ are

$$\frac{z}{1-z}, \qquad -log\,(1-z) \quad \text{and} \quad \frac{1}{2}log\left(\frac{1+z}{1-z}\right),$$

and so on. However, the familiar Koebe function is not a member of Σ. Other common examples of functions in $\mathcal{S}$ such as

$$z - \frac{z^2}{2} \quad \text{and} \quad \frac{z}{1-z^2}$$

are also not members of Σ.

In 1967, Lewin [12] investigated the bi-univalent function class Σ and showed that $|a_2| < 1.51$. Subsequently, Brannan and Clunie [4] conjectured that $|a_2| \leq \sqrt{2}$. Later, Netanyahu [16] showed that $\max |a_2| = \frac{4}{3}$ if $f \in \Sigma$. Brannan and Taha [5] introduced certain subclasses of a bi-univalent function class Σ similar to the familiar subclasses $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ of starlike and convex functions of order α $(0 \leq \alpha < 1)$, respectively (see [3]). Recently, many researchers have introduced and investigated several interesting subclasses of the bi-univalent function class Σ and they have found non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$. In fact, the aforecited work of Srivastava et al. [18] essentially revived the investigation of various subclasses of the bi-univalent function class Σ in recent years; it was followed by such works as those by Frasin and Aouf [9], Xu et al. [21], Çağlar et al. [7], and others (see, for example, [13, 17]). The coefficient estimate problem for each of the following Taylor-Maclaurin coefficients $|a_n|$ $(n \in \mathbb{N}\backslash\{1, 2\})$ for each $f \in \Sigma$ given by (1.1) is still an open problem.

This work is concerned with the coefficient bounds for the Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$. Also, we modify the definition of the classes introduced by Srivastava et la. [20] and find better estimations. At the end, several connections to some of the previous results are pointed out.

Definition 1.1. For $0 \leq \lambda \leq 1, 0 \leq \delta \leq 1, 0 \leq \mu \leq 1, 0 \leq \gamma \leq 1$, $0 < \alpha \leq 1$ and $\tau \in \mathbb{C}^* = \mathbb{C}/\{0\}$. Let $f \in \Sigma$ given by (1.1), then f is said to be in the class $\mathcal{S}_\Sigma^\alpha(\tau, \delta, \mu, \lambda, \gamma)$ if it satisfy the following conditions

$$\left|\arg\left(1 + \frac{1}{\tau}\left[\frac{(1-\delta)f(z) + \delta z f^{'}(z) + \mu z^2 f^{''}(z)}{(1-\lambda)z + \lambda(1-\gamma)f(z) + \lambda\gamma z f^{'}(z)} - 1\right]\right)\right| < \frac{\alpha\pi}{2},$$

and

$$\left|\arg\left(1 + \frac{1}{\tau}\left[\frac{(1-\delta)g(w) + \delta w g^{'}(w) + \mu w^2 g^{''}(w)}{(1-\lambda)w + \lambda(1-\gamma)g(w) + \lambda\gamma w g^{'}(w)} - 1\right]\right)\right| < \frac{\alpha\pi}{2},$$

for all $z \in \mathbb{U}$, $g = f^{-1} \in \Sigma$ given by (1.4).

Definition 1.2. For $0 \leq \lambda \leq 1, 0 \leq \delta \leq 1, 0 \leq \mu \leq 1, 0 \leq \gamma \leq 1$, $0 \leq \beta < 1$ and $\tau \in \mathbb{C}^* = \mathbb{C}/\{0\}$. Let $f \in \Sigma$ given by (1.1), then f is said to be in the class $\mathcal{S}_\Sigma(\tau, \delta, \mu, \lambda, \gamma; \beta)$ if it satisfy the following conditions

$$\Re\left(1 + \frac{1}{\tau}\left[\frac{(1-\delta)f(z) + \delta z f^{'}(z) + \mu z^2 f^{''}(z)}{(1-\lambda)z + \lambda(1-\gamma)f(z) + \lambda\gamma z f^{'}(z)} - 1\right]\right) > \beta,$$

and

$$\Re\left(1 + \frac{1}{\tau}\left[\frac{(1-\delta)g(w) + \delta w g^{'}(w) + \mu w^2 g^{''}(w)}{(1-\lambda)w + \lambda(1-\gamma)g(w) + \lambda\gamma w g^{'}(w)} - 1\right]\right) > \beta,$$

for all $z \in \mathbb{U}$, $g = f^{-1} \in \Sigma$ given by (1.4).

