Decay and blow up at infinite time of solutions for a logarithmic Petrovsky equation

Erhan Pişkin and Zeynep Çalışır

Dicle University, Department of Mathematics, Diyarbakir, Turkey
E-mail: episkin@dicle.edu.tr, zeynep.mat.zcm@gmail.com

Abstract

In this work, we consider a logarithmic Petrovsky equation with strong damping with initial and boundary conditions in a bounded domain. Under suitable conditions, we prove decay of solutions. Also, we establish the blow up at infinite time of solutions.

2000 Mathematics Subject Classification. **35B35**. 35L55.
Keywords. Petrovsky equation, logarithmic nonlinearity, energy decay, blow up at infinite time.

1 Introduction

In this work, we consider the following logarithmic Petrovsky equation

$$u_{tt} + \Delta^2 u + \Delta^2 u_t = u \ln |u|^2, \quad x \in \Omega,\, t > 0, \tag{1}$$

with the inital-boundary conditions

$$u(x,t) = \frac{\partial u(x,t)}{\partial v} = 0, \quad x \in \partial\Omega,\, t \geq 0 \tag{2}$$

and

$$u(x,0) = u_0(x),\ u_t(x,0) \ = u_1(x), \quad x \in \Omega, \tag{3}$$

where Ω is a bounded domain with smooth boundary $\partial\Omega$ in R^n $(n \geq 1)$, v is the unit outher normal to $\partial\Omega$.

This type problems arise in optics, geophysics, nuclear physics and quantum mechanics [1, 2, 3, 4].

Ma and Fang [5] investigated the following equation

$$u_{tt} - \Delta u - \Delta u_t = u \ln |u|^2.$$

They proved global existence, energy decay and blow up of solutions.

Al-Gharabli and Messaoudi [6] considered the following equation

$$u_{tt} + \Delta^2 u + u + h(u_t) = ku \ln |u|$$

where $h(s)$ is nonlinear function. They proved existence and general decay of solutions. Also, some authors studied Petrovsky equation with polynomial nonlinearity $(|u|^{p-1}u)$ without logarithmic nonlinearity $(u \ln |u|)$ (see [7, 8, 9]). Let us note here that though the logarithmic nonlinearity

Tbilisi Mathematical Journal 13(4) (2020), pp. 113–127.
Tbilisi Centre for Mathematical Sciences.
Received by the editors: 21 September 2019.
Accepted for publication: 05 October 2020 .

is somehow weaker than the polynomial nonlinearity, both the existence and stability result are not obtained by straightforward application of the method used for polynomial nonlinearity [6]. Recently, some authors studied hyperbolic type equations with logarithmic nonlinearity (see [10, 11, 13, 14, 15, 16]).

The main purpose of this work is to proved the energy decay and the blow up at infinite time of solution to the logarithmic Petrovsky equation (1).

This work is organized as follows: In Section 2, we present some material needed for the proof of our main theorems. In Section 3, we prove the polynomial and exponential decay of solutions. In Section 4, we prove the blow up at infinite time of solutions.

2 Preliminaries

In this section, we present some material needed for the proof of our main theorems. In this work, we denote $\|.\|_p = \|.\|_{L^p(\Omega)}$, $\|.\| = \|.\|_{L^2(\Omega)}$.

Lemma 2.1 [17]. (Logarithmic Sobolev inequality) Let $u \in H_0^1(\Omega)$ and a be any positive constant, then

$$\int_\Omega u^2 \ln |u| \, dx \leq \frac{1}{2} \|u\|^2 \ln \|u\|^2 + \frac{\alpha^2}{2\pi} \|\nabla u(x)\|^2 - (1 + \ln \alpha) \|u\|^2 . \tag{4}$$

