On the existence of unique range sets generated by non-critically injective polynomials and related issues

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Abstract

In this paper, we prove the existence of non-critically injective polynomials whose set of zeros form unique range sets that answers one of the most awaited and fundamental questions of uniqueness theory of entire and meromorphic functions. We also show that there exist some unique range sets and their generating polynomials which can not be characterized by any of the existing generalized results of unique range sets but as an application of our main theorems the same can be characterized. Moreover, as an application of our main results we prove that the cardinality of a unique range set does not always depend upon the number of distinct critical points of its generating polynomial.

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1 Introduction, definitions and results

Let us denote by $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ and $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. For any meromorphic function f, we always mean that it is defined on $\mathbb{C}$. Throughout the paper, we have used standard definitions and notations of Nevanlinna theory as explained in [10].

The study of unique range set in the literature of uniqueness theory was initiated in 1977 after the introduction of set sharing of meromorphic function by F. Gross in [8].

Definition 1.1. [8] For a non-constant meromorphic function f and $S \subset \overline{\mathbb{C}}$, let $E_f(S) = \bigcup_{a \in S}\{(z,p) \in \mathbb{C} \times \mathbb{N} : f(z) = a \text{ with multiplicity } p\}$ $\big(\overline{E}_f(S) = \bigcup_{a \in S}\{(z,1) \in \mathbb{C} \times \mathbb{N} : f(z) = a\}\big)$. Then we say f, g share the set S CM(IM) if $E_f(S) = E_g(S)$ $\big(\overline{E}_f(S) = \overline{E}_g(S)\big)$.

In [8], Gross proposed the following question which has a significant influence on the theory and popularly known as "Gross-Question":

Question 1.1. [8] Can one find two (or possibly even one) finite sets S_j $(j = 1, 2)$ such that any two non-constant entire functions f and g satisfying $E_f(S_j) = E_g(S_j)$ for $j = 1, 2$ must be identical?

Later analogous question for meromorphic functions was raised by various authors in the following manner.

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Question 1.2. [14] Can one find two (or possibly even one) finite sets S_j $(j = 1, 2)$ such that any two non-constant meromorphic functions f and g satisfying $E_f(S_j) = E_g(S_j)$ for $j = 1, 2$ must be identical ?

In view of these questions, the definition of unique range set appears in the literature.

Definition 1.2. [8] If for two non-constant meromorphic(entire) functions f, g and $S \subset \mathbb{C}$, $E_f(S) = E_g(S)$ implies $f \equiv g$, then S is called a unique range set for meromorphic(entire) functions or in brief URSM(URSE) .
Similarly if $\overline{E}_f(S) = \overline{E}_g(S)$ implies $f \equiv g$, then S is called URSM-IM(URSE-IM).

Since then, the notion of unique range sets has emerged as one of the most cultivated concepts of uniqueness theory. As a result various authors [9, 13, 14, 6, 7, 2, 1, 4, 5] have made significant contributions on this new direction throughout these years. Among them, Li-Yang [13] first explored the fact that the base of this finite unique range sets are the zeros of some suitable polynomials. This polynomials are said to generate URSM(URSE)'s and are named as generating polynomials. The term 'suitable polynomial' is very important in generating URSM(URSE). For example Yi's polynomial $P(z) = z^n + az^{n-m} + b$, where $\gcd(m, n) = 1$ and a, b are chosen in such a way that the polynomial has only simple zero; generates URSM for $n \geq 2m + 9$ and $m \geq 2$ but the same does not generate a URSM for any value of n, whenever $m = 1$. Moreover, from all the above discussions, it is clear to all of us that finite unique range sets are very special subsets of $\mathbb{C}$ so that two distinct non-constant meromorphic functions can never share the sets. That implies, their generating polynomials must be of very special characteristics. That is why, the classification of different polynomials has got the utmost importance in characterizing the unique range sets.

In course of time, the study of unique range sets has been divided into two problems. One is to find out the smallest cardinality of a URSM(URSE). The other is to characterize the URSM(URSE)'s under the aegis of classification of polynomials. The present paper is mainly orchestrated under the purview of the latter one. Under this perspective, the following definition steps into the literature.

Definition 1.3. [7] A polynomial P in $\mathbb{C}$ is a strong uniqueness polynomial for meromorphic (entire) functions if for any two non-constant meromorphic (entire) functions f and g, $P(f) \equiv cP(g)$ implies $f \equiv g$, where c is any non-zero constant. In this case we say P is an SUPM (SUPE) in brief.

With the help of the above notion, in 2001, H. Fujimoto [7] characterized one special kind of polynomial whose zero set is forming unique range set. Fujimoto called this kind of polynomial is to satisfy property H. Actually, Fujimoto concentrated on the polynomials which are injective on the set of its distinct critical points; i.e., distinct zeros of its derivative. That is why, property H is named as critical injection property in [4]. Below we shortly recall the definition.

Definition 1.4. [4, 7] Let $P(z)$ be a polynomial such that it has k distinct critical points given by $d_1, d_2, \ldots, d_k$ with multiplicities $q_1, q_2, \ldots, q_k$ respectively. Then $P(z)$ is said to satisfy the critical injection property if $P(d_i) \neq P(d_j)$ for $i \neq j$, where $i, j \in \{1, 2, \cdots, k\}$.

Any polynomial $P(z)$ satisfying this property is called a critically injective polynomial and not satisfying this property is called a non-critically injective polynomial.

From now on, we call critically injective polynomial and non-critically injective polynomial by CIP and NCIP respectively, for our convenience.

Using this CIP's, Fujimoto generalized and characterized all the existing results of URSM (URSE) in the following theorem.

Theorem A. [7] Let $P(z) = (z - \alpha_1)(z - \alpha_2)\dots(z - \alpha_n)$ be a CIP of degree n in $\mathbb{C}$ having only simple zeros. Let P' have k distinct zeros and either $k \geq 3$ or $k = 2$ and P' have no simple zero. Further suppose that P is an *SUPM (SUPE).* If S is the set of zeros of P, then S is a *URSM (URSE)* whenever $n > 2k + 6$ ($n > 2k + 2$) while a *URSM-IM (URSE-IM)* whenever $n > 2k + 12$ ($n > 2k + 5$).

It is clear from *Definition 1.4* that any polynomial can be either CIP or NCIP. So, according to Li-Yang's observation as mentioned earlier, every finite unique range set is generated by either a CIP or a NCIP. Since the existence of unique range sets generated by CIP's have been provided by various authors in [9, 13, 14, 6] and finally been characterised by Fujimoto [7] in 2001, so natural inquisition for the latter one is really of great importance for the characterisation of unique range sets. Hence for the last two decades the following question remains as one of the most awaited unsolved fundamental questions in the literature.

Question 1.3. Does there at all exist any NCIP generating unique range set?

Till date no NCIP have been found to generate URSM. Apropos of the above question, during the last few years, An [1], Banerjee-Lahiri [4] and Banerjee-Mallick [5] made some attempts to find out polynomials generating URSM which are not necessarily CIP. But the class of polynomials they considered are yet not determined to be CIP or NCIP or may be CIP under certain conditions and may be NCIP under some other conditions. So the answer of *Question 1.3* is yet to be particularized and hence this becomes one of the prime motivations of the paper. In fact, in this paper, we give the answer of this long unsolved problem in affirmative.

