A note on integer powers of a companion matrix and applications

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Abstract

The recursive computation of the entries of integer powers of a companion matrix are extended to the general case of an infinite-type matrix and the possibility to compute matrix functions of a companion matrix by using the Dunford-Taylor integral is shown.

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1 Introduction

The problem for computing powers of the companion matrix has been solved in a paper by W.Y.C. Chen and J.D. Louck [2], by using combinatorial methods, with the aid of the Faà di Bruno formula [5]. However, as it is well known, this formula is not convenient, by the computational point of view, as it is based on the partitions of integer numbers, while, in more recent articles [1, 3] it has been show that the same results can be obtained by using recurrence relations.
In that context, it was observed that, in order to compute the entries of the subsequent powers, the main problem is to obtain the first column, since the other ones are obtained simply by shifting the column index by one unit. In this article we write explicitly the recursions from which the general Companion matrix powers can be derived.
After introducing some basic notions, in order to make the considered recursive method more accessible to the reader, the case of a 3×3 companion matrix is first examined. Then the general case of a $r \times r$ matrix is presented and the proof of the obtained formulas is carried out.
The method is used for computing functions of the companion matrix [4] exploiting the Dunford-Taylor integral, and an extension to the infinite order companion matrix is touched on in Section 5.
In Section 6, some numerical examples have been made by the first author by using the Mathematica© computer program.

2 Basic definitions

Definition. Given the $r \times r$, matrix $\mathcal{A} = (a_{ij})$, its *characteristic polynomial* is given by

$$P(\lambda) := det(\lambda\mathcal{I} - \mathcal{A}) = \lambda^r - u_1\lambda^{r-1} - u_2\lambda^{r-2} + \cdots - u_{r-1}\lambda - u_r\,. \tag{1}$$

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and the coefficients

$$\begin{cases} u_1 \ := \ tr\ \mathcal{A} = a_{11} + a_{22} + ... + a_{rr} \\ \\ u_2 \ := \ -\sum_{i<j}^{1,r} \begin{vmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{vmatrix} \\ \\ \cdots \\ \\ u_r \ := (-1)^{r-1}\ det\ \mathcal{A} \end{cases} \tag{2}$$

are related to the *invariants* of $\mathcal{A}$, [7, 8].
The matrix

$$\mathcal{Q} := \begin{pmatrix} u_1 & 1 & 0 & \cdots & 0 & 0 \\ u_2 & 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ u_{r-2} & 0 & 0 & \cdots & 1 & 0 \\ u_{r-1} & 0 & 0 & \cdots & 0 & 1 \\ u_r & 0 & 0 & \cdots & 0 & 0 \end{pmatrix},$$

associated with the linear recurrence relation

$$X_n = u_1 X_{n-1} + u_2 X_{n-2} + \cdots + u_r X_{n-r}\,, \qquad (n \in \mathbf{Z})\,, \tag{3}$$

is called the companion matrix of (3).
Note that a matrix $\mathcal{A}$ is similar to its companion matrix if and only if $\mathcal{A}$ is *non derogatory* [10].
A general method for computing integer powers of a non-singular complex matrix can be found in [9].

3 Powers of a 3×3 companion matrix

We start by considering the 3×3 Companion matrix:

$$\mathcal{Q}_{3\times 3} = \mathcal{Q} := \begin{pmatrix} u_1 & 1 & 0 \\ u_2 & 0 & 1 \\ u_3 & 0 & 0 \end{pmatrix}. \tag{4}$$

We find:

$$\mathcal{Q}^2 := \begin{pmatrix} u_1^2 + u_2 & u_1 & 1 \\ u_1 u_2 + u_3 & u_2 & 0 \\ u_1 u_3 & u_3 & 0 \end{pmatrix}, \quad \mathcal{Q}^3 := \begin{pmatrix} u_1^3 + 2u_1 u_2 + u_3 & u_1^2 + u_2 & u_1 \\ u_2^2 + u_1^2 u_2 + u_1 u_3 & u_1 u_2 + u_3 & u_2 \\ u_1^2 u_3 + u_2 u_3 & u_1 u_3 & u_3 \end{pmatrix}, \tag{5}$$