Next, we introduce the potential energy fuctional J and Nehari functional I of problem (1)-(3):

$$J(u) = \frac{1}{2} \|u\|^2 + \frac{1}{2} \|\Delta u\|^2 - \frac{1}{2} \int_\Omega u^2 \ln u^2 dx, \tag{5}$$

$$I(u) = \|\Delta u\|^2 - \int_\Omega u^2 \ln u^2 dx \tag{6}$$

then, we have

$$J(u) = \frac{1}{2} I(u) + \frac{1}{2} \|u\|^2 . \tag{7}$$

Now, multiplying (1) by u_t and integrating over Ω, we can compute

$$\int_\Omega u_{tt} u_t dx = \int_\Omega \Delta^2 u u_t dx + \int_\Omega \Delta^2 u_t u_t dx = \int_\Omega u \ln |u|^2 u_t dx. \tag{8}$$

By a direct calculation

$$\begin{aligned} \int_\Omega u_{tt} u_t \, dx &= \frac{1}{2} \frac{d}{dt} \int_\Omega |u_t|^2 dx \\ &= \frac{1}{2} \frac{d}{dt} \|u_t\|^2 , \end{aligned} \tag{9}$$

$$\begin{aligned} \int_\Omega \Delta^2 u u_t dx &= \int_\Omega \Delta u \Delta u_t dx \\ &= \frac{1}{2} \frac{d}{dt} \|\Delta u\|^2 , \end{aligned} \tag{10}$$

and

$$\begin{aligned}\int_\Omega \Delta^2 u_t u_t dx &= \int_\Omega |\Delta u_t|^2 dx \\ &= \|\Delta u_t\|. \end{aligned} \tag{11}$$

By using integration by parts, we have

$$\int_\Omega u \ln u^2 u_t dx = \frac{1}{2}\frac{d}{dt}\left(\int_\Omega u^2 \ln u^2 dx - \|u\|^2\right) \tag{12}$$

By replacement (9), (10), (11) and (12) into (8), we obtain

$$\frac{1}{2}\frac{d}{dt}\|u_t\|^2 + \frac{1}{2}\frac{d}{dt}\|\Delta u\|^2 + \|\Delta u_t\|^2 = \frac{1}{2}\frac{d}{dt}\int_\Omega u^2 \ln u^2 dx - \frac{1}{2}\frac{d}{dt}\|u\|^2,$$

$$\frac{d}{dt}E(t) = -\|\Delta u_t\|^2,$$

$$E'(t) = -\|\Delta u_t\|^2 \le 0$$

where

$$E(t) = \frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|u\|^2 + \frac{1}{2}\|\Delta u\|^2 - \frac{1}{2}\int_\Omega u^2 \ln u^2 dx. \tag{13}$$

Now, we establish a family of potential wells for problem (1) to (3) and prove a series of relative properties. We proceed to give the definitions of Nehari manifold and potential depth, respectively, as follows:

$$\aleph = \left\{u \in H_0^2(\Omega) : \ I(u) = 0, \ \int_\Omega |\Delta u|^2 dx \neq 0\right\} \tag{14}$$

$$d = \inf\left\{\sup_{\mu\ge 0} J(\mu u) \ : \ u \in H_0^2(\Omega), \ \int_\Omega |\Delta u|^2 dx \neq 0\right\} \tag{15}$$

We also introduce the stable set W and the unstable set V

$$\begin{aligned} W &= \left\{u \in H_0^2(\Omega) \ \ I(u) > 0, \ J(u) < d\right\} \cup \{0\}, \\ V &= \left\{u \in H_0^2(\Omega) \ \ I(u) < 0, \ J(u) < d\right\}. \end{aligned}$$

Similarly, for $\delta > 0$, we also introduce

$$\begin{aligned} J_\delta(u) &= \frac{\delta}{2}\int_\Omega |\Delta u|^2 dx - \frac{1}{2}\int_\Omega u^2 \ln u^2 dx + \frac{1}{2}\int_\Omega u^2 dx, \\ I_\delta(u) &= \delta\int_\Omega |\Delta u|^2 dx - \int_\Omega u^2 \ln u^2 dx, \\ \aleph_\delta &= \left\{u \in H_0^2(\Omega) : \ I_\delta(u) = 0, \ \int_\Omega |\Delta u|^2 dx \neq 0\right\}, \\ d(\delta) &= \inf_{u\in\aleph_\delta} J(u) \end{aligned} \tag{16}$$

as well as

$$\begin{aligned} W_\delta &= \left\{u \in H_0^2(\Omega):\ I(u) > 0,\ J(u) < d(\delta)\right\} \cup \{0\} \\ V_\delta &= \left\{u \in H_0^2(\Omega):\ I(u) < 0,\ J(u) < d(\delta)\right\} \end{aligned}$$