Moreover, in [1] and [4], authors concentrated on gap polynomials where the gap between the first two terms of the polynomial is at least 3. But in our main results we show that there exists NCIP generating unique range set where the gap between the first two terms can be 1 for URSM and no gap for URSE. So for entire functions a NCIP generating a unique range set is not necessarily be a gap polynomial.

Another interesting fact is that there exist some polynomials of degree n whose zero set forms URSM(URSE) and at the same time it has $(n - 1)$ distinct critical points. For example,

$$P(z) = az^n - n(n-1)z^2 + 2n(n-2)bz - (n-1)(n-2)b^2,$$

where $n \geq 3$ is an integer and a and b are two non-zero complex numbers satisfying $ab^{n-2} \neq 2$. The zero set of the above polynomial forms a URSM for $n \geq 11$ and it has $(n - 1)$ distinct critical points [3, see section 5, concluding remark and an open question].

For this type of polynomials, if we apply the theorems proved in [1, 4, 7], we can see that $n \geq 2(n - 1) + 7 = 2n + 5$, which is an absurd inequality as n is a positive integer. So this type of polynomials can not be characterized by the results of [1, 4, 7]. Actually, in all these generalized results, the degree say 'n' of the polynomial is $\geq 2k + 7$, where k is the number of distinct critical points of the polynomial. So, for $n \geq 2k + 7$, we see $2k \leq n - 7$. Also note that $k \geq 2$ [7, see Line:- 4-8 of Page-1183]; i.e., $n \geq 11$, so k is at most $(n - 9)$ otherwise $2(n - 8) \leq n - 7$ implies $n \leq 9$. For this reason, for any polynomial having more than $(n - 9)$ distinct critical points can

not be characterized by the results of [1, 4, 7] to generate URSM(URSE). But as an application of our main results, we shall prove that it is possible to characterize polynomials whose zero set is a URSM at the same time it has more than $(n-9)$ distinct critical points. That is we have found a different way of counting the degree of the generating polynomials. In a word, we have shown that it is not always customary for the degree of a generating polynomial to be related to the number of its distinct critical points.

Before going to our main results, we recall the following notion of weighted sharing [11, 12], which is a scaling between CM and IM sharing and has an immense application in the uniqueness theory for relaxing the nature of sharing the sets.

Definition 1.5. [11, 12] Let k be a nonnegative integer or infinity. For $a \in \overline{\mathbb{C}}$ we denote by $E_k(a;f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a;f) = E_k(a;g)$, we say that f, g share the value a with weight k.

We write f, g share (a,k) to mean that f, g share the value a with weight k. Clearly if f, g share (a,k), then f, g share (a,p) for any integer p, $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a,0)$ or (a,∞) respectively.

Definition 1.6. [11] For $S \subset \overline{\mathbb{C}}$ we define $E_f(S,k) = \cup_{a\in S} E_k(a;f)$, where k is a non-negative integer $a \in S$ or infinity. Clearly $E_f(S) = E_f(S,\infty)$ and $\overline{E}_f(S) = E_f(S,0)$.

Definition 1.7. [4] A set $S \subset \overline{\mathbb{C}}$ is called a unique range set for meromorphic (entire) functions with weight k if for any two non-constant meromorphic (entire) functions f and g, $E_f(S,k) = E_g(S,k)$ implies $f \equiv g$. We write S is $URSMk$ $(URSEk)$ in short.

From the *Definition 1.7*, it is clear that every $URSMk(URSEk)$ is a $URSM(URSE)$ but the converse may not be true.

Let

$$P(z) = z^n + az^{n-m} + bz^{n-2m} + c, \tag{1.1}$$

where $a, b \in \mathbb{C}^*$ and $n, m \in \mathbb{N}$ such that $\gcd(m,n) = 1$, $n > 2m$, $a^2 = 4b$. Then

$$P^{'}(z) = z^{n-2m-1}\left(z^m + \frac{a}{2}\right)\left(nz^m + (n-2m)\frac{a}{2}\right).$$

Suppose d_i and c_i be the distinct roots of the equation $z^m = -\dfrac{(n-2m)a}{2n}$ and $z^m = -\dfrac{a}{2}$ respectively for $i = 1, 2, \ldots, m$.
Further suppose that $\lambda_i = -\left(d_i^n + ad_i^{n-m} + bd_i^{n-2m}\right)$ for $i = 1, 2, \ldots, m$. Therefore d_i's are not zeros of $P(z)$ if $c \neq \lambda_i$. Also note that as $a^2 = 4b$, so

$$P(z) = z^{n-2m}\left(z^m + \frac{a}{2}\right)^2 + c = z^{n-2m}\prod_{i=1}^{m}(z-c_i)^2 + c.$$

Hence $z = 0, c_i$ are not zeros of $P(z)$ if $c \neq 0$. So clearly $P(z)$ has no multiple zero if $c \neq 0, \lambda_i$.
Observe that $P(0) = P(c_i) = c$, which implies that $P(z)$ is a NCIP.
Now we shall show that the zero set of this polynomial forms a unique range set and thus obtain the answer of *Question 1.3* successfully.

Theorem 1.1. Let $P(z)$ be defined by (1.1) and $S = \{z : P(z) = 0\}$, with $m \geq 2$ and $c \in \mathbb{C} - \{0, \lambda_i\}$. Then S is a URSM2 (URSE2) for $n \geq 4m + 9$ ($n \geq 4m + 5$).

We observe that in *Theorem 1.1*, $m \geq 2$. That is S is the zero set of a NCIP having at least 1 gap between the first two terms of the polynomial. So it would be a natural query to know about the results while dealing the above problem under the assumption that there is no gap between the first two terms of the polynomial. That is we have the following question.

Question 1.4. Does the zero set of $P(z)$ defined by (1.1) form a URSM if $m = 1$?

In order to answer this question, we have obtained the following result.

Theorem 1.2. Let $P(z)$ be defined by (1.1) and $S = \{z : P(z) = 0\}$, with $m = 1$, $n \geq 13$ and $c \in \mathbb{C} - \{0, \lambda_1\}$. Then for two non-constant meromorphic functions f, g satisfying $E_f(S, 2) = E_g(S, 2)$ the following cases hold.

i) If n is even with $n = 2l$ for some positive integer l, then

$$g = -\frac{a}{2}\frac{(h^{2l-2} + h^{2l-3} + \ldots + h^2 + h + 1) \pm h^{l-1}}{h^{2l-1} + h^{2l-2} + h^{2l-3} + \ldots + h^3 + h^2 + h + 1}$$

and

$$f = -\frac{a}{2}\frac{(h^{2l-1} + h^{2l-2} + \ldots + h^3 + h^2 + h) \pm h^l}{h^{2l-1} + h^{2l-2} + h^{2l-3} + \ldots + h^3 + h^2 + h + 1},$$

where $h = \dfrac{f}{g}$. The sign before the last term of the numerator is taken same for f and g at the same time.

ii) If n is odd, then

$$g = -\frac{a}{2}\frac{(k^{2n-4} + \ldots + k^2 + 1) \pm k^{n-2}}{k^{2n-2} + k^{2n-4} + \ldots + k^2 + 1}$$

and

$$f = -\frac{a}{2}\frac{(k^{2n-2} + \ldots + k^4 + k^2) \pm k^n}{k^{2n-2} + k^{2n-4} + \ldots + k^2 + 1},$$

where k is a non-constant meromorphic function with $k^2 = \frac{f}{g}$. The sign before the last term of the numerator is taken same for f and g at the same time.

iii) In particular, if f and g have no zero, then $f \equiv g$ for $n \geq 9$.