Remark 1.3. If we put $\delta = 1$ in the Definitions (1.1) and (1.2), we modify the definitions of the classes $\mathcal{H}_\Sigma(\tau, \mu, \lambda, \gamma; \alpha)$ and $\mathcal{H}_\Sigma(\tau, \mu, \lambda, \gamma; \beta)$ which are introduced by Srivastava el at. [20].

Remark 1.4. For special choices of the parameters $\tau, \delta, \mu, \lambda, \gamma$, we can obtain the following subclasses as a special case of our main classes defined above:

1. $\mathcal{S}_\Sigma^\alpha(1, \lambda, \delta, 0, \gamma) = \mathcal{N}_\Sigma(\alpha, \lambda, \delta)$ which introduced by Bulut [6].
2. $\mathcal{S}_\Sigma^\alpha(1, 1, 0, \lambda, 0) = \mathcal{S}_\Sigma^{a,1,a}(\alpha, \lambda)$ and $\mathcal{S}_\Sigma(1, 1, 0, \lambda, 0; \beta) = \mathcal{M}_\Sigma^{a,1,a}(\beta, \lambda)$ which were introduced by Srivastava et al. [19].
3. $\mathcal{S}_\Sigma^\alpha(1, 1, 0, 1, \lambda) = \mathcal{G}_\Sigma(\alpha, \lambda)$ and $\mathcal{S}_\Sigma(1, 1, 0, 1, \lambda; \beta) = \mathcal{M}_\Sigma(\beta, \lambda)$ which introduced by Murugusundaramoorthy et al. [15].
4. $\mathcal{S}_\Sigma^\alpha(1, 1, \lambda, 1, \lambda) = \mathfrak{B}_\Sigma(\alpha, \lambda)$ and $\mathcal{S}_\Sigma(1, 1, \lambda, 1, \lambda; \beta) = \mathcal{N}_\Sigma(\beta, \lambda)$ which introduced by Keerthi and Raja [11].
5. $\mathcal{S}_\Sigma^\alpha(1, \lambda, 0, 0, \gamma) = \mathcal{B}_\Sigma(\alpha, \lambda)$ and $\mathcal{S}_\Sigma(1, \lambda, 0, 0, \gamma; \beta) = \mathcal{B}_\Sigma(\beta, \lambda)$ which introduced by Frasin and Aouf [9].
6. $\mathcal{S}_\Sigma^\alpha(1, 1, \beta, 0, \gamma) = \mathcal{H}_\Sigma(\alpha, \beta)$ and $\mathcal{S}_\Sigma(1, 1, \beta, 0, \gamma; \gamma) = \mathcal{H}_\Sigma(\gamma, \beta)$ which introduced by Frasin [10].
7. $\mathcal{S}_\Sigma^\alpha(1, 1, 0, 0, \gamma) = \mathcal{H}_\Sigma^\alpha$ and $\mathcal{S}_\Sigma(1, 1, 0, 0, \gamma; \beta) = \mathcal{H}_\Sigma(\beta)$ which introduced by Srivastava et al. [18].