Then we can define

$$\begin{aligned} \partial W_\delta &= \left\{u \in H_0^2(\Omega):\ I(u) = 0,\ \|\Delta u\| \neq 0\right\} \cup \left\{u \in H_0^2(\Omega)/J(u) = d(\delta)\right\} \\ \partial V_\delta &= \left\{u \in H_0^2(\Omega):\ I_\delta(u) = 0\right\} \cup \left\{u \in H_0^2(\Omega)/J(u) = d(\delta)\right\}. \end{aligned}$$

Thus, we have the following properties.

Lemma 2.2. If $0 < \|u\| < r(\delta)$, then $I_\delta(u) > 0$, where $r(\delta) = (\pi\delta)^{\frac{n}{4}} e^{\frac{n}{2}}$.

Proof. Using the logarithmic Sobolev inequality, for any $a > 0$, we have

$$\begin{aligned} I_\delta(u) &= \delta \int_\Omega |\Delta u|^2 dx - \int_\Omega u^2 \ln u^2 dx \\ &= \delta \int_\Omega |\Delta u|^2 dx - \int_\Omega u^2 \ln \|u\|^2 dx + \int_\Omega u^2 \ln \|u\|^2 dx - \int_\Omega u^2 \ln u^2 dx \\ &\quad \delta \int_\Omega |\Delta u|^2 dx - \int_\Omega u^2 \ln \|u\|^2 - \int_\Omega u^2 \ln \frac{u^2}{\|u\|^2} dx \\ &= \delta \int_\Omega |\Delta u|^2 dx - 2 \int_\Omega u^2 \left(\ln \frac{u}{\|u\|} + \ln \|u\|\right) dx \\ &\geq \delta \int_\Omega |\Delta u|^2 dx - \frac{\alpha^2}{\pi} \int_\Omega |\Delta u|^2 dx + n\left(1 + \ln \alpha\right) \|u\|^2 - \int_\Omega |u|^2 \ln \|u\|^2 dx \\ &= \left(\delta - \frac{\alpha^2}{\pi}\right) \int_\Omega |\Delta u|^2 dx + n\left(1 + \ln \alpha\right) \|u\|^2 - 2 \|u\|^2 \ln \|u\|. \end{aligned}$$

Taking $\alpha = \sqrt{\pi\delta}$, we get

$$I_\delta(u) \geq \left[n\left(1 + \ln \sqrt{\pi\delta}\right) - 2 \ln \|u\|\right] \|u\|^2 \tag{17}$$

If $0 < \|u\| < r(\delta)$, then $r(\delta) = (\pi\delta)^{\frac{n}{4}} e^{\frac{n}{2}}$, which implies that

$$\|u\| < (\pi\delta)^{\frac{n}{4}}\, e^{\frac{n}{2}}. \tag{18}$$

Finally, inserting (18) into (17), we get $I_\delta(u) > 0$.

Lemma 2.3. For any $u \in H_0^2(\Omega), \|u\| \neq 0$ and let $g(\mu) = J(\mu u)$. Then we have

$$I(\mu u) = \mu g'(\mu) \begin{cases} > 0, & 0 < \mu < \mu^*, \\ = 0, & \mu = \mu^* \\ < 0, & \mu^* < \mu < \infty \end{cases}, \tag{19}$$

where

$$\mu^* = \exp\left[\frac{\|\Delta u\|^2 - \int\limits_\Omega |u|^2 \ln |u|^2 \, dx}{2\|u\|^2}\right].$$