$$\mathcal{Q}^4(x) = \begin{pmatrix} u_1^4 + 3u_1^2 u_2 + u_2^2 + 2u_1 u_3 & u_1^3 + 2u_1 u_2 + u_3 & u_1^2 + u_2 \\ u_1^3 u_2 + 2u_2^2 u_1 + u_1^2 u_3 + 2u_2 u_3 & u_2^2 + u_1^2 u_2 + u_1 u_3 & u_1 u_2 + u_3 \\ u_1^3 u_3 + u_3^2 + 2u_1 u_2 u_3 & u_1^2 u_3 + u_2 u_3 & u_1 u_3 \end{pmatrix}, \tag{6}$$

Putting:

$$\mathcal{Q}^n := \begin{pmatrix} a_{1,1}^{(n)} & a_{1,2}^{(n)} & a_{1,3}^{(n)} \\ a_{2,1}^{(n)} & a_{2,2}^{(n)} & a_{2,3}^{(n)} \\ a_{3,1}^{(n)} & a_{3,2}^{(n)} & a_{3,3}^{(n)} \end{pmatrix}, \tag{7}$$

we can prove the following result.

Theorem 1. The entires of the integer powers of matrix (4) can be computed by the recursions. We find, for the first column:

$$\begin{cases} a_{1,1}^{(n+1)} = u_1 a_{1,1}^{(n)} + u_2 a_{1,2}^{(n-1)} + u_3 a_{1,3}^{(n-2)} \\ a_{2,1}^{(n+1)} = u_1 a_{2,1}^{(n)} + u_2 a_{2,2}^{(n-1)} + u_3 a_{2,3}^{(n-2)} \\ a_{3,1}^{(n+1)} = u_1 a_{3,1}^{(n)} + u_2 a_{3,2}^{(n-1)} + u_3 a_{3,3}^{(n-2)} . \end{cases} \tag{8}$$

For the second and third columns:

$$\begin{cases} a_{1,2}^{(n+1)} = a_{1,1}^{(n)} \\ a_{2,2}^{(n+1)} = a_{2,1}^{(n)} \\ a_{3,2}^{(n+1)} = a_{3,1}^{(n)} \end{cases} \qquad \begin{cases} a_{1,3}^{(n+1)} = a_{1,2}^{(n)} \\ a_{2,3}^{(n+1)} = a_{2,2}^{(n)} \\ a_{3,3}^{(n+1)} = a_{3,2}^{(n)} . \end{cases} \tag{9}$$

Further recursions are given by the equations:

$$\begin{cases} a_{2,1}^{(n)} = -u_2 a_{1,1}^{(n)} + u_3 a_{1,1}^{(n-1)} \\ a_{3,1}^{(n)} = u_3 a_{1,2}^{(n)} \\ a_{3,2}^{(n)} = u_3 a_{1,3}^{(n)} . \end{cases} \tag{10}$$

Proof. The recursions in equations (8)-(10) are just obtained by applying the rules for computing these entires in the matrix product $\mathcal{Q}^n \mathcal{Q}$.

3.1 The general companion matrix

The result of Theorem 4 can be extended in a straightforward way to the general Companion matrix as follows.
Consider the $r \times r$ Companion matrix

$$\mathcal{Q}_{r\times r} = \mathcal{Q} := \begin{pmatrix} u_1 & 1 & 0 & \cdots & 0 & 0 \\ u_2 & 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ u_{r-2} & 0 & 0 & \cdots & 1 & 0 \\ u_{r-1} & 0 & 0 & \cdots & 0 & 1 \\ u_r & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \tag{11}$$

associated with the linear recurrence relation (3).
The invariants of the matrix (11) are given by

$$u_1,\ \ -u_2,\ \ u_3,\ \ \ldots,\ \ (-1)^{r-1}u_r\,,$$

so that its characteristic polynomial $P(\lambda)$ is given by equation (1), and the eigenvalues are solution of the equation

$$\lambda^r + u_1\lambda^{r-1} + u_2\lambda^{r-2} + \cdots + u_{r-1}\lambda + u_r = 0\,. \tag{12}$$

Let:

$$\mathcal{Q}^n := \begin{pmatrix} a_{1,1}^{(n)} & a_{1,2}^{(n)} & \cdots & a_{1,r}^{(n)} \\ a_{2,1}^{(n)} & a_{2,2}^{(n)} & \cdots & a_{2,r}^{(n)} \\ \cdots & \cdots & \cdots & \cdots \\ a_{r,1}^{(n)} & a_{r,2}^{(n)} & \cdots & a_{r,r}^{(n)} \end{pmatrix}. \tag{13}$$

Then the following result holds.