Lemma 1.5 (Carathéodory's Lemma)**.** [8] If $h \in \mathcal{P}$, then the estimates $|c_n| \leq 2$, $n = 1, 2, 3, ...$ are sharp, where $\mathcal{P}$ is the family of all functions h which are analytic in $\mathbb{U}$ for which $h(0) = 1$ and $\Re(h(z)) > 0 (z \in \mathbb{U})$ where

$$h(z) = 1 + c_1 z + c_2 z^2 + \cdots, \quad z \in \mathbb{U}.$$

2 Coefficient bounds for the function class $\mathcal{S}_\Sigma^\alpha(\tau, \delta, \mu, \lambda, \gamma)$

In this section, we establish coefficient bounds for the Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ of the function $f \in \mathcal{S}_\Sigma^\alpha(\tau, \delta, \lambda, \gamma)$.

Theorem 2.1. Let f defined by (1.1) belonging to the class $\mathcal{S}_{\Sigma}^{\alpha}(\tau,\delta,\mu,\lambda,\gamma)$, then

$$|a_2| \leq \frac{2\alpha|\tau|}{\sqrt{|2\alpha\tau\triangle(\mu,\delta,\lambda,\gamma)+(1-\alpha)(1+\delta+2\mu-\lambda-\gamma\lambda)^2|}}. \tag{2.1}$$

and

$$|a_3| \leq \min\left\{\frac{4\alpha^2|\tau|^2}{(1+\delta+2\mu-\lambda-\gamma\lambda)^2}+\frac{2\alpha|\tau|}{|1+2\delta+6\mu-\lambda-2\gamma\lambda|},\frac{2\alpha|\tau|}{|\triangle(\mu,\delta,\lambda,\gamma)|}\right\} \tag{2.2}$$

where

$$\triangle(\mu,\delta,\lambda,\gamma)=1+2\delta+6\mu-2\lambda-3\gamma\lambda-\lambda\delta-2\mu\lambda+\lambda^2+2\gamma\lambda^2-\delta\gamma\lambda-2\mu\gamma\lambda+\gamma^2\lambda^2. \tag{2.3}$$

Proof. Since $f\in\mathcal{S}_{\Sigma}^{\alpha}(\tau,\delta,\lambda,\gamma)$, then there exist two analytic functions $p(z)=1+\sum_{n=1}^{\infty}p_nz^n$ and $q(w)=1+\sum_{n=1}^{\infty}q_nw^n$ with positive real part in the unit disc U such that

$$1+\frac{1}{\tau}\left(\frac{(1-\delta)f(z)+\delta zf^{'}(z)+\mu z^2f^{''}(z)}{(1-\lambda)z+\lambda(1-\gamma)f(z)+\lambda\gamma zf^{'}(z)}-1\right)=(p(z))^{\alpha}, \tag{2.4}$$

and

$$1+\frac{1}{\tau}\left(\frac{(1-\delta)g(w)+\delta wg^{'}(w)+\mu w^2g^{''}(w)}{(1-\lambda)w+\lambda(1-\gamma)g(w)+\lambda\gamma wg^{'}(w)}-1\right)=(q(w))^{\alpha}. \tag{2.5}$$

Now by comparing the coefficients of powers z,z^2,w and w^2 in boss sides of equations (2.4) and (2.5), we obtain

$$\frac{(1+\delta+2\mu-\lambda-\gamma\lambda)}{\tau}=\alpha p_1 \tag{2.6}$$

$$\frac{(1+2\delta+6\mu-\lambda-2\gamma\lambda)a_3-(1+\delta+2\mu-\lambda-\gamma\lambda)(\lambda+\gamma\lambda)a_2^2}{\tau}=\alpha p_2+\frac{\alpha(\alpha-1)}{2}p_1^2 \tag{2.7}$$

$$-\frac{(1+\delta+2\mu-\lambda-\gamma\lambda)}{\tau}=\alpha q_1 \tag{2.8}$$

$$\frac{(1+2\delta+6\mu-\lambda-2\gamma\lambda)(2a_2^2-a_3)-(1+\delta+2\mu-\lambda-\gamma\lambda)(\lambda+\gamma\lambda)a_2^2}{\tau}=\alpha q_2+\frac{\alpha(\alpha-1)}{2}q_1^2. \tag{2.9}$$