Proof. By the definition of $J(u)$,

$$\begin{aligned} g(\mu) &= J(\mu u) \\ &= \frac{\mu^2}{2}\int_\Omega |\Delta u|^2 \, dx - \frac{\mu^2}{2}\int_\Omega |u|^2 \ln(\mu u)^2 dx + \frac{\mu^2}{2}\int_\Omega |u|^2 \, dx \\ &\quad + \frac{\mu^2}{2}\int_\Omega |u|^2 \ln(\mu)^2 \, dx - \frac{\mu^2}{2}\int_\Omega |u|^2 \ln \mu^2 dx \\ &= \frac{\mu^2}{2}\int_\Omega |\Delta u|^2 \, dx + \frac{\mu^2}{2}\int_\Omega |u|^2 \, dx \\ &\quad - \frac{\mu^2}{2}\int_\Omega |u|^2 \left(\ln(\mu u)^2 - \ln \mu^2\right) dx - \frac{\mu^2}{2}\int_\Omega |u|^2 \ln \mu^2 dx \\ &= \frac{\mu^2}{2}\int_\Omega |\Delta u|^2 \, dx + \frac{\mu^2}{2}\int_\Omega |u|^2 \, dx \\ &\quad - \frac{\mu^2}{2}\int_\Omega |u|^2 \ln |u|^2 \, dx - \frac{\mu^2}{2}\ln \mu^2 \int_\Omega |u|^2 \, dx. \end{aligned} \tag{20}$$

Since $\|u\| \neq 0$,(20) then $\lim\limits_{\mu\to 0} g(\mu) = 0$ and $\lim\limits_{\mu\to+\infty} g(\mu) = -\infty$, Next, by the definition of $I(u)$, we get

$$\begin{aligned} I(\mu u) &= \mu g'(\mu) \\ &= \mu^2 \int_\Omega |\Delta u|^2 \, dx - \mu^2 \int_\Omega |u|^2 \ln |u|^2 \, dx - 2\mu^2 \ln \mu \int_\Omega |u|^2 \, dx. \end{aligned} \tag{21}$$

Then we can derive $I(\mu^* u) = 0$ when

$$\mu^* = \exp \frac{\|\Delta u\|^2 - \int\limits_\Omega |u|^2 \ln |u|^2 \, dx}{2\|u\|^2}$$

by (21). Hence, we have

$$I(\mu u) = \mu g'(\mu) \begin{cases} > 0, & 0 < \mu < \mu^*, \\ = 0, & \mu = \mu^*, \\ < 0, & \mu^* < \mu < \infty. \end{cases}$$

Lemma 2.4. For the potential depth d, we have

$$d \geq M = \frac{\pi^{\frac{n}{2}} e^n}{2}$$

Proof. It follows form Lemma 2.2 and Lemma 2.3 that

$$\begin{aligned} \sup_{\mu>0} J(\mu u) &= J(\mu^* u) \\ &= \frac{1}{2} I(\mu^* u) + \frac{1}{2}(\mu^*)^2 \|u\|^2 \\ &\geq \frac{1}{2}(\mu^*)^2 \|u\|^2, \end{aligned} \tag{22}$$

and

$$\begin{aligned} I_\delta(u) &= (n(1+\ln\sqrt{\pi\delta}) - 2\ln\|u\|)\|u\|^2 \\ &= \left[\frac{n(2+\ln\pi)}{2} - 2\ln\|u\|\right]\|u\|^2. \end{aligned} \tag{23}$$

Further, we obtain

$$\begin{aligned} 0 &= I(\mu^* u) \\ &\geq \left[\frac{n(2+\ln\pi)}{2} - 2\ln\|\mu^* u\|\right]\|\mu^* u\|^2 \end{aligned} \tag{24}$$

Then, we get

$$\mu^* \|u\| \geq e^{\frac{n}{2}} \pi^{\frac{n}{4}} \tag{25}$$

Finally, using (15), (22) and (25) imply

$$d \geq M = \frac{1}{2}\pi^{\frac{n}{2}} e^n.$$

3 Energy decay

In this section, we discuss the energy decay of solutions of the problem (1)-(3).