From *Theorem 1.2*, we get that the answer of *Question 1.4* is in general negative. The answer is positive for a special class of meromorphic functions; i.e., the class of meromorphic functions having no zero. So the next question arises "What happens if we consider the class of meromorphic functions having no pole; i.e., the class of entire functions?" In a word this thought implies the following question.

Question 1.5. Is it possible that the zero set of $P(z)$ defined by (1.1) forms a URSE if $m = 1$?

We have also got the answer of this question in affirmative which is as follows.

Theorem 1.3. Let $P(z)$ be defined by (1.1) and $S = \{z : P(z) = 0\}$, with $m = 1$, $n \geq 9$ and $c \in \mathbb{C} - \{0, \lambda_1\}$. Then S is a URSE2.

Now we need the following definitions to proceed further.

Definition 1.8. [12] Suppose l be a positive integer. We denote by $\overline{N}(r, a; f \mid\geq l)$, the counting function of those a-points of f whose multiplicities are greater than or equal to l, where each a-point is counted only once.

Definition 1.9. [12] We denote by $N_2(r, a; f) = \overline{N}(r, a; f) + \overline{N}(r, a; f \mid\geq 2)$.

On the way of proving the main theorems we shall need the following Lemmas.

2 Lemmas

Lemma 2.1. [12] Let f and g be two non-constant meromorphic functions sharing $(1, 2)$. Then one of the following three cases holds:

(i) $T(r, f) \leq N_2(r, 0; f)+N_2(r, \infty; f)+N_2(r, 0; g)+N_2(r, \infty; g)+S(r, f)+S(r, g)$, the same inequality holds for $T(r, g)$;

(ii) $fg = 1$;

(iii) $f \equiv g$.

Lemma 2.2. Let $F = \prod_{i=1}^{t}(f - \alpha_i)^{m_i}$ and $G = \prod_{i=1}^{t}(g - \alpha_i)^{m_i}$, where f and g be two non-constant meromorphic functions, $t \geq 3$ and $\sum_{i=1}^{t} m_i = n$. If

(i) $t \geq 4$ or

(ii) $t = 3$ and $\gcd(m_i, n) = 1$ for at least one of the m_i's such that $m_i \geq 2$,
then $FG \neq a$, where a is any non-zero complex number.

Proof. If possible suppose that $FG = a$. Then

$$\prod_{i=1}^{t}(f - \alpha_i)^{m_i} \prod_{i=1}^{t}(g - \alpha_i)^{m_i} = a. \tag{2.1}$$

It is clear from (2.1) that each α_i-point of f is a pole of g.

(i) Suppose that z_0 be a α_i-point of f of multiplicity p and a pole of g of multiplicity q. Then $m_i p = nq$; i.e., $m_i p \geq n$; i.e., $\frac{1}{p} \leq \frac{m_i}{n}$. Since α_i-point of f is chosen arbitrarily, so we would get similar inequality for every α_i-point of f. Now using $\sum_{i=1}^{t} m_i = n$ and the second fundamental

theorem for these α_i-points of f, we get

$$\begin{aligned}(t-2)T(r,f) &\leq \sum_{i=1}^{t}\overline{N}(r,\alpha_i;f)+S(r,f)\\ &\leq \sum_{i=1}^{t}\frac{m_i}{n}T(r,f)+S(r,f)\\ &\leq T(r,f)+S(r,f),\end{aligned}$$

which is a contradiction for $t \geq 4$.

(ii) In this part we have $t = 3$. First of all let us suppose that there exist only one m_i say $m_2 \geq 2$ such that $\gcd(m_2, n) = 1$. Suppose z_0 be any α_2-point of f of multiplicity r and a pole of g of multiplicity s. Then $m_2 r = ns$. Since $\gcd(m_2, n) = 1$, so n divides r; i.e., $r = nl$ for some $l \in \mathbb{N}$. Hence $r \geq n$ i.e., $\frac{1}{r} \leq \frac{1}{n}$. Now proceeding as the above part we shall get

$$\begin{aligned}T(r,f) &\leq \sum_{i=1}^{3}\overline{N}(r,\alpha_i;f)+S(r,f)\\ &\leq \frac{m_1}{n}T(r,f)+\frac{1}{n}T(r,f)+\frac{m_3}{n}T(r,f)+S(r,f)\\ &\leq \frac{n-m_2+1}{n}T(r,f)+S(r,f)\\ &\leq (1-\frac{m_2-1}{n})T(r,f)+S(r,f),\end{aligned}$$

which is a contradiction as $m_2 \geq 2$. Now if we have more than one m_i's such that $m_i \geq 2$ and $\gcd(m_i, n) = 1$, then proceeding in a similar fashion we arrive at a contradiction. Q.E.D.

3 Proof of the Theorems

Proof of Theorem 1.1. Suppose f and g be two non-constant meromorphic functions such that $E_f(S, 2) = E_g(S, 2)$. Further suppose that

$$F = \frac{f^n + af^{n-m} + bf^{n-2m}}{-c}, \tag{3.1}$$

$$G = \frac{g^n + ag^{n-m} + bg^{n-2m}}{-c}. \tag{3.2}$$

Since $a^2 = 4b$, so (3.1) and (3.2) reduces to

$$F = \frac{f^{n-2m}(f^m + \frac{a}{2})^2}{-c}, \tag{3.3}$$

$$G = \frac{g^{n-2m}(g^m + \frac{a}{2})^2}{-c}. \tag{3.4}$$

Clearly F and G share $(1, 2)$. Hence according to *Lemma 2.1*, one of the following cases holds.

Case-1. Let

$$T(r,F) \leq N_2(r,0;F) + N_2(r,\infty;F) + N_2(r,0;G) + N_2(r,\infty;G) + S(r,F) + S(r,G). \tag{3.5}$$

Since the same inequality holds for $T(r,G)$, so with out loss of generality we may assume that $T(r,G) \leq T(r,F)$. Then clearly from (3.1) and (3.2), we get

$$T(r,g) \leq T(r,f) \quad and \quad S(r,F) + S(r,G) = S(r,f) + S(r,g).$$

Further (3.3) and (3.4) implies:

$$N_2(r,0;F) = 2\overline{N}(r,0;f) + 2\sum_{i=1}^{m}\overline{N}(r,c_i;f),$$

$$N_2(r,\infty;F) = 2\overline{N}(r,\infty;f),$$

$$N_2(r,0;G) = 2\overline{N}(r,0;g) + 2\sum_{i=1}^{m}\overline{N}(r,c_i;g),$$

$$N_2(r,\infty;G) = 2\overline{N}(r,\infty;g).$$

So in view of (3.5), we have

$$\begin{aligned} nT(r,f) &\leq 2\overline{N}(r,0;f) + 2\sum_{i=1}^{m}\overline{N}(r,c_i;f) + 2\overline{N}(r,\infty;f) \\ &\quad +2\overline{N}(r,0;g) + 2\sum_{i=1}^{m}\overline{N}(r,c_i;g) + 2\overline{N}(r,\infty;g) \\ &\quad +S(r,f) + S(r,g) \\ &\leq (4m+8)T(r,f) + S(r,f) + S(r,g), \end{aligned}$$

which is a contradiction for $n \geq 4m+9$.