From equations (2.6) and (2.8), we deduce

$$p_1=-q_1, \tag{2.10}$$

and

$$a_2=\frac{\alpha\tau p_1}{1+\delta+2\mu-\lambda-\gamma\lambda}. \tag{2.11}$$

By adding equation (2.7) to (2.9), we obtain

$$2\frac{\triangle(\mu,\delta,\lambda,\gamma)}{\tau}a_2^2=\alpha(p_2+q_2)+\frac{\alpha(\alpha-1)}{2}(p_1^2+q_1^2) \tag{2.12}$$

where $\triangle(\mu,\delta,\lambda,\gamma)$ are given by (2.3).
Using equation (2.10) and substituting the value of a_2 from equation (2.11) into equation (2.12), we deduce

$$p_1^2=\frac{(p_2+q_2)(1+\delta+2\mu-\lambda-\gamma\lambda)^2}{2\alpha\tau\triangle(\mu,\delta,\lambda,\gamma)+(1-\alpha)(1+\delta+2\mu-\lambda-\gamma\lambda)^2}, \tag{2.13}$$

therefore, by applying Lemma (1.5) in the equation (2.13), then

$$|p_1|\leq\frac{2|1+\delta+2\mu-\lambda-\gamma\lambda|}{\sqrt{|2\alpha\tau\triangle(\mu,\delta,\lambda,\gamma)+(1-\alpha)(1+\delta+2\mu-\lambda-\gamma\lambda)^2|}}. \tag{2.14}$$

By substituting from equation (2.14) into (2.11) and using Lemma (1.5), we conclude the desired estimate of a_2 given by (2.1).
On the other hand, to investigate the bounds of $|a_3|$, subtracting equation (2.9) from (2.7) we obtain

$$2\frac{(1+2\delta+6\mu-\lambda-2\gamma\lambda)}{\tau}(a_3-a_2^2)=\alpha(p_2-q_2)+\frac{\alpha(\alpha-1)}{2}(p_1^2-q_1^2). \tag{2.15}$$

By using equation (2.10) into (2.15), we obtain

$$a_3=a_2^2+\frac{\alpha\tau}{2(1+2\delta+6\mu-\lambda-2\gamma\lambda)}(p_2-q_2). \tag{2.16}$$

Using Lemma (1.5) and substituting of the value of a_2 from (2.11) into (2.16), we conclude one of the desired estimates of $|a_3|$.
Now, by substituting the value of a_2^2 from equation (2.12) into (2.16), we obtain

$$a_3=\frac{\alpha\tau}{2}\left[p_2\left(\frac{1}{\triangle(\mu,\delta,\lambda,\gamma)}+\frac{1}{(1+2\delta+6\mu-\lambda-2\gamma\lambda)}\right)+q_2\left(\frac{1}{\triangle(\mu,\delta,\lambda,\gamma)}-\frac{1}{(1+2\delta+6\mu-\lambda-2\gamma\lambda)}\right)\right]. \tag{2.17}$$

Applying Lemma (1.5) for the coefficients p_2,q_2, we conclude the other desired estimation of $|a_3|$.
Q.E.D.

3 Coefficient bounds for the function class $\mathcal{S}_\Sigma(\tau,\delta,\mu,\lambda,\gamma;\beta)$

In this section, we establish coefficient bounds for the Maclaurin coefficients $|a_2|$ and $|a_3|$ of the function $f\in\mathcal{S}_\Sigma(\tau,\delta,\mu,\lambda,\gamma;\beta)$.

