Comparison rate of convergence and data dependence for a new iteration method

Samet Maldar[1], Yunus Atalan[2] and Kadri Dogan[3]

[1], [2] Department of Mathematics, Aksaray University
[3] Department of Computer Engineering, Artvin Coruh University
E-mail: sametmaldarr10@gmail.com[1], yunusatalann@gmail.com[2], dogankadri@hotmail.com[3]

Abstract

In this paper, we have defined hyperbolic type of some iteration methods. The new iteration has been investigated convergence for mappings satisfying certain condition in hyperbolic spaces. It has been proved that this iteration is equivalent in terms of convergence with another iteration method in the same spaces. The rate of convergence of these two iteration methods have been compared. We have investigated data dependence result using hyperbolic type iteration. Finally, we have given numerical examples about rate of convergence and data dependence.

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1 Introduction

Fixed point theory has generally developed in two directions, metric spaces and normed linear spaces. All studies in this theory, such as defined iteration methods, convergence, stability results, equivalence of convergence, data dependence results and rate of convergence have depended on the structure of space. Geometric structure of space, that is to say convex structure, has been an important concept in fixed point theory. In Banach spaces, the results have been obtained from the studies largely depend on the linear structure of the space. Therefore, Banach space has been given a convex structure. As a result of extensive studies in Banach space, an exhaustive literature has been obtained (see [1],[7],[12],[20],[21]). However, due to the limited number of linear structures, nonlinear structures have gained importance. Therefore, the significance of the studies in metric spaces have been revealed. But, it is not always easy to obtain a convex structure in metric spaces. The studies on obtaining a convex structure on metric space were first studied by Takashi [24]. In 2004, hyperbolic spaces have been defined by Kohlenbach [17]. Hyperbolic space, which has a wide field of research in fixed point theory, is a nonlinear and geometric structure in metric spaces. Kohlenbach [17]'s definition of hyperbolic space is as follows:

Definition 1.1. A (X, d, Q) is called a hyperbolic space if (X, d) is a metric space and $Q : X \times X \times I \to X$ satisfying

(Q_i) $d(u, Q(a, b, \alpha)) \leq \alpha d(u, a) + (1 - \alpha)d(u, b)$;

(Q_{ii}) $d(Q(a, b, \alpha), Q(a, b, \beta)) = |\alpha - \beta|\, d(a, b)$;

(Q_{iii}) $Q(a, b, \alpha) = Q(b, a, 1 - \alpha)$;

(Q_{iv}) $d(Q(a, c, \alpha), Q(b, w, \alpha)) \leq (1 - \alpha)d(a, b) + \alpha d(c, w)$;

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for all $a, b, c, w \in X$ and $\alpha, \beta \in [0, 1]$.

Many researchers have been worked on hyperbolic spaces (see [2],[5],[8]-[10],[14]-[16]).

The following conditions can be written using hyperbolic spaces.

(i) If (X, d) is a convex metric space that satisfies only condition (Q_i), then it is defined by Takahashi ([24]).

(ii) If (X, d) satisfies conditions $(Q_i) - (Q_{iii})$, then it is hyperbolic type in Gobel and Kirk ([6]).

(iii) If (X, d) satisfies conditions (Q_{iv}), then it is class hyperbolic type in Reich et al. ([22]).

Iteration methods appear to be a useful method for investigating the fixed point of a mapping (see [11],[12],[13],[19]).

In this contex, the iteration method defined in 2019 is as follow:

$$\begin{cases} x_0 \in C \\ x_{n+1} = \frac{(1-\alpha_n)}{k} T x_n + \left(1 - \frac{(1-\alpha_n)}{k}\right) T y_n \\ y_n = \frac{(1-\beta_n)}{k} T x_n + \left(1 - \frac{(1-\beta_n)}{k}\right) T z_n \\ z_n = \frac{(1-\gamma_n)}{k} x_n + \left(1 - \frac{(1-\gamma_n)}{k}\right) T x_n \end{cases} \tag{1.1}$$

where C is a nonempty subset of a Banach space X, T is a self-mapping on C and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are real sequences in $[0, 1]$ and k is the natural number (see [18]).

In 2018, a new type Mann iteration method was introduced as follows:

$$\begin{cases} x_0 \in C \\ x_{n+1} = \frac{(1-\alpha_n)}{k} x_n + \left(1 - \frac{(1-\alpha_n)}{k}\right) T x_n \end{cases} \tag{1.2}$$

where $\{\alpha_n\}$ is real sequence in $[0, 1]$ and k is the natural number [4].