Theorem 2. The entires of the integer powers of matrix (11) can be computed by recursions. Assuming as initial conditions the entries of the matrix $\mathcal{Q}$, that is:

$$\begin{array}{lllll} a_{1,1}^{(1)} = u_1\,, & a_{1,2}^{(1)} = 1\,, & a_{1,3}^{(1)} = 0\,, & \cdots, & a_{1,r-1}^{(1)} = 0\,, \quad a_{1,r}^{(1)} = 0 \\ a_{2,1}^{(1)} = u_2\,, & a_{2,2}^{(1)} = 0\,, & a_{2,3}^{(1)} = 1\,, & \cdots, & a_{2,r-1}^{(1)} = 0\,, \quad a_{2,r}^{(1)} = 0 \\ \cdots & & & & \\ a_{r-2,1}^{(1)} = u_{r-2}\,, & a_{r-2,2}^{(1)} = 0\,, & a_{r-2,3}^{(1)} = 0\,, & \cdots, & a_{r-2,r-1}^{(1)} = 1\,, \quad a_{r-2,r}^{(1)} = 0 \\ a_{r-1,1}^{(1)} = u_{r-1}\,, & a_{r-1,2}^{(1)} = 0\,, & a_{r-1,3}^{(1)} = 0\,, & \cdots, & a_{r-1,r-1}^{(1)} = 0\,, \quad a_{r-1,r}^{(1)} = 1 \\ a_{r,1}^{(1)} = u_r\,, & a_{r,2}^{(1)} = 0\,, & a_{r,3}^{(1)} = 0\,, & \cdots, & a_{r,r}^{(1)} = 0\,, \quad a_{r,r}^{(1)} = 0\,, \end{array} \tag{14}$$

we find the following recursions.
For the first column:

$$\begin{cases} a_{1,1}^{(n+1)} = u_1 a_{1,1}^{(n)} + u_2 a_{1,2}^{(n-1)} + \cdots + u_r a_{1,r}^{(n-r+1)} \\ a_{2,1}^{(n+1)} = u_1 a_{2,1}^{(n)} + u_2 a_{2,2}^{(n-1)} + \cdots + u_r a_{2,r}^{(n-r+1)} \\ \cdots \\ a_{r-1,1}^{(n+1)} = u_1 a_{r-1,1}^{(n)} + u_2 a_{r-1,2}^{(n-1)} + \cdots + u_r a_{r-1,r}^{(n-r+1)} \\ a_{r,1}^{(n+1)} = u_1 a_{r,1}^{(n)} + u_2 a_{r,2}^{(n-1)} + \cdots + u_r a_{r,r}^{(n-r+1)}\,. \end{cases} \tag{15}$$

For the second, third, ..., rth column:

$$\begin{cases} a_{1,k}^{(n+1)} = a_{1,k-1}^{(n)} \\ a_{2,k}^{(n+1)} = a_{2,k-1}^{(n)} \\ \cdots \\ a_{r,k}^{(n+1)} = a_{r,k-1}^{(n)} \end{cases} \qquad (k = 2, 3, \ldots, r)\,,$$

or, in more concise form:

$$a_{h,k}^{(n+1)} = a_{h,k-1}^{(n)} \,, \quad (h = 1, 2, \ldots, r;\; k = 2, 3, \ldots, r) \,. \tag{16}$$