Theorem 3.1. Let f defined by (1.1) belonging to the class $\mathcal{S}_\Sigma(\tau,\delta,\lambda,\gamma;\beta)$, then

$$|a_2|\leq\min\left\{\frac{2(1-\beta)|\tau|}{|1+\delta+2\mu-\lambda-\gamma\lambda|},\sqrt{\frac{2|\tau|(1-\beta)}{|\triangle(\mu,\delta,\lambda,\gamma)|}}\right\}, \tag{3.1}$$

and

$$|a_3|\leq\min\left\{\frac{4(1-\beta)^2|\tau|^2}{|1+\delta+2\mu-\lambda-\gamma\lambda|^2}+\frac{2(1-\beta)|\tau|}{|1+2\delta+6\mu-\lambda-2\gamma\lambda|},\frac{2(1-\beta)|\tau|}{|\triangle(\mu,\delta,\lambda,\gamma)|}\right\}. \tag{3.2}$$

Proof. Since $f\in\mathcal{S}_\Sigma(\tau,\delta,\lambda,\gamma;\beta)$, then there exist two analytic functions $u(z)=\sum_{n=1}^{\infty}\rho_n z^n$ and $v(w)=\sum_{n=1}^{\infty}\nu_n w^n$ with positive real part in the unit disc such that

$$1+\frac{1}{\tau}\left(\frac{(1-\delta)f(z)+\delta zf^{'}(z)+\mu z^2f^{''}(z)}{(1-\lambda)z+\lambda(1-\gamma)f(z)+\lambda\gamma zf^{'}(z)}-1\right)=\beta+(1-\beta)u(z) \tag{3.3}$$

and

$$1+\frac{1}{\tau}\left(\frac{(1-\delta)g(w)+\delta wg^{'}(w)+\mu w^{2}g^{''}(w)}{(1-\lambda)w+\lambda(1-\gamma)g(w)+\lambda\gamma wg^{'}(w)}-1\right)=\beta+(1-\beta)v(w) \tag{3.4}$$

Now by comparing the coefficients of powers z, z^2, w and w^2 in boss sides of equations (3.3) and (3.4), we obtain

$$\frac{(1+\delta+2\mu-\lambda-\gamma\lambda)}{\tau}=(1-\beta)\rho_1 \tag{3.5}$$

$$\frac{(1+2\delta+6\mu-\lambda-2\gamma\lambda)a_3-(1+\delta+2\mu-\lambda-\gamma\lambda)(\lambda+\gamma\lambda)a_2^2}{\tau}=(1-\beta)\rho_2 \tag{3.6}$$

$$-\frac{(1+\delta+2\mu-\lambda-\gamma\lambda)}{\tau}=(1-\beta)\nu_1 \tag{3.7}$$

$$\frac{(1+2\delta+6\mu-\lambda-2\gamma\lambda)(2a_2^2-a_3)-(1+\delta+2\mu-\lambda-\gamma\lambda)(\lambda+\gamma\lambda)a_2^2}{\tau}=(1-\beta)\nu_2. \tag{3.8}$$

From equations (3.5) and (3.7), we deduce

$$\rho_1=-\nu_1, \tag{3.9}$$

and

$$a_2=\frac{(1-\beta)\tau\rho_1}{1+\delta+2\mu-\lambda-\gamma\lambda}. \tag{3.10}$$

By adding equation (3.6) to (3.8), we obtain

$$\frac{2\triangle(\mu,\delta,\lambda,\gamma)a_2^2}{\tau}=(1-\beta)(\rho_2+\nu_2), \tag{3.11}$$

where $\triangle(\mu,\delta,\lambda,\gamma)$ are given by (2.3).
Applying Lemma (1.5), we obtain

$$|a_2|\leq\sqrt{\frac{2|\tau|(1-\beta)}{|\triangle(\mu,\delta,\lambda,\gamma)|}}. \tag{3.12}$$

On the other hand, from equation (3.10) we can deduce also that

$$|a_2|\leq\frac{2(1-\beta)|\tau|}{|1+\delta+2\mu-\lambda-\gamma\lambda|}. \tag{3.13}$$

Combining this with inequality (3.12), we obtain the desired estimate on the coefficient $|a_2|$ which given by (3.1).
In order to deduce an estimation of bounds of $|a_3|$, subtracting equation (3.8) from (3.6), we get

$$a_3=a_2^2+\frac{\tau(1-\beta)(\rho_2-\nu_2)}{2(1+2\delta+6\mu-\lambda-2\gamma\lambda)}. \tag{3.14}$$