The iteration method (1.2) can be pointed in hyperbolic as

$$\begin{cases} x_0 \in C \\ x_{n+1} = Q\left(x_n, T x_n, \frac{(1-\alpha_n)}{k}\right) \end{cases} \tag{1.3}$$

The expression of the iteration method (1.1) in hyperbolic can be:

$$\begin{cases} x_0 \in C \\ x_{n+1} = Q\left(T x_n, T y_n, \frac{(1-\alpha_n)}{k}\right) \\ y_n = Q\left(T x_n, T z_n, \frac{(1-\beta_n)}{k}\right) \\ z_n = Q\left(x_n, T x_n, \frac{(1-\gamma_n)}{k}\right) \end{cases} \tag{1.4}$$

Definition 1.2 ([3])**.** Let T with a fixed point p be a self operator on a metric space X and satisfying the condition

$$d(p, Tx) \leq \delta d(p, x), \tag{1.5}$$

for some $0 \leq \delta < 1$ and for each $x \in X$.

From (1.5) , we obtain

$$\begin{aligned} d(Tx,Ty) &\leq d(Tx,p)+d(p,Ty) \\ &\leq \delta d(p,x)+\delta d(p,y) \\ &\leq \delta d(x,y)+2\delta d(p,y) \end{aligned} \tag{1.6}$$

Definition 1.3 ([21]). Let $\{b_n\}_{n=0}^{\infty}$ and $\{c_n\}_{n=0}^{\infty}$be two sequences with limits the same point p. We say that $\{b_n\}_{n=0}^{\infty}$ converges faster than $\{c_n\}_{n=0}^{\infty}$ to p, if

$$\lim_{n\to\infty}\frac{d(b_n,p)}{d(c_n,p)}=0.$$

Lemma 1.4 ([25]). Let $\{b_n\}_{n=0}^{\infty}$ and $\{d_n\}_{n=0}^{\infty}$ be nonnegative real sequences satisfying the following inequality:

$$b_{n+1}\leq(1-r_n)b_n+d_n,$$

where $r_n\in(0,1)$ for all $n\in\mathbb{N}$, $\sum\limits_{n=0}^{\infty}r_n=\infty$ and $\lim_{n\to\infty}\frac{d_n}{r_n}=0$ then $b_n\to 0$ as $n\to\infty$.

Lemma 1.5 ([23]). Let $\{b_n\}_{n=0}^{\infty}$ be a nonnegative real sequence and there exists $n_0\in\mathbb{N}$ such that for all $n\geq n_0$ *satisfying*

$$b_{n+1}\leq(1-\gamma_n)b_n+\gamma_n\mu_n$$

where $\gamma_n\in(0,1)$ such that $\sum\limits_{n=0}^{\infty}\gamma_n=\infty$ and $\{\mu_n\}_{n=0}^{\infty}\geq 0$. Then the following inequality holds:

$$0\leq\limsup_{n\to\infty}b_n\leq\limsup_{n\to\infty}\mu_n.$$

Definition 1.6 ([23]). L*et $T,S:C\to C$ be two operators. We say that S is an approximate operator of T for all $x\in C$ and a fixed $\varepsilon>0$ if $d(Tx,Sx)\leq\varepsilon$.*

2 Main results

Theorem 2.1. Let C be a nonempty, closed, convex subset of a hyperbolic metric space X and $T:C\to C$ be an operator satisfying (1.5). Let $\{x_n\}_{n=0}^{\infty}$ be the iterative sequence (1.4) *with a real sequence* $\{\alpha_n\}_{n=0}^{\infty},\{\beta_n\}_{n=0}^{\infty}$ *and* $\{\gamma_n\}_{n=0}^{\infty}\in[0,1]$ *satisfying* $\sum\limits_{n=0}^{\infty}\alpha_n=\infty$. Then $\{x_n\}_{n=0}^{\infty}$ converges to p.

Proof. By (Q_i), (1.5) and (1.4), we have

$$\begin{aligned} d(x_{n+1},p) &= d\left(Q\left(Tx_n,Ty_n,\frac{(1-\alpha_n)}{k}\right),p\right) \\ &\leq \left(1-\frac{(1-\alpha_n)}{k}\right)d(Tx_n,p)+\frac{(1-\alpha_n)}{k}d(Ty_n,p) \\ &\leq \delta\left(1-\frac{(1-\alpha_n)}{k}\right)d(x_n,p)+\delta\frac{(1-\alpha_n)}{k}d(y_n,p) \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} d(y_n,p) &= d\left(Q\left(Tx_n,Tz_n,\frac{(1-\beta_n)}{k}\right),p\right) \\ &\leq \left(1-\frac{(1-\beta_n)}{k}\right)d(Tx_n,p)+\frac{(1-\beta_n)}{k}d(Tz_n,p) \\ &\leq \left(1-\frac{(1-\beta_n)}{k}\right)\delta d(x_n,p)+\frac{(1-\beta_n)}{k}\delta d(z_n,p) \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} d(z_n,p) &= d\left(Q\left(x_n,Tx_n,\frac{(1-\gamma_n)}{k}\right),p\right) \\ &\leq \left(1-\frac{(1-\gamma_n)}{k}\right)d(x_n,p)+\frac{(1-\gamma_n)}{k}d(Tx_n,p) \\ &\leq \left(1-\frac{\gamma_n(1-\delta)}{k}\right)d(x_n,p) \end{aligned} \tag{2.3}$$