Proof. The result simply follows from the rules for computing the matrix product $\mathcal{Q}_{r\times r}^{n}\mathcal{Q}_{r\times r}$. The entry $a_{h,1}^{(n+1)}$, $(h = 1, 2, \ldots, r)$, derives from the scalar product

$$(a_{1,1}^{(n)}, a_{1,2}^{(n-1)}, \ldots, a_{1,r}^{(n-r+1)}) \cdot (u_1, u_2, \ldots, u_r),$$

so that equations in (15) hold.
Furthermore, equation (16) $(h = 1, 2, \ldots, r;\; k = 2, 3, \ldots, r)$ holds since the entry $a_{h,k}^{(n+1)}$ is obtained by performing the scalar product

$$(a_{h,1}^{(n)}, a_{h,2}^{(n-1)}, \ldots, a_{h,r}^{(n-r+1)}) \cdot (0, \ldots, 0, 1, 0, \ldots, 0),$$

where the number 1 is placed at the $(k-1)$-place.

4 Matrix functions

Consider a $r \times r$ matrix $\mathcal{A} = \{a_{h,k}\}$, with characteristic polynomial (1) and invariants given by equation (2). Let f be a function holomorphic in an open set Δ, containing all the eigenvaues of $\mathcal{A}$. Then, the matrix functions $f(\mathcal{A})$ are given by the Dunford-Taylor (also called Riesz-Fantappiè) integral [6]:

$$f(\mathcal{A}) = \frac{1}{2\,\pi\,\mathrm{i}} \left[\sum_{k=1}^{r} \oint_{\gamma} \frac{f(\lambda)\, \sum_{h=0}^{k-1} (-1)^h u_h\, \lambda^{k-h-1}}{P(\lambda)}\, d\lambda\, \mathcal{A}^{r-k} \right] \,, \tag{17}$$

where γ denotes a simple contour enclosing all the zeros of $P(\lambda)$.
In the case of the companion matrix (11), using the matrix powers (13) with entries given by equations (15)-(16), we find

$$f(\mathcal{Q}) = \frac{1}{2\,\pi\,\mathrm{i}} \left[\sum_{k=1}^{r} \oint_{\gamma} \frac{f(\lambda)\, \sum_{h=0}^{k-1} (-1)^h u_h\, \lambda^{k-h-1}}{P(\lambda)}\, d\lambda\, \mathcal{Q}^{r-k} \right] \,. \tag{18}$$

Some worked examples are given in Section 6.

5 The infinite companion matrix

In the infinite case, it is usual to consider the companion matrix

$$\mathcal{Q} := \begin{pmatrix} u_1 & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots \\ u_2 & 0 & 1 & \cdots & 0 & 0 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ u_{r-1} & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots \\ u_r & 0 & 0 & \cdots & 0 & 0 & 1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix} , \tag{19}$$

associated with the linear recurrence :

$$X_n = u_1 X_{n-1} + u_2 X_{n-2} + \cdots + u_r X_{n-r} + \cdots , \qquad (n \in \mathbf{Z}) , \tag{20}$$

The invariants of the matrix (19) are given by

$$u_1, \ -u_2, \ u_3, \ \ldots, \ (-1)^{r-1} u_r , \cdots ,$$

that is the coefficients of the function $E(\lambda)$:

$$E(\lambda) \ := 1 + u_1\lambda + u_2\lambda^2 - u_3\lambda^3 + \cdots + u_r\lambda^r + \cdots , \tag{21}$$

and the eigenvalues are the zeros of the entire function $E(\lambda)$:

$$1 + u_1\lambda + u_2\lambda^2 + u_3\lambda^3 + \cdots + u_r\lambda^r + \cdots = 0 . \tag{22}$$

Putting:

$$\mathcal{Q}^n := \begin{pmatrix} a_{1,1}^{(n)} & a_{1,2}^{(n)} & a_{1,3}^{(n)} & \cdots & a_{1,r}^{(n)} & \cdots \\ a_{2,1}^{(n)} & a_{2,2}^{(n)} & a_{2,3}^{(n)} & \cdots & a_{2,r}^{(n)} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{r,1}^{(n)} & a_{r,2}^{(n)} & a_{r,3}^{(n)} & \cdots & a_{r,r}^{(n)} & \cdots \\ a_{r+1,1}^{(n)} & a_{r+1,2}^{(n)} & a_{r+1,3}^{(n)} & \cdots & a_{r+1,r}^{(n)} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix} , \tag{23}$$