By substituting the value of a_2 from equation (3.10) with applying Lemma (1.5) for the coefficients p_1, p_2 and q_2, we obtain

$$|a_3| \leq \frac{4(1-\beta)^2|\tau|^2}{(1+\delta+2\mu-\lambda-\gamma\lambda)^2} + \frac{2|\tau|(1-\beta)}{|1+2\delta+6\mu-\lambda-2\gamma\lambda|}. \tag{3.15}$$

Now, by substituting the value of a_2^2 from equation (3.11) into equation (3.14) with applying Lemma (1.5) for the coefficients ρ_2 and ν_2, we obtain the other estimation of $|a_3|$ which given by (3.2). Q.E.D.

4 Some corollaries and consequences

In this section, we introduced and modified some previous known results as an immediate consequences of Theorem (2.1) and (3.1).

Let us set $\delta = 1$ in the above Theorems, we improve the results given by Srivstava el at.

Corollary 4.1. [20] Let f given by (1.1) be in the class $\mathcal{H}_\Sigma(\tau, \mu, \lambda, \gamma; \alpha)$, then

$$|a_2| \leq \frac{2\alpha|\tau|}{\sqrt{|2\alpha\tau\nabla(\mu,\lambda,\gamma) + (1-\alpha)(2+2\mu-\lambda-\gamma\lambda)^2|}},$$

$$|a_3| \leq \min\left\{ \frac{4\alpha^2|\tau|^2}{(2+2\mu-\lambda-\gamma\lambda)^2} + \frac{2\alpha|\tau|}{|3+6\mu-\lambda-2\gamma\lambda|}, \frac{2\alpha|\tau|}{|\nabla(\mu,\lambda,\gamma)|} \right\},$$

where

$$\nabla(\mu,\lambda,\gamma) = \gamma^2\lambda^2 + 2\gamma\lambda^2 - 2\gamma\lambda\mu - 4\gamma\lambda + \lambda^2 - 2\lambda\mu - 3\lambda + 6\mu + 3$$

Corollary 4.2. [20] Let f given by (1.1) be in the class $\mathcal{H}_\Sigma(\tau, \mu, \lambda, \gamma; \beta)$, then

$$|a_2| \leq \min\left\{ \frac{2(1-\beta)|\tau|}{|2+2\mu-\lambda-\gamma\lambda|}, \sqrt{\frac{2|\tau|(1-\beta)}{|\tilde{\Omega}|}} \right\},$$

$$|a_3| \leq \min\left\{ \frac{4(1-\beta)^2|\tau|^2}{|2+2\mu-\lambda-\gamma\lambda|^2} + \frac{2(1-\beta)|\tau|}{|3+6\mu-\lambda-2\gamma\lambda|}, \frac{2(1-\beta)|\tau|}{|\tilde{\Omega}|} \right\}.$$

By putting $\tau = 1, \lambda = 0$ in the above Theorems, we conclude

Corollary 4.3. [6] Let f given by (1.1) be in the class $\mathcal{N}_\Sigma(\beta, \delta, \mu)$, then

$$|a_2| \leq \min\left\{ \frac{2(1-\beta)}{1+\delta+2\mu}, \sqrt{\frac{2(1-\beta)}{1+2\delta+6\mu}} \right\},$$

$$|a_3| \leq \frac{2(1-\beta)}{1+2\delta+6\mu}.$$

Put $\tau = \delta = \lambda = 1$ and $\mu = \gamma$, we obtain

Corollary 4.4. [11] Let f given by (1.1) be in the class $\mathfrak{B}_\Sigma(\alpha,\gamma)$ then,

$$|a_2| \leq \frac{2\alpha}{\sqrt{|4\alpha(1+2\gamma)+(1-3\alpha)(1+\gamma)^2|}},$$

$$|a_3| \leq \min\left\{\frac{4\alpha^2}{(1+\gamma)^2}+\frac{\alpha}{1+2\gamma}, \frac{2\alpha}{1+2\gamma-\gamma^2}\right\}.$$