from (2.3), (2.2) and (2.1), we have

$$d(x_{n+1},p)\leq\delta\left(1-\frac{\alpha_n(1-\delta)}{k}\right)d(x_n,p).$$

Hence, we get

$$\begin{aligned} d(x_n,p) &\leq \delta\left(1-\frac{\alpha_{n-1}(1-\delta)}{k}\right)d(x_{n-1},p) \\ d(x_{n-1},p) &\leq \delta\left(1-\frac{\alpha_{n-2}(1-\delta)}{k}\right)d(x_{n-2},p) \\ &\vdots \\ d(x_1,p) &\leq \delta\left(1-\frac{\alpha_0(1-\delta)}{k}\right)d(x_0,p) \end{aligned}$$

and we obtain

$$d(x_{n+1},p)\leq\delta^{n+1}\prod_{i=0}^{n}[1-\frac{\alpha_i(1-\delta)}{k}]d(x_0,p) \tag{2.4}$$

from (2.4), we get

$$\begin{aligned} d(x_{n+1},p) &\leq \delta^{n+1}\prod_{i=0}^{n}e^{-\frac{\alpha_i(1-\delta)}{k}}d(x_0,p) \\ &= \delta^{n+1}e^{-\frac{(1-\delta)}{k}\sum_{i=0}^{n}\alpha_i}d(x_0,p). \end{aligned} \tag{2.5}$$

Taking the limit inequality (2.5), it can be seen that $x_n\to p$ as $n\to\infty$. Q.E.D.

Theorem 2.2. Let C, Q and T with fixed point p be the same as in Theorem 2.1. Suppose that $\{x_n\}_{n=0}^{\infty}$ to be defined by the iteration method (1.4) for $x_0 \in C$ and let $\{u_n\}_{n=0}^{\infty}$ be defined by the iteration method (1.3) for $u_0 \in C$. Then the following are equivalent:

(i) $\{x_n\}_{n=0}^{\infty}$ converges to p

(ii) $\{u_n\}_{n=0}^{\infty}$ converges to p.

Proof. We will prove (i)⇒(ii). Suppose that $\{x_n\}_{n=0}^{\infty}$ converges to p. By using (Q_i), (Q_{iv}), (1.6), (1.5), (1.4), (1.3), we obtain

$$\begin{aligned} d(x_{n+1}, u_{n+1}) &\leq d\left(Q\left(Tx_n, Ty_n, \frac{(1-\alpha_n)}{k}\right), Q\left(u_n, Tu_n, \frac{(1-\alpha_n)}{k}\right)\right) \\ &\leq \left(1 - \frac{(1-\alpha_n)}{k}\right) d(Tx_n, u_n) + \frac{(1-\alpha_n)}{k} d(Tu_n, Ty_n) \\ &\leq \left(1 - \frac{(1-\alpha_n)}{k}\right) d(Tx_n, x_n) + \left(1 - \frac{(1-\alpha_n)}{k}\right) d(x_n, u_n) \\ &\quad + \delta\frac{(1-\alpha_n)}{k} d(u_n, y_n) + 2\delta\frac{(1-\alpha_n)}{k} d(y_n, p) \end{aligned} \tag{2.6}$$

and also

$$\begin{aligned} d(y_n, u_n) &= d\left(Q\left(Tx_n, Tz_n, \frac{(1-\beta_n)}{k}\right), u_n\right) \\ &\leq \frac{(1-\beta_n)}{k} d(u_n, Tx_n) + \left(1 - \frac{(1-\beta_n)}{k}\right) d(u_n, Tz_n) \\ &\leq \frac{(1-\beta_n)}{k} d(u_n, x_n) + \frac{(1-\beta_n)}{k} d(x_n, Tx_n) + \left(1 - \frac{(1-\beta_n)}{k}\right) d(u_n, z_n) \\ &\quad + \left(1 - \frac{(1-\beta_n)}{k}\right) d(z_n, Tz_n) \end{aligned} \tag{2.7}$$

we obtain

$$\begin{aligned} d(u_n, z_n) &= d\left(u_n, Q\left(x_n, Tx_n, \frac{(1-\gamma_n)}{k}\right)\right) \\ &\leq \frac{(1-\gamma_n)}{k} d(u_n, x_n) + \left(1 - \frac{(1-\gamma_n)}{k}\right) d(u_n, Tx_n) \\ &\leq d(u_n, x_n) + \left(1 - \frac{(1-\gamma_n)}{k}\right) d(x_n, Tx_n). \end{aligned} \tag{2.8}$$