we find the following recursions.
For the first column:

$$\begin{cases} a_{1,1}^{(n+1)} = u_1 a_{1,1}^{(n)} + u_2 a_{1,2}^{(n-1)} + \cdots + u_r a_{1,r}^{(n-r+1)} + \ldots \\ a_{2,1}^{(n+1)} = u_1 a_{2,1}^{(n)} + u_2 a_{2,2}^{(n-1)} + \cdots + u_r a_{2,r}^{(n-r+1)} + \ldots \\ \cdots \\ a_{r-1,1}^{(n+1)} = u_1 a_{r-1,1}^{(n)} + u_2 a_{r-1,2}^{(n-1)} + \cdots + u_r a_{r-1,r}^{(n-r+1)} + \ldots \\ a_{r,1}^{(n+1)} = u_1 a_{r,1}^{(n)} + u_2 a_{r,2}^{(n-1)} + \cdots + u_r a_{r,r}^{(n-r+1)} + \ldots \\ \cdots \end{cases} \tag{24}$$

For the second, third, ..., and all other columns:

$$\begin{cases} a_{1,k}^{(n+1)} = a_{1,k-1}^{(n)} \\ a_{2,k}^{(n+1)} = a_{2,k-1}^{(n)} \\ \cdots \\ a_{r,k}^{(n+1)} = a_{r,k-1}^{(n)} , \\ \cdots \end{cases} \qquad (k = 2, 3, \ldots, r, \ldots) .$$

6 Numerical examples

6.1 Exponential of a 3×3 non-singular companion matrix

Let us consider the companion matrix:

$$\mathcal{Q} = \left\{ \begin{array}{ccc} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & 0 \end{array} \right\} . \tag{25}$$

The relevant invariants are:

$$u_1 = 1\,, \quad -u_2 = -1\,, \quad u_3 = 1\,, \tag{26}$$

whereas the corresponding eigenvalues are found to be:

$$\lambda_1 = -\mathrm{i}\,, \quad \lambda_2 = \mathrm{i}\,, \quad \lambda_4 = 1\,. \tag{27}$$

Therefore, by making use of the Dunford-Taylor integral (18) in combination with the recursive algorithm for the evaluation of the powers of $\mathcal{Q}$ as detailed in Section 3, the exponential of $\mathcal{Q}$ is readily computed as:

$$e^{\mathcal{Q}} = \left\{ \begin{array}{ccc} \frac{1}{2}\left[e + \sin(1) + \cos(1)\right] & \frac{1}{2}\left[e + \sin(1) - \cos(1)\right] & \frac{1}{2}\left[e - \sin(1) - \cos(1)\right] \\ -\sin(1) & \cos(1) & \sin(1) \\ \frac{1}{2}\left[e + \sin(1) - \cos(1)\right] & \frac{1}{2}\left[e - \sin(1) - \cos(1)\right] & \frac{1}{2}\left[e - \sin(1) + \cos(1)\right] \end{array} \right\} . \tag{28}$$

6.2 Fifth power of a 4×4 non-singular companion matrix

Let us consider the companion matrix:

$$\mathcal{Q} = \left\{ \begin{array}{cccc} 10 & 1 & 0 & 0 \\ -35 & 0 & 1 & 0 \\ 50 & 0 & 0 & 1 \\ -24 & 0 & 0 & 0 \end{array} \right\} . \tag{29}$$

The relevant invariants are:

$$u_1 = 10\,, \quad -u_2 = -35\,, \quad u_3 = 50\,, \quad -u_4 = -24\,, \tag{30}$$

whereas the corresponding eigenvalues are found to be:

$$\lambda_1 = 4\,, \quad \lambda_2 = 3\,, \quad \lambda_3 = 2\,, \quad \lambda_4 = 1\,. \tag{31}$$