Theorem 3.1. Let $u_0(x) \in H_0^2(\Omega)$, $u_1(x) \in L^2(\Omega)$, $E(0) \leq M$ and $I(u_0) \geq 0$. Then problem (1)-(3) has a global weak solution $u \in C\left(0,\infty;\ H_0^2(\Omega)\right)$ with $u_t \in C\left(0,\infty;\ H_0^2(\Omega)\right)$. Moreover, we have the following decay estimation

(i) (Polynomial decay)

$$E(t) \leq \frac{c_0}{1+t}, \quad \text{for all } t \geq 0, \tag{26}$$

(ii) (Exponential decay) Further that $E(0) \leq \varsigma M$ for $0 < \varsigma < 1$, then

$$E(t) \leq c_1 e^{-c_2 t}, \quad \text{for all } t \geq 0, \tag{27}$$

where c_0, c_1, c_2 are positive constans.

Proof. We divide the proof into three steps.

Step 1: Global existence. The proof can be obtained in a similar way as done in [5, 6, 12].

Step 2: Polynomial decay. By the definitions of (7), (13), we have

$$\begin{aligned} E(0) &= E(t)+\int_0^t\int_\Omega |\Delta u_t|^2\,dxd\tau, \\ &= \frac{1}{2}\|u_t\|^2+\frac{1}{2}\|u\|^2+\frac{1}{2}\|\Delta u\|^2-\frac{1}{2}\int_\Omega u^2\ln u^2dx+\int_0^t\int_\Omega |\Delta u_\tau|^2\,dxd\tau \\ &\geq \frac{1}{2}\int_\Omega |u|^2\,dx+\frac{1}{2}\int_\Omega |u_t|^2\,dx+\int_0^t\int_\Omega |\Delta u_t|^2\,dxdt \end{aligned} \tag{28}$$

since $I(u_0)\geq 0$. By combining (28) with $E(0)\leq M$ and $\|u\|^2\leq \lambda_1\|\Delta u\|^2$ yields that

$$\int_0^t\int_\Omega |u_\tau|^2\,dxd\tau\leq \frac{1}{\lambda_1}\int_0^t\int_\Omega |\Delta u_\tau|^2\,dxd\tau\leq \frac{M}{\lambda_1}. \tag{29}$$

Now, multiply (1) by u and integrate over $\Omega\times(0,t)$, we obtain

$$\int_0^t\int_\Omega uu_{tt}dxdt+\int_0^t\int_\Omega u\Delta^2udxdt+\int_0^t\int_\Omega u\Delta^2u_tdxdt=\int_0^t\int_\Omega u^2\ln|u|^2\,dxdt. \tag{30}$$

By a direct calculation

$$\int_0^t\int_\Omega uu_{tt}dxdt=\int_\Omega uu_tdx-\int_\Omega u_0u_1dx-\int_0^t\int_\Omega |u_t|^2\;\;dxdt, \tag{31}$$

and

$$\int_0^t\int_\Omega u\Delta^2udxdt=\int_0^t\int_\Omega |\Delta u|^2\,dxdt. \tag{32}$$

And by using integration by parts, we have

$$\begin{aligned} \int_0^t\int_\Omega u\Delta^2u_tdxdt &= \int_0^t\int_\Omega \Delta u_t\Delta udxdt \\ &= \frac{1}{2}\frac{d}{dt}\int_0^t\int_\Omega |\Delta u|^2\,dxdt \\ &= \frac{1}{2}\int_\Omega |\Delta u|^2\,dxdt-\frac{1}{2}\int_\Omega |\Delta u_0|^2\,dxdt. \end{aligned} \tag{33}$$

By replacement (31), (32) and (33) into (30), we obtain

$$\begin{aligned}\int_0^t \int_\Omega \left(|\Delta u|^2 - u^2 \ln u^2 \right) dxdt &= -\int_\Omega u u_t dx + \int_\Omega u_0 u_1 dx \\ &\quad + \int_0^t \int_\Omega |u_t|^2 dxdt + \frac{1}{2}\int_\Omega |\Delta_0 u|^2 dxdt \\ &\quad - \frac{1}{2}\int_\Omega |\Delta u|^2 dxdt,\end{aligned}$$

$$\begin{aligned}\int_0^t I(u)dt &= -\int_\Omega u u_t dx + \int_\Omega u_0 u_1 dx + \int_0^t \int_\Omega |u_\tau|^2 dxdt \\ &\quad + \frac{1}{2}\int_\Omega |\Delta_0 u|^2 dx - \frac{1}{2}\int_\Omega |\Delta u|^2 dx \end{aligned} \tag{34}$$