Moreover, we get

$$\begin{aligned} d(x_n, Tx_n) &\leq d(x_n, p) + d(Tx_n, p) \\ &\leq (1+\delta) d(x_n, p) \end{aligned} \tag{2.9}$$

and

$$\begin{aligned} d(Tz_n, z_n) &\leq d(Tz_n, p) + d(z_n, p) \\ &\leq (1+\delta) d(z_n, p) \end{aligned} \tag{2.10}$$

and

$$\begin{aligned} d(Tz_n,p) &\leq \delta d(z_n,p) \\ &\leq \delta d\left(Q\left(x_n,Tx_n,\frac{(1-\gamma_n)}{k}\right),p\right) \\ &\leq \delta\frac{(1-\gamma_n)}{k}d(x_n,p)+\delta\left(1-\frac{(1-\gamma_n)}{k}\right)d(Tx_n,p) \\ &\leq \delta\left(1-\frac{\gamma_n(1-\delta)}{k}\right)d(x_n,p) \end{aligned} \tag{2.11}$$

and

$$\begin{aligned} d(y_n,p) &\leq d\left(Q\left(Tx_n,Tz_n,\frac{(1-\beta_n)}{k}\right),p\right) \\ &\leq \frac{(1-\beta_n)}{k}d(Tx_n,p)+\left(1-\frac{(1-\beta_n)}{k}\right)d(Tz_n,p) \\ &\leq \left(1-\frac{\beta_n(1-\delta)}{k}\right)d(x_n,p) \end{aligned} \tag{2.12}$$

and substituting (2.7)-(2.8) in (2.6) and using (2.9), (2.10), (2.11), (2.12), $\delta\in(0,1)$, we obtain

$$\begin{aligned} d(x_{n+1},u_{n+1}) \leq\ & \left[\left(1-\frac{(1-\alpha_n)}{k}\right)+\delta\frac{(1-\alpha_n)}{k}\right]d(x_n,u_n) \\ &+\left[\begin{array}{c}\left(1-\frac{(1-\alpha_n)}{k}\right)+\delta\frac{(1-\alpha_n)}{k}\frac{(1-\beta_n)}{k} \\ +\delta\frac{(1-\alpha_n)}{k}\left(1-\frac{(1-\beta_n)}{k}\right)\left(1-\frac{(1-\gamma_n)}{k}\right)\end{array}\right]d(Tx_n,x_n) \\ &+\delta\frac{(1-\alpha_n)}{k}\left(1-\frac{(1-\beta_n)}{k}\right)d(z_n,Tz_n)+2\delta\frac{(1-\alpha_n)}{k}\left(1-\frac{\beta_n(1-\delta)}{k}\right)d(x_n,p). \end{aligned}$$

Denote that

$$\begin{aligned} b_n &= d(x_n,u_n), \\ r_n &= \left(\frac{\alpha_n(1-\delta)}{k}\right), \\ d_n &= \left[\begin{array}{c}\left(1-\frac{(1-\alpha_n)}{k}\right)+\delta\frac{(1-\alpha_n)}{k}\frac{(1-\beta_n)}{k} \\ +\delta\frac{(1-\alpha_n)}{k}\left(1-\frac{(1-\beta_n)}{k}\right)\left(1-\frac{(1-\gamma_n)}{k}\right)\end{array}\right]d(Tx_n,x_n) \\ &\quad +\delta\frac{(1-\alpha_n)}{k}\left(1-\frac{(1-\beta_n)}{k}\right)d(z_n,Tz_n)+2\delta\frac{(1-\alpha_n)}{k}\left(1-\frac{\beta_n(1-\delta)}{k}\right)d(x_n,p) \end{aligned} \tag{2.13}$$

It is clear that (2.13) satisfy all the conditions in Lemma 1.4 and, hence, it follows from its conclusion that $\lim_{n\to\infty}d(x_n,u_n)=0$. Hence, we obtain $\lim_{n\to\infty}d(u_n,p)=0$.

We will prove (ii)⇒(i). Suppose that $\{u_n\}_{n=0}^{\infty}$ converges to p. It follows from (1.3), (1.4), (1.6), (Q_i) and (Q_{iv}) that

$$\begin{aligned} d(u_{n+1}, x_{n+1}) &= d\left(Q\left(u_n, Tu_n, \frac{(1-\alpha_n)}{k}\right), Q\left(Tx_n, Ty_n, \frac{(1-\alpha_n)}{k}\right)\right) \\ &\leq \left(1 - \frac{(1-\alpha_n)}{k}\right) d(u_n, Tx_n) + \frac{(1-\alpha_n)}{k} d(Tu_n, Ty_n) \\ &\leq \delta\left(1 - \frac{(1-\alpha_n)}{k}\right) d(u_n, x_n) + \left(1 - \frac{(1-\alpha_n)}{k}\right) d(u_n, Tu_n) \\ &\quad + \delta\frac{(1-\alpha_n)}{k} d(u_n, y_n) + 2\delta d(u_n, p) \end{aligned}$$