Case-2. Let $FG = 1$. Therefore we have

$$\left[f^n + af^{n-m} + bf^{n-2m}\right]\left[g^n + ag^{n-m} + bg^{n-2m}\right] = c^2;$$

i.e.,

$$\left[f^{n-2m}\prod_{i=1}^{m}(f-c_i)^2\right]\left[g^{n-2m}\prod_{i=1}^{m}(g-c_i)^2\right] = c^2. \tag{3.6}$$

Now if $m \geq 3$, then in view of *Lemma 2.2*, we get from (3.6) that $t \geq 4$, which implies that this case is impossible.
If $m = 2$, then from the assumption of the theorem we have $\gcd(n,2) = 1$ and at the same time from (3.6) we have $t = 3$. We also observe from (3.6) that there exist two m_i's such that $m_i = 2$.

Hence $\gcd(m_i, n) = 1$ for at least two $m_i \geq 2$. Therefore in view of *Lemma 2.2*, we again get a contradiction.

Case-3. Let $F \equiv G$. So we have

$$f^n + af^{n-m} + bf^{n-2m} = g^n + ag^{n-m} + bg^{n-2m}. \tag{3.7}$$

Now suppose that $h = \dfrac{f}{g}$ and putting this value in (3.7), we get

$$g^n(h^n - 1) + ag^{n-m}(h^{n-m} - 1) + bg^{n-2m}(h^{n-2m} - 1) = 0;$$

i.e.,

$$g^{2m}(h^n - 1) + ag^m(h^{n-m} - 1) + \frac{a^2}{4}(h^{n-2m} - 1) = 0. \tag{3.8}$$

Next let us consider the following subcases.

Subcase-3.1. Suppose that h is non-constant. Then from (3.8), we get

$$\begin{aligned}\left(g^m + \frac{a}{2}\frac{h^{n-m} - 1}{h^n - 1}\right)^2 &= \frac{a^2}{4}\left(\frac{h^{n-m} - 1}{h^n - 1}\right)^2 - \frac{a^2}{4}\left(\frac{h^{n-2m} - 1}{h^n - 1}\right) \\ &= \frac{a^2}{4}\frac{(h^{n-m} - 1)^2 - (h^{n-2m} - 1)(h^n - 1)}{(h^n - 1)^2} \\ &= \frac{a^2}{4}\frac{(h^n - 2h^{n-m} + h^{n-2m})}{(h^n - 1)^2} \\ &= \frac{a^2}{4}\frac{h^{n-2m}(h^m - 1)^2}{(h^n - 1)^2}.\end{aligned} \tag{3.9}$$

Now we need to consider the following subcases.

Subcase-3.1.1. Suppose that n is even. Then $n = 2l$ for some positive integer l. Hence (3.9) becomes

$$\left(g^m + \frac{a}{2}\frac{h^{2l-m} - 1}{h^{2l} - 1}\right)^2 = \frac{a^2}{4}\frac{[h^{l-m}(h^m - 1)]^2}{(h^{2l} - 1)^2}$$

$$\implies \left(g^m + \frac{a}{2}\frac{h^{2l-m} - 1}{h^{2l} - 1}\right) = \pm\frac{a}{2}\frac{h^{l-m}(h^m - 1)}{(h^l - 1)}$$

$$\implies g^m = -\frac{a}{2}\left(\frac{h^{2l-m} - 1}{h^{2l} - 1} \mp \frac{h^{l-m}(h^m - 1)}{(h^{2l} - 1)}\right)$$

$$\implies g^m = -\frac{a}{2}\left(\frac{(h^{2l-m} - 1) \mp h^{l-m}(h^m - 1)}{h^{2l} - 1}\right);$$

which implies

$$g^m = -\frac{a}{2}\frac{(h^{2l-m-1}+h^{2l-m-2}+\ldots+h+1)\mp h^{l-m}(h^{m-1}+h^{m-2}+\ldots+h+1)}{h^{2l-1}+h^{2l-2}+h^{2l-3}+\ldots+h^3+h^2+h+1}; \tag{3.10}$$

i.e.,

$$g^m = -\frac{a}{2}\frac{(h^{2l-m-1}+h^{2l-m-2}+\ldots+h+1)+h^{l-m}(h^{m-1}+h^{m-2}+\ldots+h+1)}{h^{2l-1}+h^{2l-2}+h^{2l-3}+\ldots+h^3+h^2+h+1} \tag{3.11}$$

or

$$g^m = -\frac{a}{2}\frac{(h^{2l-m-1}+h^{2l-m-2}+\ldots+h+1)-h^{l-m}(h^{m-1}+h^{m-2}+\ldots+h+1)}{h^{2l-1}+h^{2l-2}+h^{2l-3}+\ldots+h^3+h^2+h+1}. \tag{3.12}$$

From (3.11), we see that the numerator and denominator in the right hand side may have common factors say $(h-\zeta_i)$'s for $i=1,2,\ldots,t$ iff

$$\zeta_i^{l-m}\left(\zeta_i^{m-1}+\zeta_i^{m-2}+\ldots+\zeta_i+1\right)=\zeta_i^{2l-1}+\ldots+\zeta_i^{2l-m};$$

i.e.,

$$\zeta_i^{l-m}\left(\zeta_i^{m-1}+\ldots+\zeta_i+1\right)\left(\zeta_i^{l}-1\right)=0.$$

Since $\zeta_i=0,1$; are neither zeros of

$$(z^{2l-m-1}+z^{2l-m-2}+\ldots+z+1)+z^{l-m}(z^{m-1}+z^{m-2}+\ldots+z+1)$$

nor zeros of

$$z^{2l-1}+z^{2l-2}+z^{2l-3}+\ldots+z^3+z^2+z+1,$$

so the value of t is at most $(2l-1)-\{(l-m)+1\}=l+m-2$; i.e., the numerator and the denominator may have at most $l+m-2$ factors in common. It is obvious that the denominator has only simple factors. Hence the denominator has at least $l-m+1$ distinct non-common factors. Suppose $(h-\eta_i)$'s for $i=\{1,2,\ldots,l-m+1\}$ are the distinct non-common factors of the denominator. Clearly each η_i-point of h is a pole of g of multiplicity at least m. So using second fundamental theorem on these points, we get

$$\begin{aligned}(l-m-1)T(r,h) &\leq \sum_{i=1}^{l-m+1}\overline{N}(r,\eta_i;h)+S(r,h)\\ &\leq \frac{1}{m}\sum_{i=1}^{l-m+1}N(r,\eta_i;h)+S(r,h)\\ &\leq \frac{l-m+1}{m}T(r,h)+S(r,h),\end{aligned}$$

which is a contradiction for $l>\frac{m^2+1}{m-1}=\frac{(m-1)^2+2(m-1)+2}{m-1}=m+1+\frac{2}{m-1}$; i.e., a contradiction for $n\geq 2m+8$. Similarly we can arrive at a contradiction if we consider (3.12) instead of (3.11).