Using (29) and $u \in L^\infty\left(0,\infty;\ H_0^2(\Omega)\right)$, then (34) yields

$$\int_0^t I(u(\tau))d\tau < c, \quad \text{for all } t > 0. \tag{35}$$

Since

$$\frac{d}{dt}\left[(1+t)E(t)\right] = (1+t)E'(t) + E(t) \le E(t), \tag{36}$$

then integrate (36) over $(0,t)$, we get

$$\int_0^t \frac{d}{dt}\left[(1+t)E(t)\ \right] dt \le \int_0^t E(t)dt,$$

$$(1+t)E(t) - E(0) \le \int_0^t E(t)dt,$$

$$(1+t)E(t) \le E(0) + \frac{1}{2}\int_0^t I(u(\tau))d\tau + \frac{1}{2}\int_0^t \|u_\tau\|^2 d\tau + \frac{1}{2}\int_0^t \|u\|^2 d\tau. \tag{37}$$

Thus, applying $E(0) \le M$, (29), (35) and $u \in L^\infty\left(0,\infty;\ H_0^2(\Omega)\right)$ to (37), we can derive

$$E(t) \le \frac{c_0}{1+t},$$

where c_0 is positive constant.

Step 3: Exponential decay. Let

$$L(t) = E(t) + \varepsilon \int_\Omega u u_t dx \tag{38}$$

where $\varepsilon > 0$ will be determined in the later. Thanks to Young's inequality

$$\begin{aligned}\left| \int_\Omega u u_t dx \right| &\le \int_\Omega |u u_t| dx \\ &\le \frac{1}{2}\|u\|^2 + \frac{1}{2}\|u_t\|^2,\end{aligned}$$

we can know that there exist two positive constants α_1 and α_2 such that for $t \geq 0$

$$\alpha_1 E(t) \leq L(t) \leq \alpha_2 E(t). \tag{39}$$

That is to say, $L(t)$ and $E(t)$ are equivalent.

Direct differentiatin of $L(t)$ and using (1), yields

$$\begin{aligned} L'(t) &= E'(t) + \varepsilon \int_\Omega u_t u_t dx + \varepsilon \int_\Omega u u_{tt} dx \\ &= -\int_\Omega |\Delta u_t|^2 dx + \varepsilon \int_\Omega |u_t|^2 dx + \varepsilon \int_\Omega u^2 \ln |u|^2 dx \\ &\quad -\varepsilon \int_\Omega u \Delta^2 u dx - \varepsilon \int_\Omega u \Delta^2 u_t dx \\ &= \varepsilon \int_\Omega |u_t|^2 dx - \varepsilon \int_\Omega |\Delta u|^2 dx + \varepsilon \int_\Omega u^2 \ln |u|^2 dx \\ &\quad -\varepsilon \int_\Omega \Delta u \Delta u_t dx - \int_\Omega |\Delta u_t|^2 dx \\ &\leq \varepsilon \int_\Omega |u_t|^2 dx + \varepsilon(\sigma - 1) \int_\Omega |\Delta u|^2 dx \\ &\quad +\varepsilon \int_\Omega u^2 \ln |u|^2 dx + \Big(\frac{\varepsilon}{4\sigma} - 1\Big) \int_\Omega |\Delta u_t|^2 dx, \end{aligned} \tag{40}$$

where σ is positive constant to be chose later. Add and subtract $B\varepsilon E(t)$ to the right of (40), we obtain