and

$$\begin{aligned} d(u_n, y_n) &= d\left(u_n, Q\left(Tx_n, Tz_n, \frac{(1-\beta_n)}{k}\right)\right) \\ &\leq \frac{(1-\beta_n)}{k} d(u_n, Tx_n) + \left(1 - \frac{(1-\beta_n)}{k}\right) d(u_n, Tz_n) \\ &\leq \delta\frac{(1-\beta_n)}{k} d(u_n, x_n) + \frac{(1-\beta_n)}{k} d(u_n, Tu_n) + 2\delta\frac{(1-\beta_n)}{k} d(u_n, p) \\ &\quad + \delta\left(1 - \frac{(1-\beta_n)}{k}\right) d(u_n, z_n) + 2\delta\left(1 - \frac{(1-\beta_n)}{k}\right) d(u_n, p). \end{aligned}$$

Also, using (1.5) and (Q_i), we get

$$\begin{aligned} d(u_n, z_n) &= d\left(u_n, Q\left(x_n, Tx_n, \frac{(1-\gamma_n)}{k}\right)\right) \\ &\leq \frac{(1-\gamma_n)}{k} d(u_n, x_n) + \left(1 - \frac{(1-\gamma_n)}{k}\right) d(u_n, Tx_n) \\ &\leq \frac{(1-\gamma_n)}{k} d(u_n, x_n) + \left(1 - \frac{(1-\gamma_n)}{k}\right) d(u_n, Tu_n) \\ &\quad + \delta\left(1 - \frac{(1-\gamma_n)}{k}\right) d(u_n, x_n) + 2\delta\left(1 - \frac{(1-\gamma_n)}{k}\right) d(u_n, p) \end{aligned}$$

combining the above inequalities we obtain

$$\begin{aligned} d(u_{n+1},x_{n+1}) &= d\left(Q\left(u_n,Tu_n,\frac{(1-\alpha_n)}{k}\right),Q\left(Tx_n,Ty_n,\frac{(1-\alpha_n)}{k}\right)\right) \\ &\leq \delta\left[\begin{array}{c}\delta\frac{(1-\alpha_n)}{k}\frac{(1-\beta_n)}{k}+\delta\frac{(1-\alpha_n)}{k}\left(1-\frac{(1-\beta_n)}{k}\right)\frac{(1-\gamma_n)}{k}\\ +\delta^2\frac{(1-\alpha_n)}{k}\left(1-\frac{(1-\beta_n)}{k}\right)\left(1-\frac{(1-\gamma_n)}{k}\right)+\left(1-\frac{(1-\alpha_n)}{k}\right)\end{array}\right]d(u_n,x_n) \\ &+\left[\begin{array}{c}\delta\frac{(1-\alpha_n)}{k}\frac{(1-\beta_n)}{k}+\left(1-\frac{(1-\alpha_n)}{k}\right)\\ +\delta^2\frac{(1-\alpha_n)}{k}\left(1-\frac{(1-\beta_n)}{k}\right)\left(1-\frac{(1-\gamma_n)}{k}\right)\end{array}\right]d(u_n,Tu_n) \\ &+\delta\left[\begin{array}{c}\delta\frac{(1-\alpha_n)}{k}\left(1-\frac{(1-\beta_n)}{k}\right)\frac{(1-\gamma_n)}{k}\\ +2+2\delta\frac{(1-\alpha_n)}{k}\left(1-\frac{(1-\beta_n)}{k}\right)\end{array}\right]d(u_n,p) \end{aligned}$$

This can be applied as in (2.13). It is clear that the above equalities satisfies all the conditions in Lemma 1.4 and, hence, it follows from its conclusion that $\lim_{n\to\infty} d(u_n,x_n)=0$. Hence, we obtain $\lim_{n\to\infty} d(x_n,p)=0$. Q.E.D.

Theorem 2.3. Let C,X, and T with fixed point p be the same as in Theorem 2.1. Let $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ be real sequences in $[0,1]$ satisfying $\alpha_1\leq\alpha_n<1$. *For given $x_0=u_0\in C$, consider the iterative sequences $\{x_n\}_{n=0}^{\infty}$ and $\{u_n\}_{n=0}^{\infty}$ defined by (1.4) and (1.3) respectively. Then, $\{x_n\}_{n=0}^{\infty}$ converges to p faster than $\{u_n\}_{n=0}^{\infty}$.*

Proof. From (2.4), we have

$$d(x_{n+1},p)\leq\delta^{n+1}\prod_{i=0}^{n}[1-\frac{\alpha_i(1-\delta)}{k}]d(x_0,p) \tag{2.14}$$

Also, using iterative method (1.3) and (1.5), we obtain

$$\begin{aligned} d(u_{n+1},p) &= d\left(Q\left(u_n,Tu_n,\frac{(1-\alpha_n)}{k}\right),p\right) \\ &\leq \left(1-\frac{(1-\alpha_n)}{k}\right)d(u_n,p)+\frac{(1-\alpha_n)}{k}d(Tu_n,p) \\ &\leq \left(1-\frac{\alpha_n(1-\delta)}{k}\right)d(u_n,p) \end{aligned} \tag{2.15}$$