Subcase-3.1.2. Suppose that n is odd. Then $n = 2p + 1$ for some positive integer p. Hence $n - 2m = 2(p - m) + 1$. Now it is to be noted that from (3.7), we get

$$f^{n-2m}(f^m + \frac{a}{2})^2 = g^{n-2m}(g^m + \frac{a}{2})^2$$

$$\Longrightarrow \frac{f^{n-2m}}{g^{n-2m}} = \left(\frac{g^m + \frac{a}{2}}{f^m + \frac{a}{2}}\right)^2$$

$$\Longrightarrow h^{n-2m} = \left(\frac{g^m + \frac{a}{2}}{f^m + \frac{a}{2}}\right)^2$$

$$\Longrightarrow h^{2(p-m)+1} = \left(\frac{g^m + \frac{a}{2}}{f^m + \frac{a}{2}}\right)^2$$

$$\Longrightarrow h = \left(\frac{g^m + \frac{a}{2}}{h^{(p-m)}(f^m + \frac{a}{2})}\right)^2.$$

Since f, g and h are meromorphic functions, so is $f^2+\frac{a}{2}$, $g^2+\frac{a}{2}$ and $h^{(p-m)}$ and so is $\left(\frac{g^2 + \frac{a}{2}}{h^{(p-m)}(f^2 + \frac{a}{2})}\right)$.

Now let us denote $\left(\frac{g^2 + \frac{a}{2}}{h^{(p-m)}(f^2 + \frac{a}{2})}\right) = k$. Hence $h = k^2$ for some meromorphic function k.

Now putting $h = k^2$ in (3.9), we get

$$\left(g^m + \frac{a}{2}\frac{(k^2)^{n-m} - 1}{(k^2)^n - 1}\right)^2 = \frac{a^2}{4}\frac{[k^{(n-2m)}((k^2)^m - 1)]^2}{((k^2)^n - 1)^2}.$$

Then proceeding similarly like the above subcase, we get

$$g^m = -\frac{a}{2}\frac{(k^{2n-2m-2} + k^{2n-2m-4} + \ldots + k^2 + 1) \mp k^{n-2m}(k^{2m-2} + k^{2m-4} + \ldots + k^2 + 1)}{k^{2n-2} + k^{2n-4} + k^{2n-6} + \ldots + k^6 + k^4 + k^2 + 1};$$

i.e.,

$$g^m = -\frac{a}{2}\frac{(k^{2n-2m-2} + \ldots + k^2 + 1) + k^{n-2m}(k^{2m-2} + \ldots + k^2 + 1)}{k^{2n-2} + k^{2n-4} + k^{2n-6} + \ldots + k^6 + k^4 + k^2 + 1} \tag{3.13}$$

or,

$$g^m = -\frac{a}{2}\frac{(k^{2n-2m-2} + \ldots + k^2 + 1) - k^{n-2m}(k^{2m-2} + \ldots + k^2 + 1)}{k^{2n-2} + k^{2n-4} + k^{2n-6} + \ldots + k^6 + k^4 + k^2 + 1}. \tag{3.14}$$

From (3.13), we observe that the numerator and denominator in the right hand side may have common factors say $(k - \beta_i)$'s for $i = 1, 2, \ldots, t$ iff

$$\beta_i^{n-2m}(\beta_i^{2m-2} + \ldots + \beta_i^2 + 1) = \beta_i^{2n-2} + \ldots + \beta_i^{2n-2m};$$

i.e.,

$$\beta_i^{n-2m}(\beta_i^{2m-2} + \ldots + \beta_i^2 + 1)(\beta_i^n - 1) = 0.$$

Since $\beta_i = 0$ is neither the zero of

$$(z^{2n-2m-2} + \ldots + z^2 + 1) + z^{n-2m}(z^{2m-2} + \ldots + z^2 + 1)$$

nor the zero of

$$z^{2n-2}+z^{2n-4}+z^{2n-6}+\ldots+z^6+z^4+z^2+1,$$

so the value of t is at most $(2n-2)-(n-2m)=n+2m-2$; i.e., the numerator and the denominator may have at most $n+2m-2$ factors in common. It is easy to show that the denominator has only simple factors, so we do not give the calculation here. Hence the denominator has at least $n-2m$ distinct non-common factors. Suppose $(k-\gamma_i)$'s for $i=\{1,2,\ldots,n-2m\}$ are the distinct non-common factors of the denominator. Clearly each γ_i-point of k is a pole of g of multiplicity at least m. So using second fundamental theorem on these points, we get

$$\begin{aligned}(n-2m-2)T(r,k) &\leq \sum_{i=1}^{n-2m}\overline{N}(r,\gamma_i;k)+S(r,k)\\ &\leq \frac{1}{m}\sum_{i=1}^{n-2m}N(r,\gamma_i;k)+S(r,k)\\ &\leq \frac{n-2m}{m}T(r,k)+S(r,k),\end{aligned}$$

which is a contradiction for $n>\frac{2m^2}{m-1}=\frac{2(m-1)^2+4(m-1)+2}{m-1}=2m+2+\frac{2}{m-1}$; i.e., a contradiction for $n\geq 2m+5$.
We can similarly arrive at a contradiction for equation (3.14). So we deal the next subcase.

Subcase-3.2. Suppose that h is constant. Since g is non-constant, so from (3.8), we get $h^n=h^{n-m}=h^{n-2m}=1$. Since $\gcd(m,n)=1$, so $h=1$; i.e., $f\equiv g$. Q.E.D.

Proof of Theorem 1.2. i) Suppose that

$$F=\frac{f^n+af^{n-1}+bf^{n-2}}{-c}=\frac{f^{n-2}(f+\frac{a}{2})^2}{-c}, \tag{3.15}$$

$$G=\frac{g^n+ag^{n-1}+bg^{n-2}}{-c}=\frac{g^{n-2}(g+\frac{a}{2})^2}{-c}. \tag{3.16}$$

Clearly F and G share $(1,2)$. Hence according to *Lemma 2.1*, one of the following cases hold.

Case-1. Let

$$T(r,F)\leq N_2(r,0;F)+N_2(r,F)+N_2(r,0;G)+N_2(r,G)+S(r,F)+S(r,G). \tag{3.17}$$

Proceeding in a similar fashion as Case-1 of Proof of *Theorem 1.1*, we obtain a contradiction for $n\geq 13$.

Case-2. Let $FG=1$. Therefore we have

$$\left[f^{n-2}(f+\frac{a}{2})^2\right]\left[g^{n-2}(g+\frac{a}{2})^2\right]=c^2. \tag{3.18}$$

Now each 0-point and $-\frac{a}{2}$-point of f is a pole of g and vice-versa. Suppose z_0 be a zero of f of multiplicity p and a pole of g of multiplicity q. Then we have

$$(n-2)p = nq. \tag{3.19}$$

Now let us consider the following subcases.

Subcase-2.1. Suppose that n is odd. Then $\gcd(n,2) = 1$ and so $\gcd(n, n-2) = 1$. Therefore in view of (3.19), we get $p \geq n$; i.e., $\frac{1}{p} \leq \frac{1}{n}$. Hence,

$$\overline{N}(r,0;f) \leq \frac{1}{n}T(r,f).$$

Similarly, we can show that

$$\overline{N}(r,-\frac{a}{2};f) \leq \frac{1}{n}T(r,f).$$

Using the similar arguments we can also show

$$\overline{N}(r,0;g) \leq \frac{1}{n}T(r,g),$$

$$\overline{N}(r,-\frac{a}{2};g) \leq \frac{1}{n}T(r,g).$$

On the other hand

$$\overline{N}(r,\infty;f) = \overline{N}(r,0;g) + \overline{N}(r,-\frac{a}{2};g) \leq \frac{2}{n}T(r,g).$$

Now using the first fundamental theorem, we get from (3.18) that

$$T(r,f) = T(r,g) + O(1). \tag{3.20}$$

Hence using the second fundamental theorem in view of (3.20), we get

$$\begin{aligned} T(r,f) &\leq \overline{N}(r,0;f) + \overline{N}(r,-\frac{a}{2};f) + \overline{N}(r,\infty;f) + S(r,f) \\ &\leq \frac{2}{n}T(r,f) + \frac{2}{n}T(r,g) + S(r,f) \\ &\leq \frac{4}{n}T(r,f) + S(r,f), \end{aligned}$$

which is a contradiction for $n \geq 5$.