$$\begin{aligned} L'(t) &\leq -B\varepsilon E(t) + \varepsilon \int_\Omega |u_t|^2 dx + \varepsilon(\sigma - 1) \int_\Omega |\Delta u|^2 dx \\ &\quad +\varepsilon \int_\Omega u^2 \ln |u|^2 dx + \Big(\frac{\varepsilon}{4\sigma} - 1\Big) \int_\Omega |\Delta u_t|^2 dx \\ &\quad +\frac{B\varepsilon}{2} \int_\Omega |u_t|^2 dx + \frac{B\varepsilon}{2} \int_\Omega |\Delta u|^2 dx \\ &\quad -\frac{B\varepsilon}{2} \int_\Omega u^2 \ln |u|^2 dx + \frac{B\varepsilon}{2} \int_\Omega |u|^2 dx \end{aligned} \tag{41}$$

by virtue of $\|u_t\|^2 \leq c\|\Delta u_t\|^2$, (41) turns into

$$\begin{aligned} L'(t) &\leq -B\varepsilon E(t) + \varepsilon c\int_\Omega |\Delta u_t|^2\,dx + \varepsilon(\sigma-1)\int_\Omega |\Delta u|^2\,dx \\ &\quad +\varepsilon\int_\Omega u^2\ln|u|^2\,dx + \left(\frac{\varepsilon}{4\sigma}-1\right)\int_\Omega |\Delta u_t|^2\,dx \\ &\quad +\frac{B\varepsilon c}{2}\int_\Omega |\Delta u_t|^2\,dx + \frac{B\varepsilon}{2}\int_\Omega |\Delta u|^2\,dx \\ &\quad -\frac{B\varepsilon}{2}\int_\Omega u^2\ln|u|^2\,dx + \frac{B\varepsilon}{2}\int_\Omega |u|^2\,dx \\ &\leq -B\varepsilon E(t) + \left(\varepsilon c + \frac{\varepsilon}{4\sigma} - 1 + \frac{B\varepsilon c}{2}\right)\int_\Omega |\Delta u_t|^2\,dx \\ &\quad +\left(\varepsilon(\sigma-1)+\frac{B\varepsilon}{2}\right)\int_\Omega |\Delta u|^2\,dx \\ &\quad +\left(\varepsilon-\frac{B\varepsilon}{2}\right)\int_\Omega u^2\ln|u|^2\,dx + \frac{B\varepsilon}{2}\int_\Omega |u|^2\,dx. \end{aligned} \tag{42}$$

Next, applying logarithmic Sobolev inequallity (4) to the fourth term on the right side of (42), we obtain

$$\begin{aligned} L'(t) &\leq -B\varepsilon E(t) + \left(\frac{B\varepsilon c}{2} + \varepsilon c + \frac{\varepsilon}{4\sigma} - 1\right)\int_\Omega |\Delta u_t|\,dx \\ &\quad +\left[\frac{B\varepsilon}{2}+\varepsilon(\sigma-1)\right]\int_\Omega |\Delta u|^2\,dx + \frac{B\varepsilon}{2}\int_\Omega |u|^2\,dx \\ &\quad +\left(\varepsilon-\frac{B\varepsilon}{2}\right)\left(\frac{\alpha^2}{\pi}\int_\Omega |\Delta u|^2\,dx + u^2\ln\|u\|^2 - n\left(1+\ln\alpha\right)\|u\|^2\right) \\ &= -B\varepsilon E(t) + \left[\frac{B\varepsilon}{2}+\varepsilon(\sigma-1)\right]\int_\Omega |\Delta u|^2\,dx \\ &\quad +\frac{B\varepsilon}{2}\int_\Omega |u|^2\,dx + \left(\frac{B\varepsilon c}{2} + \varepsilon c + \frac{\varepsilon}{4\sigma} - 1\right)\int_\Omega |\Delta u_t|^2\,dx \\ &\quad +\left(\varepsilon-\frac{B\varepsilon}{2}\right)\frac{\alpha^2}{\pi}\int_\Omega |\Delta u|^2\,dx + \left(\varepsilon-\frac{B\varepsilon}{2}\right)\int_\Omega u^2\ln\|u\|^2\,dx \\ &\quad -\left(\varepsilon-\frac{B\varepsilon}{2}\right)n\left(1+\ln\alpha\right)\|u\|^2 \\ &= -B\varepsilon E(t) + \left(\frac{B\varepsilon c}{2} + \varepsilon c + \frac{\varepsilon}{4\sigma} - 1\right)\int_\Omega |\Delta u_t|^2\,dx \\ &\quad +\varepsilon\left[\frac{B}{2}+(\sigma-1)+\left(1-\frac{B}{2}\right)\frac{\alpha^2}{\pi}\right]\int_\Omega |\Delta u|^2\,dx \\ &\quad +\varepsilon\left[\frac{B}{2}-\left(1-\frac{B}{2}\right)n\left(1+\ln\alpha\right)+\left(1-\frac{B}{2}\right)\ln\|u\|^2\right]\|u\|^2 \end{aligned} \tag{43}$$