If the process is repeated n times, we get

$$\begin{aligned} d(u_n,p) &\leq \left(1-\frac{\alpha_{n-1}(1-\delta)}{k}\right)d(u_{n-1},p) \\ d(u_{n-1},p) &\leq \left(1-\frac{\alpha_{n-2}(1-\delta)}{k}\right)d(u_{n-2},p) \\ &\vdots \\ d(u_1,p) &\leq \left(1-\frac{\alpha_0(1-\delta)}{k}\right)d(u_0,p) \end{aligned}$$

From the above inequalities, we have

$$d(u_{n+1},p) \leq \prod_{i=0}^{n}[\left(1-\frac{\alpha_i(1-\delta)}{k}\right)d(u_0,p) \tag{2.16}$$

Applying assumptions $\alpha_1 < \alpha_n \leq 1$ to (2.16) respectively, we obtain

$$\begin{aligned} a_n &= \delta^{n+1}\prod_{i=0}^{n}\left(1-\frac{\alpha_i(1-\delta)}{k}\right)d(x_0,p) \\ b_n &= \prod_{i=0}^{n}\left(1-\frac{\alpha_i(1-\delta)}{k}\right)d(u_0,p) \end{aligned}$$

$$\begin{aligned} \Delta_n &= \frac{a_n}{b_n} = \frac{\delta^{n+1}\prod\limits_{i=0}^{n}\left(1-\frac{\alpha_i(1-\delta)}{k}\right)d(x_0,p)}{\prod\limits_{i=0}^{n}\left(1-\frac{\alpha_i(1-\delta)}{k}\right)d(u_0,p)} \\ &= \delta^{n+1} < 1. \end{aligned}$$

Therefore, $\lim_{n\to\infty}\Delta_n = 0$. Hence from Definition 1.3, we obtain that $\{x_n\}_{n=1}^{\infty}$converges to p faster than $\{u_n\}_{n=1}^{\infty}$. Q.E.D.

Example 2.4. Let $C = [0,1]$ be endowed with the usual metric. Define operator $T : C \to C$ by $Tx = e^{\frac{2}{6+x^2}} + \frac{1}{5}\sin 2x - 1$ with a unique fixed point $p = 0,569233171573655$. The operator T satisfies the condition (1.5) with $\delta \in [0.31, 1)$. For $x_0 = u_0 = 1$, $k = 80$ and $\alpha_n = \beta_n = \gamma_n = 0.40$ for all $n \in \mathbb{N}$, the following table shows that iteration process (1.4) converges to p faster than iteration process (1.3).

TABLE 1. Comparison the rate of convergence of iteration method (1.4) and iteration method (1.3).

Number of iter.	**iteration (1.4)**	**iteration (1.3)**
x_0	1	1
x_1	0,56791674445939	0,51622739519139
x_2	0,56922995170152	0,56155071613645
$\vdots$	$\vdots$	$\vdots$
x_8	0,56923317157366	0,56923155795148
x_9	0,56923317157366	0,56923297904622
$\vdots$	$\vdots$	$\vdots$
x_{17}		0,56923317157365
x_{18}		0,56923317157366

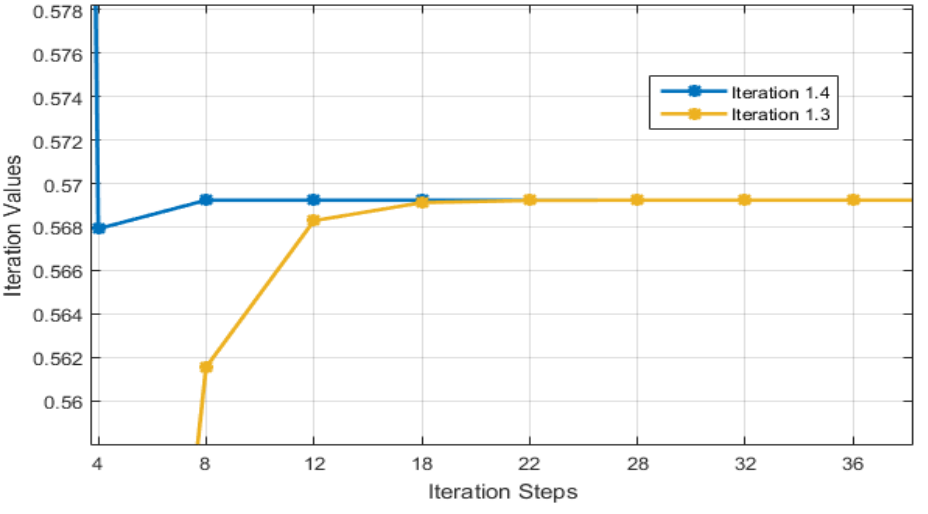


FIGURE 1. Graph of Iteration (1.3) and (1.4)