Corollary 4.5. [11] Let f given by (1.1) be in the class $\mathcal{N}_\Sigma(\beta,\gamma)$ then,

$$|a_2| \leq \sqrt{\frac{2(1-\beta)}{1+2\gamma-\gamma^2}},$$

$$|a_3| \leq \min\left\{\frac{2(1-\beta)}{1+2\gamma-\gamma^2}, \frac{4(1-\beta)^2}{(1+\gamma)^2}+\frac{1-\beta}{1+2\gamma}\right\}.$$

Put $\tau=\delta=\lambda=1$ and $\mu=0$, we obtain

Corollary 4.6. [15] Let f given by (1.1) be in the class $\mathcal{G}_\Sigma(\alpha,\gamma)$, then

$$|a_2| \leq \frac{2\alpha}{(1-\gamma)\sqrt{1+\alpha}},$$

$$|a_3| \leq \min\left\{\frac{4\alpha^2}{(1-\gamma)^2}+\frac{\alpha}{1-\gamma}, \frac{2\alpha}{(1-\gamma)^2}\right\}.$$

Corollary 4.7. [15] Let f given by (1.1) be in the class $\mathcal{M}_\Sigma(\beta,\gamma)$, then

$$|a_2| \leq \frac{\sqrt{2(1-\beta)}}{1-\gamma},$$

$$|a_3| \leq \min\left\{\frac{2(1-\beta)}{(1-\gamma)^2}, \frac{4(1-\beta)^2}{(1-\gamma)^2}+\frac{1-\beta}{1-\gamma}\right\}.$$

By putting $\tau=\delta=1, \lambda=0$ in the above Theorems, we obtain

Corollary 4.8. [10] Let f given by (1.1) be in the class $\mathcal{H}_\Sigma(\alpha,\mu)$, then

$$|a_2| \leq \frac{2\alpha}{\sqrt{|2(2+\alpha)+4\mu(\alpha+\mu+2-\alpha\mu)|}},$$

$$|a_3| \leq \frac{2\alpha}{3(1+2\mu)}.$$

Corollary 4.9. [10] Let f given by (1.1) be in the class $\mathcal{H}_\Sigma(\beta,\mu)$, then

$$|a_2| \leq \sqrt{\frac{2(1-\beta)}{3(1+2\mu)}} \quad \text{and} \quad |a_3| \leq \frac{2(1-\beta)}{3(1+2\mu)}.$$

Putting $\tau = 1, \lambda = \mu = 0$, we obtain

Corollary 4.10. [9] Let f given by (1.1) be in the class $\mathcal{B}_\Sigma(\alpha, \delta)$ then,

$$|a_2| \leq \frac{2\alpha}{\sqrt{|\alpha(1+2\delta-\delta^2)+(1+\delta)^2}},$$

$$|a_3| \leq \frac{2\alpha}{1+2\delta}.$$

Corollary 4.11. [9] Let f given by (1.1) be in the class $\mathcal{B}_\Sigma(\beta, \delta)$, then

$$|a_2| \leq \sqrt{\frac{2(1-\beta)}{1+2\delta}},$$

$$|a_3| \leq \frac{2(1-\beta)}{1+2\delta}.$$

Remark 4.12. The results investigated in some above Corollaries from (4.1) to (4.11); represented an improvement of the estimate of $|a_3|$ of the earlier corresponding results.

Acknowledgements

The authors would like to express sincere thanks to the worthy referees for careful reading and suggestions which helped us to improve our study.

References

[1] A.A. Amourah, F. Yousef, T. Al-Hawary, M. Darus, *On $H_3(p)$ Hankel determinant for certain subclass of p-valent functions*, Ital. J. Pure Appl. Math., **37**(2017), 611-618.