Subcase-2.2. Suppose that n is even. Then n and $n-2$ may have only 2 as the common factor. Hence from (3.19), we get $p \geq \frac{n}{2}$; i.e., $\frac{1}{p} \leq \frac{2}{n}$. Now proceeding similarly like the above subcase, we get

$$\overline{N}(r,0;f) \leq \frac{2}{n}T(r,f).$$

$$\overline{N}(r,-\frac{a}{2};f) \leq \frac{2}{n}T(r,f).$$

$$\overline{N}(r,0;g) \leq \frac{2}{n}T(r,g),$$
$$\overline{N}(r,-\frac{a}{2};g) \leq \frac{2}{n}T(r,g).$$

and

$$\overline{N}(r,\infty;f) \leq \overline{N}(r,0;g) + \overline{N}(r,-\frac{a}{2};g) \leq \frac{4}{n}T(r,g).$$

Then using the second fundamental theorem same as the above subcase, we get a contradiction for $n \geq 9$.

Case-3. Let $F \equiv G$. So we have

$$f^n + af^{n-1} + bf^{n-2} = g^n + ag^{n-1} + bg^{n-2}. \tag{3.21}$$

Now suppose that $h = \frac{f}{g}$ and putting this value in (3.21), we get

$$g^n(h^n - 1) + ag^{n-1}(h^{n-1} - 1) + bg^{n-2}(h^{n-2} - 1) = 0;$$

i.e.,

$$g^2(h^n - 1) + ag(h^{n-1} - 1) + \frac{a^2}{4}(h^{n-2} - 1) = 0. \tag{3.22}$$

Then proceeding similarly like Subcase-3.1 of *Theorem 1.1*, we get

$$g = -\frac{a}{2}\frac{(h^{2l-2} + h^{2l-3} + \ldots + h^2 + h + 1) + h^{l-1}}{h^{2l-1} + h^{2l-2} + h^{2l-3} + \ldots + h^3 + h^2 + h + 1} \tag{3.23}$$

and

$$f = -\frac{a}{2}\frac{(h^{2l-1} + h^{2l-2} + \ldots + h^3 + h^2 + h) + h^l}{h^{2l-1} + h^{2l-2} + h^{2l-3} + \ldots + h^3 + h^2 + h + 1}; \tag{3.24}$$

or,

$$g = -\frac{a}{2}\frac{(h^{2l-2} + h^{2l-3} + \ldots + h^2 + h + 1) - h^{l-1}}{h^{2l-1} + h^{2l-2} + h^{2l-3} + \ldots + h^3 + h^2 + h + 1} \tag{3.25}$$

and

$$f = -\frac{a}{2}\frac{(h^{2l-1} + h^{2l-2} + \ldots + h^3 + h^2 + h) - h^l}{h^{2l-1} + h^{2l-2} + h^{2l-3} + \ldots + h^3 + h^2 + h + 1}, \tag{3.26}$$

where h is a non-constant meromorphic function satisfying $h = \frac{f}{g}$ and n is even with $n = 2l$ for some positive integer l.

For n is odd, proceeding similarly like Subcase-3.2 of *Theorem 1.1*, we get

$$g = -\frac{a}{2}\frac{(k^{2n-4} + \ldots + k^2 + 1) + k^{n-2}}{k^{2n-2} + k^{2n-4} + \ldots + k^2 + 1} \tag{3.27}$$

and

$$f = -\frac{a}{2}\frac{(k^{2n-2} + \ldots + k^4 + k^2) + k^n}{k^{2n-2} + k^{2n-4} + \ldots + k^2 + 1}; \tag{3.28}$$

or,

$$g = -\frac{a}{2}\frac{(k^{2n-4}+\ldots+k^2+1)-k^{n-2}}{k^{2n-2}+k^{2n-4}+\ldots+k^2+1} \tag{3.29}$$

and

$$f = -\frac{a}{2}\frac{(k^{2n-2}+\ldots+k^4+k^2)-k^n}{k^{2n-2}+k^{2n-4}+\ldots+k^2+1}, \tag{3.30}$$

where k is a non-constant meromorphic function with $k^2 = \frac{f}{g}$.

ii) In particular, when f and g have no zero, then in Case-1 of this theorem we would get a contradiction for $n \geq 9$. In Case-2, we get contradiction automatically for $n \geq 9$. In Case-3, we again get equations (3.23), (3.24), (3.25), (3.26) or (3.27), (3.28), (3.29), (3.30).

From (3.23), we see that the numerator and the denominator in the right hand side may have some common factors. It is obvious that the numerator may have only simple factors. So, let us suppose that the common factors are $(h-\alpha_1), (h-\alpha_2), \ldots, (h-\alpha_t)$. Then this α_i's for $i = 1, 2, \ldots, t$ are zeros of the polynomial

$$(z^{2l-2}+z^{2l-3}+\ldots+z^2+z+1)+z^{l-1} \tag{3.31}$$

and

$$z^{2l-1}+z^{2l-2}+z^{2l-3}+\ldots+z^3+z^2+z+1. \tag{3.32}$$

Now by some simple factorization we can write (3.31) and (3.32) as

$$(z^{l-1}+1)(z^{l-1}+z^{l-2}+\ldots+z+1) \tag{3.33}$$

and

$$(z^l+1)(z^{l-1}+z^{l-2}+\ldots+z+1) \tag{3.34}$$

respectively. Hence α_i's are the roots of the equation

$$z^{l-1}+z^{l-2}+\ldots+z+1=0.$$

Clearly $z^{l-1}+1$ and z^l+1 can not have any common zero. So if δ is any root of the equation

$$z^{l-1}+1=0,$$

then $(h-\delta)$ is a non-common factor of the numerator. Then each δ-point of h is a zero of g in view of (3.23). We know that g has no zero by the condition of the theorem. Therefore h must omit the point δ. Observe that (3.21) implies f and g share ∞ CM. By the assumption of the theorem they also share 0 CM. Hence h does not have any zero or pole; i.e., h omits three points $0, \infty, \delta$; which contradicts the fact that h is non-constant.