Theorem 2.5. *S is an approximate operator of T. Let $\{x_n\}_{n=1}^{\infty}$ be an iterative sequence generated by (1.4) for T and define an iterative sequence $\{u_n\}_{n=1}^{\infty}$ as follows:*

$$\begin{cases} u_0 \in K \\ u_{n+1} = Q\left(Su_n, Sv_n, \frac{(1-\alpha_n)}{k}\right) \\ v_n = Q\left(Su_n, Sw_n, \frac{(1-\beta_n)}{k}\right) \\ w_n = Q\left(u_n, Su_n, \frac{(1-\gamma_n)}{k}\right) \end{cases} \tag{2.17}$$

where $\{\alpha_n\}_{n=1}^{\infty}$ is a real sequence in $[0,1]$ satisfying $\frac{1}{2} \leq \frac{\alpha_n}{k}$ for all $n \in \mathbb{N}$. If $Tp = p$ and $Sq = q$ such that $u_n \to q$ as $n \to \infty$, then we have

$$d(p,q) \leq \frac{6\varepsilon}{1-\delta}$$

where $\varepsilon > 0$ is a fixed number.

Proof. It follows from (Q_{iv}), (1.5), (1.4) that

$$\begin{aligned} d(x_{n+1},u_{n+1}) &\leq d\left(Q\left(Su_n,Sv_n,\frac{(1-\alpha_n)}{k}\right),Q\left(Tx_n,Ty_n,\frac{(1-\alpha_n)}{k}\right)\right) \\ &\leq \left(1-\frac{(1-\alpha_n)}{k}\right)d(Su_n,Tx_n)+\frac{(1-\alpha_n)}{k}d(Sv_n,Ty_n) \\ &\leq \delta\left(1-\frac{(1-\alpha_n)}{k}\right)d(u_n,x_n)+2\delta\left(1-\frac{(1-\alpha_n)}{k}\right)d(x_n,p)+\varepsilon \\ &\quad +\frac{(1-\alpha_n)}{k}d(v_n,y_n)+2\delta\frac{(1-\alpha_n)}{k}d(y_n,p) \end{aligned}$$

It can be observed in (2.12).

$$d(y_n,p)\leq\left(1-\frac{\beta_n(1-\delta)}{k}\right)d(x_n,p) \tag{2.18}$$

and

$$\begin{aligned} d(v_n,y_n) &= d\left(Q\left(Su_n,Sw_n,\frac{(1-\beta_n)}{k}\right),Q\left(Tx_n,Tz_n,\frac{(1-\beta_n)}{k}\right)\right) \\ &\leq \left(1-\frac{(1-\beta_n)}{k}\right)d(Su_n,Tx_n)+\frac{(1-\beta_n)}{k}d(Sw_n,Tz_n) \\ &\leq \delta\left(1-\frac{(1-\beta_n)}{k}\right)d(u_n,x_n)+2\delta\left(1-\frac{(1-\beta_n)}{k}\right)d(x_n,p)+\varepsilon \\ &\quad +\delta\frac{(1-\beta_n)}{k}d(w_n,z_n)+2\delta\frac{(1-\beta_n)}{k}d(z_n,p) \end{aligned} \tag{2.19}$$

Also it can be observed in (2.11).

$$d(z_n,p)\leq\left(1-\frac{\gamma_n(1-\delta)}{k}\right)d(x_n,p)$$

and

$$\begin{aligned} d(w_n,z_n) &= d\left(Q\left(u_n,Su_n,\frac{(1-\gamma_n)}{k}\right),Q\left(x_n,Tx_n,\frac{(1-\gamma_n)}{k}\right)\right) \\ &\leq \left(1-\frac{(1-\gamma_n)}{k}\right)d(u_n,x_n)+\frac{(1-\gamma_n)}{k}d(Su_n,Tx_n) \\ &\leq \left(1-\frac{\gamma_n(1-\delta)}{k}\right)d(u_n,x_n)+\frac{(1-\gamma_n)}{k}\varepsilon \\ &\quad +2\delta\frac{(1-\gamma_n)}{k}d(x_n,p) \end{aligned}$$

using $\delta \in (0,1)$, and $1-\frac{\alpha_n}{k} \leq \frac{\alpha_n}{k}$, we get