So, by making use of the Dunford-Taylor integral (18) in combination with the recursive algorithm for the evaluation of the powers of $\mathcal{Q}$ as detailed in Section 3, $\mathcal{Q}^5$ is readily computed as:

$$\mathcal{Q}^5 = \left\{ \begin{array}{cccc} 7770 & 1701 & 350 & 65 \\ -43595 & -9240 & -1799 & -300 \\ 76650 & 15940 & 3010 & 476 \\ -40824 & -8400 & -1560 & -240 \end{array} \right\} . \tag{32}$$

6.3 Sine of a 4×4 non-singular companion matrix

Let us consider the companion matrix:

$$\mathcal{Q} = \left\{ \begin{array}{cccc} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \pi^4 & 0 & 0 & 0 \end{array} \right\} . \tag{33}$$

The relevant invariants are:

$$u_1 = -u_2 = u_3 = 0\,, \quad -u_4 = \pi^4\,, \tag{34}$$

whereas the corresponding eigenvalues are found to be:

$$\lambda_1 = -\mathrm{i}\pi\,, \quad \lambda_2 = \mathrm{i}\pi\,, \quad \lambda_3 = -\pi\,, \quad \lambda_4 = \pi\,. \tag{35}$$

So, by making use of the Dunford-Taylor integral (18) in combination with the recursive algorithm for the evaluation of the powers of $\mathcal{Q}$ as detailed in Section 3, the sine of $\mathcal{Q}$ is readily computed as:

$$\sin \mathcal{Q} = \left\{ \begin{array}{cccc} 0 & \frac{\sinh(\pi)}{2\pi} & 0 & -\frac{\sinh(\pi)}{2\pi^3} \\ -\frac{1}{2}\pi \sinh(\pi) & 0 & \frac{\sinh(\pi)}{2\pi} & 0 \\ 0 & -\frac{1}{2}\pi \sinh(\pi) & 0 & \frac{\sinh(\pi)}{2\pi} \\ \frac{1}{2}\pi^3 \sinh(\pi) & 0 & -\frac{1}{2}\pi \sinh(\pi) & 0 \end{array} \right\} . \tag{36}$$

6.4 Eighth power of a 5×5 non-singular companion matrix

Let us consider the companion matrix:

$$\mathcal{Q} = \left\{ \begin{array}{ccccc} 5 & 1 & 0 & 0 & 0 \\ -10 & 0 & 1 & 0 & 0 \\ 10 & 0 & 0 & 1 & 0 \\ -9 & 0 & 0 & 0 & 1 \\ 5 & 0 & 0 & 0 & 0 \end{array} \right\} . \tag{37}$$

The relevant invariants are:

$$u_1 = 5\,, \quad -u_2 = -10\,, \quad u_3 = 10\,, \quad -u_4 = -9\,, \quad u_5 = 5 \tag{38}$$

whereas the corresponding eigenvalues are found to be:

$$\lambda_1 = 2 - \mathrm{i}\,, \quad \lambda_2 = 2 + \mathrm{i}\,, \quad \lambda_3 = -\mathrm{i}\,, \quad \lambda_4 = \mathrm{i}\,, \quad \lambda_5 = 1\,. \tag{39}$$

Therefore, by making use of the Dunford-Taylor integral (18) in combination with the recursive algorithm for the evaluation of the powers of $\mathcal{Q}$ as detailed in Section 3, $\mathcal{Q}^8$ is readily computed as:

$$\mathcal{Q}^8 = \left\{ \begin{array}{ccccc} -1469 & -330 & 30 & 90 & 66 \\ 3120 & 181 & -480 & -420 & -240 \\ -3120 & -180 & 481 & 420 & 240 \\ 3120 & 180 & -480 & -419 & -240 \\ -1650 & 150 & 450 & 330 & 175 \end{array} \right\} . \tag{40}$$

7 Conclusion

The entries of integer powers of a companion matrix, associated to a linear recurrece relation, can be computed by simple recursions, which are write down in an explicit form. The results are easily extended to the general case of an infinite order matrix. The Dunford-Taylor formula permits to derive functions of companion matrices without knowing the eigenvalues of the considered matrices. Applications of this formula permits to recover the entries of powers of a given companion matrix. Furthermore, the exponential function and other elementary functions of a companion matrix have been computed by using the Mathematica© program.

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