Similarly from (3.27), we see that the numerator and the denominator in the right hand side may have common factors say $(k-\mu_i)$'s for $i = 1, 2, \ldots, t$ iff

$$(\mu_i^{2n-4}+\ldots+\mu_i^2+1)+\mu_i^{n-2} = \mu_i^{2n-2}+\mu_i^{2n-4}+\ldots+\mu_i^2+1;$$

i.e.,

$$\mu_i^{n-2} = \mu_i^{2n-2};$$

i.e.,

$$\mu_i^{n-2}(\mu_i^n - 1) = 0.$$

Since $\mu_i = 0, 1$ are neither zeros of

$$(z^{2n-4} + \ldots + z^2 + 1) + z^{n-2}$$

nor zeros of

$$z^{2n-2} + z^{2n-4} + \ldots + z^2 + 1,$$

so the numerator and the denominator may have at most $(2n-2) - \{(n-2)+1\} = (n-1)$ factors in common. Further note that

$$z^{2n-2} + z^{2n-4} + \ldots + z^2 + 1,$$

can have only simple zeros and $2n-4 > n-1$. So the numerator has at least one factor say $(k-\xi)$ which can not be canceled out by the factors of the denominator. Hence each ξ-point of k must be a zero of g. Now proceeding same as above, we can show that k omits three points $0, \infty, \xi$; which is again a contradiction.
We can similarly arrive at a contradiction for equations (3.25), (3.29).
Therefore h is constant. Since g is non-constant, so from (3.22), we get $h^n = h^{n-1} = h^{n-2} = 1$, which implies $h = 1$; i.e., $f \equiv g$. Q.E.D.

Proof of Theorem 1.3. Let f and g be two non-constant entire functions satisfying $E_f(S, 2) = E_g(S, 2)$. Further suppose that

$$F = \frac{f^n + af^{n-1} + bf^{n-2}}{-c} = \frac{f^{n-2}(f + \frac{a}{2})^2}{-c}, \tag{3.35}$$

$$G = \frac{g^n + ag^{n-1} + bg^{n-2}}{-c} = \frac{g^{n-2}(g + \frac{a}{2})^2}{-c}. \tag{3.36}$$

Clearly F and G share $(1, 2)$. Now proceeding similarly like the proof of *Theorem 1.2* we would get a contradiction for $n \geq 9$ in Case-1.

In Case-2, we would have

$$f^{n-2}(f + \frac{a}{2})^2 g^{n-2}(g + \frac{a}{2})^2 = c^2.$$

Since f and g have no poles, so above equation implies f omits $0, -\frac{a}{2}$; which is a contradiction as f is non-constant.

In Case-3, we would get equations (3.23), (3.24), (3.25), (3.26) or (3.27), (3.28), (3.29), (3.30). Then proceeding in a similar fashion like part-ii of *Theorem 1.2*, we get that from (3.23) we can have l non- common factors in the denominator. Suppose they are $(h - \rho_i)$ for $i = 1, 2, \ldots, l$. Then

each ρ_i-point of h is a pole of g but g has no pole, so h omits l points. Since $n \geq 9$, so l is at least 5. This implies h omits at least 5 points, a contradiction.

From (3.27), we would get that the denominator has at least $n-1$ distinct factors. Then using similar arguments as above, we get a contradiction for k to be non-constant; i.e., h to be non-constant. We would also get contradiction for (3.25) and (3.29) using similar arguments. Hence h is constant and so $h^n = h^{n-1} = h^{n-2} = 1$. This implies $h = 1$; i.e., $f \equiv g$.

Q.E.D.

Remark 3.1. Observe that if we combine *Theorem 1.1* and *Theorem 1.3*, then we get that the zero set of $P(z)$ always forms a URSE2 for $n \geq 4m+5$, where $m \geq 1$.

Remark 3.2. Since the class of entire functions is a subset of the class of meromorphic functions, so every URSM is a URSE but the converse may not be always true. The set used in *Theorem 1.3* is an example of such kind.

Remark 3.3. The polynomial we have used in *Theorem 1.1* and *Theorem 1.3* may have 1 or no gap between the first two terms of the polynomial where as in [1, 4], the authors considered polynomials having the gap at least 3. That is our theorems show that taking 1 or no gap it is possible to find NCIP which generates unique range set.

4 Applications

In this application part, at first we prove one theorem as an application of our main results. After that, we shall discuss different applications of this theorem and also show that it is possible to find the cardinality of a URSM(URSE) without considering the number of distinct critical points of its generating polynomial.

Let $Q(w) = cw^n + bw^{2m} + aw^m + 1$, where $n, m \in \mathbb{N}$ and $a, b, c \in \mathbb{C}^*$ be such that $n > 2m$, $\gcd(n, m) = 1$, $a^2 = 4b$ and $c \neq -\dfrac{mae_i^m\left(\frac{a}{2}e_i^m + 1\right)}{ne_i^n} (= \sigma_i)$ with e_i to be the roots of the equation

$$w^m = -\frac{2n}{(n-2m)a}. \tag{4.1}$$

Now, $Q(w) = cw^n + \left(\frac{a}{2}w^m + 1\right)^2$. Then, $Q^{'}(w) = ncw^{n-1} + maw^{m-1}\left(\frac{a}{2}w^m + 1\right)$.
Therefore, zeros of $Q^{'}(w)$ are roots of $ncw^{n-1} + maw^{m-1}\left(\frac{a}{2}w^m + 1\right) = 0$.
Clearly, for any zero 's' of $Q^{'}(w)$ we have

$$ncs^{n-1} + mas^{m-1}\left(\frac{a}{2}s^m + 1\right) = 0 \tag{4.2}$$

From (4.2), it is obvious that $s^m \neq -\frac{2}{a}$. For if $s^m = -\frac{2}{a}$, then $ncs^n = 0$, which is impossible.
Hence, (4.2) implies $cs^n = -\dfrac{mas^m\left(\frac{a}{2}s^m + 1\right)}{n}$, where $s^m \neq -\frac{2}{a}$.
Now for $s = 0$

$$Q(0) = 1 \neq 0$$

and for $s \neq 0$

$$\begin{aligned} Q(s) &= -\frac{mas^m\left(\frac{a}{2}s^m+1\right)}{n}+\left(\frac{a}{2}s^m+1\right)^2 \\ &= \left(\frac{a}{2}s^m+1\right)\left(-\frac{ma}{n}s^m+\frac{a}{2}s^m+1\right) \\ &= \left(\frac{a}{2}s^m+1\right)\left(\frac{(n-2m)a}{2n}s^m+1\right). \end{aligned}$$

So, 's' is a zero of $Q(w)$, if $s^m = \dfrac{-2n}{(n-2m)a}$; i.e., if $s \in \{e_1, e_2, \ldots, e_m\}$. But then we would have $ce_i^n = -\dfrac{mae_i^m\left(\frac{a}{2}e_i^m+1\right)}{n}$ for $i \in \{1, 2, \ldots, m\}$, which is a contradiction as $c \neq \sigma_i = -\dfrac{(2mbe_i^{2m}+ame_i^m)}{ne_i^n}$. Hence, $Q(w)$ has only simple zeros.

Theorem 4.1. Let $S^* = \{w : Q(w) = 0\}$, with $m \geq 2$ and $c \in \mathbb{C} - \{0, \sigma_i\}$. Then S^* is a $URSM2(URSE2)$ for $n \geq 4m+9$ $(n \geq 4m+5)$.