[2] T. Al-Hawary, B.A. Frasin, F. Yousef, *Coefficients estimates for certain classes of analytic functions of complex order*, Afr. Mat., **29**(2018), No. 7-8, 1265-1271.

[3] D.A. Brannan, J.G. Clunie, W.E. Kirwan, *Coefficient estimates for a class of star-like functions*, Canad. J. Math., **22**(1970), 476-485.

[4] D.A. Brannan, J.G. Clunie, *Aspects of contemporary complex analysis*, Proceedings of the NATO Advanced Study Institute held at the University of Durham, Durham, Academic Press, New York and London, 1980.

[5] D.A. Brannan, D.L. Tan, *On some classes of bi-univalent functions*, Stud. Univ. Babes-Bolyai Math., **31**(1986), No. 2, 70-77.

[6] S. Bulut, *Faber polynomial coefficient estimates for a subclass of analytic bi-univalent functions*, Filomat, **30**(2016), No. 6, 1567-1575.

[7] M. Çağlar, H. Orhan, N. Yağmur, *Coefficient bounds for new subclasses of bi-univalent functions*, Filomat, **27**(2013), No. 7, 1165-1171.

[8] P.L. Duren, *Univalent functions*, Grundlehren der Mathematischen Wissenschaften, Band 259, Springer-Verlag, New York, Berlin, Heidelberg and Tokyo, 1983.

[9] B.A. Frasin, M.K. Aouf, *New subclasses of bi-univalent functions*, Appl. Math. Lett., **24**(2011), No. 9, 1569-1573.

[10] B.A. Frasin, *Coefficient bounds for certain classes of bi-univalent functions*, Hact. J. Math. Stat., **43**(2014), No. 3, 383-389.

[11] B. Srutha Keerthi, Bhuvaneswari Raja, *Coefficient Inequality for Certain New Sub-classes of Analytic Bi-univalent Functions*, Abstr. Appl. Anal., **3**(2013), No. 1, 1-10.

[12] M. Lewin, *On a coefficient problem for bi-univalent functions*, Proc. Amer. Math. Soc., **18**(1967), 63-68.

[13] X.-F. Li, A.-P. Wang, *Two new subclasses of bi-univalent functions*, Int. Math. Forum, **7**(2012), 1495-1504.

[14] N. Magesh, J. Yamini, *Coefficient bounds for a certain subclass of bi-univalent functions*, Int. Math. Forum, **8**(2013), No. 27, 1337-1344.

[15] G. Murugusundaramoorthy, N. Magesh, V. Prameela, *Coefficient Bounds for Certain Subclasses of Bi-Univalent Function*, Abstr. Appl. Anal., **2013**, Article ID 573017, 3 pages.

[16] E. Netanyahu, *The minimal distance of the image boundary from the origin and the second coefficient of a univalent function in* $|z| < 1$, Arch. Ration. Mech. Anal., **32**(1969), 100-112.

[17] S. Porwal, M. Darus, *On a new subclass of bi-univalent functions*, J. Egypt. Math. Soc., **21**(2013), No. 3, 190-193.

[18] H.M. Srivastava, A.K. Mishra, P. Gochhayat, *Certain subclasses of analytic and bi-univalent functions*, Appl. Math. Lett., **23**(2010), No. 10, 1188-1192.

[19] H.M. Srivastava, G. Murugusundaramoorthy, N. Magesh, *Certain sub classes of bi-univalent functions associated with the Hohlov operator*, Global Journal of Mathematical Analysis, **1**(2013), No. 2, 67-73.

[20] H.M. Srivastava, S. Gaboury, F. Ghanim, *Coefficient estimates for some general subclasses of analytic and bi-univalent functions*, Afr. Mat., **28**(2016), No. 5-6, 693-706.

[21] Q.-H. Xu, Y.-C. Gui, H.M. Srivastava, *Coefficient estimates for a certain subclass of analytic and bi-univalent functions*, Appl. Math. Lett., **25**(2012), 990-994.