$$\begin{aligned} d(x_{n+1},u_{n+1}) &\leq \delta \left\{ \begin{array}{c} \left(1-\frac{(1-\alpha_n)}{k}\right)+\frac{(1-\alpha_n)}{k}\left(1-\frac{(1-\beta_n)}{k}\right) \\ +\frac{(1-\alpha_n)}{k}\frac{(1-\beta_n)}{k}\left(1-\frac{\gamma_n(1-\delta)}{k}\right) \end{array} \right\} d(u_n,x_n) \\ &\quad +2\delta \left\{ \begin{array}{c} \left(1-\frac{(1-\alpha_n)}{k}\right)+\frac{(1-\alpha_n)}{k}\left(1-\frac{(1-\beta_n)}{k}\right) \\ +\frac{(1-\alpha_n)}{k}\frac{(1-\beta_n)}{k}\left(1-\frac{\gamma_n(1-\delta)}{k}\right) \\ +\delta\frac{(1-\alpha_n)}{k}\frac{(1-\beta_n)}{k}\frac{(1-\gamma_n)}{k} \\ +\frac{(1-\alpha_n)}{k}\left(1-\frac{\beta_n(1-\delta)}{k}\right) \end{array} \right\} d(x_n,p) \\ &\quad +\varepsilon+\frac{(1-\alpha_n)}{k}\varepsilon+\delta\frac{(1-\alpha_n)}{k}\frac{(1-\beta_n)}{k}\frac{(1-\gamma_n)}{k}\varepsilon \\ &\leq \left(1-\frac{\alpha_n(1-\delta)}{k}\right)d(u_n,x_n) \\ &\quad +\frac{\alpha_n(1-\delta)}{k}\left\{\frac{8\delta d(x_n,p)+6\varepsilon}{(1-\delta)}\right\} \end{aligned}$$

from the above inequality it can be easily seen that all conditions in Lemma 1.5 are satisfied. Since $x_n \to p$ and $u_n \to q$ as $n \to \infty$, then we have

$$d(p,q) \leq \frac{6\varepsilon}{1-\delta}.$$

Q.E.D.

Example 2.6. Let $C=[0,1]$ be endowed with the usual metric. Define operator $T:[0,1]\to[0,1]$ by $Tx=\frac{1}{5}\sin(2x)$ with a unique fixed point $p=0$. It is easy to show that T satisfies (1.5) with $\delta \in [0.40,1)$. Define operator $S:[0,1]\to[0,1]$ by

$$Su=\frac{2(u+0.5)}{50}-\frac{4(u^3-0.01)}{45}-\frac{4(u^5+0.03)}{145}+\frac{8(u^7-0.02)}{1850} \tag{2.20}$$

we get $\max\limits_{x\in Q}|T-S|=0.1699$ by using Wolfram Mathematica 9. Hence for all $x\in C$ and for a fixed $\varepsilon=0.1699>0$, we get $|Tx-Sx|\leq 0.1699$. Thus S is an approximate operator of T in the sense of Definition 1.6. Also $q=0.02080626610575$ is the unique fixed point for the operator S in $C=[0,1]$. Therefore, $|p-q|=0.02080626610575$. If we put $\alpha_n=\beta_n=\gamma_n=0.70$, $k=100$ in (2.17) for the approximate operator S (2.20), we obtain

$$\left\{ \begin{array}{c} u_{n+1}=(0.03)\left(\frac{2(u_n+0.5)}{50}-\frac{4(u_n^3-0.01)}{45}-\frac{4(u_n^5+0.03)}{145}+\frac{8(u_n^7-0.02)}{1850}\right) \\ +(0.97)\left(\frac{2(v_n+0.5)}{50}-\frac{4(v_n^3-0.01)}{45}-\frac{4(v_n^5+0.03)}{145}+\frac{8(v_n^7-0.02)}{1850}\right) \\ v_n=(0.03)\left(\frac{2(u_n+0.5)}{50}-\frac{4(u_n^3-0.01)}{45}-\frac{4(u_n^5+0.03)}{145}+\frac{8(u_n^7-0.02)}{1850}\right) \\ +(0.97)\left(\frac{2(w_n+0.5)}{50}-\frac{4(w_n^3-0.01)}{45}-\frac{4(w_n^5+0.03)}{145}+\frac{8(w_n^7-0.02)}{1850}\right) \\ w_n=(0.03)\,u_n \\ +(0.97)\left(\frac{2(u_n+0.5)}{50}-\frac{4(u_n^3-0.01)}{45}-\frac{4(u_n^5+0.03)}{145}+\frac{8(u_n^7-0.02)}{1850}\right) \end{array} \right.$$

Then we can find the following estimate

$$|p-q| = 0.02080626610575 \leq \frac{6(0.1699)}{1-0.40} = 1.699.$$

The following table shows that the sequence $\{u_n\}_{n=0}^{\infty}$ generated by (2.17) converges to the fixed point $q = 0.02080626610575$.

TABLE 2. Convergence of iteration sequence $\{u_n\}_{n=0}^{\infty}$

Number of iteration	**iteration** $\{u_n\}_{n=0}^{\infty}$
x_0	1
x_1	$0,01846990908229$
x_2	$0,02080312126007$
$\vdots$	$\vdots$
x_7	$0,02080626610574$
x_8	$0,02080626610575$

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