Proof. Let f and g be two non-constant meromorphic functions such that $E_f(S^*, 2) = E_g(S^*, 2)$. We have $S^* = \{w : cw^n + bw^{2m} + aw^m + 1 = 0\}$ and suppose that
$S = \{z : z^n + az^{n-m} + bz^{n-2m} + c = 0\}$ with the same condition on a, b, c, n, m as given in the theorem.
Observe that

$$\begin{aligned} S^* &= \{w : cw^n + bw^{2m} + aw^m + 1 = 0\} \\ &= \{\frac{1}{z} : \frac{z^n + az^{n-m} + bz^{n-2m} + c}{z^n} = 0\} \\ &= \{\frac{1}{z} : z^n + az^{n-m} + bz^{n-2m} + c = 0\}. \end{aligned}$$

Suppose $w_1, w_2, \ldots, w_n$ are the distinct roots of $cw^n + bw^{2m} + aw^m + 1 = 0$ and $z_1, z_2, \ldots, z_n$ are distinct roots of $z^n + az^{n-m} + bz^{n-2m} + c = 0$. Clearly, $w_i = \dfrac{1}{z_j}$ for $i, j \in \{1, 2, \ldots, n\}$. Therefore without loss of generality we can write $w_i = \dfrac{1}{z_i}$. Further suppose $f_1 = \frac{1}{f}$, $g_1 = \frac{1}{g}$.

Let a_{ij} be any w_i point of f. Then a_{ij} is $\dfrac{1}{w_i}$ point of $\dfrac{1}{f}$ i.e., a_{ij} is z_i point of f_1 and vice-versa.

Now $E_f(S^*, 2) = \bigcup_{w_i \in S^*} E_2(w_i, f) = \bigcup_{z_i \in S} E_2(z_i, f_1) = E_{f_1}(S, 2)$.
So, $E_f(S^*, 2) = E_g(S^*, 2)$ implies $E_{f_1}(S, 2) = E_{g_1}(S, 2)$.

Again from the construction of $Q(w)$ we have e_i's for $\{i = 1, 2, \ldots, m\}$ are the distinct m-th roots of the equation

$$w^m = -\frac{2n}{(n-2m)a}. \tag{4.3}$$

Suppose that d_i's for $\{i = 1, 2, \ldots, m\}$ are the distinct m-th roots of the equation

$$z^m = -\frac{(n-2m)a}{2n}. \tag{4.4}$$

Then clearly putting $z = \dfrac{1}{w}$ in (4.4), we get (4.3). Then by similar argument as resorted above to find the relation between w_i and z_i we can write $e_i = \dfrac{1}{d_i}$ for $\{i = 1, 2, \ldots, m\}$.
Now observe that $d_i^m + \frac{a}{2} = -\frac{(n-2m)a}{2n} + \frac{a}{2} = \frac{ma}{n}$. So,

$$\sigma_i = -\frac{mae_i^m\left(\frac{a}{2}e_i^m + 1\right)}{ne_i^n} = -\frac{mad_i^{n-2m}\left(\frac{a}{2} + d_i^m\right)}{n} = -d_i^{n-2m}\left(\frac{a}{2} + d_i^m\right)^2 = \lambda_i.$$

Hence all the conditions of this theorem coincides with all the conditions of *Theorem 1.1.* Therefore $E_{f_1}(S, 2) = E_{g_1}(S, 2)$ implies $f_1 = g_1$. Hence $\frac{1}{f} = \frac{1}{g}$ i.e., $f \equiv g$. Q.E.D.

Remark 4.1. Similar conclusion can be made for *Theorem 1.3.*

Remark 4.2. Note that S^* is the zero set of $Q(w)$ for $m \geq 2$ and S is the zero set of $P(z)$ for $m \geq 2$, where $Q(w)$ is the reverse polynomial of $P(z)$. We have proved S^* is a URSM2 as an application of the fact that S is a URSM2. But the contrast is that $P(z)$ is a NCIP whereas we are not sure about the reverse polynomial $Q(w)$ to be CIP or NCIP.

Applications of Theorem 4.1.

Application-1. It is obvious that as an application of *Theorem 4.1*, $z^{17} + z^4 + 2z^2 + 1$ generates a URSM2. One can compute that the derivative of this polynomial has 16 distinct zeros. Now observe that this polynomial satisfies all the properties of *Theorem 1.3.* of [4] and *Theorem 1.1* of [1] except the value of n; i.e., the degree of the polynomial. According to the above mentioned results of [1, 4], the degree of any generating polynomial having 16 distinct critical points should be at least $2.16 + 7 = 39$ but the degree of this polynomial is only 17. Not only that, we would have the same problem in computing the degree of this polynomial if we try to apply the result of [7]. This is because here number of distinct critical points of the polynomial is $16 = n - 1 > n - 9$. So the zero set of this type of polynomials can not be characterized for unique range sets by the results of [1, 4, 7]. But as an application of *Theorem 4.1*, we have been able to characterize this type of polynomials to generate unique range sets and in that case we do not need to know the number of distinct critical points of the polynomial. In this direction *Theorem 1.2* of [1] is a result which does not relate the degree of the generating polynomial with the number of its distinct critical points. In that case also our result is better because as an application of *Theorem 4.1* we would be able to produce a URSM of cardinality 17 whereas according to *Theorem 1.2* of [1], $m \geq 5$; i.e., the cardinality of the URSM would be at least 19.

Application-2. As we see in [3, see section 5, concluding remark and an open question], Banerjee discussed about a polynomial of degree n which is not certain to be CIP or NCIP but its zero set generates URSM. This polynomial has $(n-1)$ distinct critical points. So from *Application-1* or the discussion made before *Definition 1.5*, it is clear that this polynomial can not be characterized by

the results of [1, 4, 7].
But using the same method as adopted to prove *Theorem 4.1*, we can prove that the reverse polynomial of that used in *Theorem 1.2.* of [5] generates a URSM for $n \geq 2m+9$, where $m \geq 1$ and $\gcd(m, n) = 1$. Not only that, for some certain values of co-efficients of the reversed polynomial we would get the polynomial discussed in [3, section 5, concluding remark and an open question] and the zero set of the same is a URSM for $n \geq 11$.

Application-3. In [4], authors assumed that the term a_3z^3 must be missing in their polynomial because otherwise $\gcd(p, 3) = 3 \neq 1$, which violates the conditions of their theorems. As an application of *Theorem 4.1*, one can show that it is possible to consider a polynomial having a_3z^3 as one of its term whose zero set is a unique range set. For example $S = \{z : cw^{21} + bw^6 + aw^3 + 1\}$ is a URSM2, where a, b, c are chosen as per the condition of *Theorem 4.1.* So, this example also shows the effectiveness of *Theorem 4.1* in the line of characterizations of unique range sets under the aegis of classification of polynomials.

5 Conclusion and some open questions

After the whole discussion of the paper, the existence of unique range sets generated by non-critically injective polynomials are confirmed. So, one can definitely conclude that critical injection property is only a particular criterion, not a fundamental criterion to be satisfied by a polynomial to generate a unique range set. Furthermore, it is clear that in all the generalized results of unique range sets [1, 4, 7], there are some restrictions on the number of distinct critical points of the generating polynomials. At the same time, it is also possible to characterize a unique range set and its cardinality without considering the distinct critical points of the generating polynomial. These facts ensure that critical injection property and distinct critical points of the generating polynomials are not the only components to characterize unique range sets, rather we need to concentrate on the other components of the generating polynomials such as non-critically injective case to solve the problems of unique range sets. In this perspective we propose the following open problems:

Question 5.1. What are the general criterion's for the class of non-critically injective polynomials to generate unique range sets and what is the least cardinality of the generated unique range sets?

Question 5.2. What are the general criterion's to be satisfied by the generating polynomial of a unique range set?

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