Since $I(u) \geq 0$ and $E(t) \leq E(0) \leq \varsigma M$, we obtain

$$\begin{aligned} \ln \|u\|^2 &\leq \ln (2J(u)) \\ &\leq \ln (2E(t)) \\ &\leq \ln (2E(0)) \\ &\leq \ln (2\varsigma M) \end{aligned} \tag{44}$$

Taking $\varsigma^{\frac{1}{n}}\pi < \alpha < \pi^{\frac{1}{2}}$, $0 < B < 1$, and $\sigma > 0$ small enough such that

$$\frac{B}{2} - \left(1 - \frac{B}{2}\right) n (1 + \ln \alpha) \ln (2\varsigma M) + \left(1 - \frac{B}{2}\right) \ln (2\varsigma M) < 0.$$

Then, choosing $\varepsilon > 0$ small sufficiently such that

$$\frac{B\varepsilon c}{2} + \varepsilon c + \frac{\varepsilon}{4\sigma} - 1 < 0,$$

and

$$\frac{B\varepsilon}{2} + \varepsilon(\sigma - 1) + \varepsilon \left(1 - \frac{B}{2}\right) \frac{\alpha^2}{\pi} < 0.$$

Then, using (39) and (43) leads to

$$L'(t) \leq -\beta\varepsilon E(t) \leq -\frac{\beta\varepsilon}{\alpha_2} L(t). \tag{45}$$

Finally, integrating (45) over $(0, t)$, we get

$$L(t) \leq L(0) e^{-\frac{\beta\varepsilon}{\alpha_2} t}$$

where we can deduce that there exist $c_1 = L(0)$ and $c_2 = \frac{\beta\varepsilon}{\alpha_2}$ such that $t \geq 0$

$$E(t) \leq c_1 e^{-c_2 t}$$

This completes the proof of the theorem.

4 Blow up at infinite time

In this section, we discuss the blow up property in infinite time of solutions of the problem (1)-(3).

Theorem 4.1. Let $u_0 \in H_0^2(\Omega)$, $u_1 \in L^2(\Omega)$, $E(0) \leq 0$, and $I(u_0) < 0$. Then the weak solution of the problem (1)-(3) grows up as an exponential function as time goes to inifinity.

Proof. Let

$$\begin{aligned} G(t) &= \|u(t)\|^2 + \int_0^t \|\Delta u(\tau)\|^2 d\tau + (T - t) \|\Delta u_0\|^2 \\ &= \int_\Omega |u(t)|^2 dx + \int_0^t \int_\Omega |\Delta u(\tau)|^2 dx d\tau + (T - t) \int_\Omega |\Delta u_0|^2 dx \end{aligned}$$

for any $0 \leq t < T$.

Direct computations show that

$$G'(t) = 2\int_\Omega u_t u dx + 2\int_0^t \int_\Omega \Delta u(\tau)\Delta u_\tau(\tau) dx d\tau - \|\Delta u_0\|^2. \tag{46}$$

Therefore, by the equation (1), and the definition of $I(u)$, we get

$$\begin{aligned} G''(t) &= 2\int_\Omega (u_{tt}u + |u_t|^2) dx + 2\int_\Omega \Delta u \Delta u_\tau dx d\tau \\ &= 2\int_\Omega \left[\left(u\ln|u|^2 - \Delta^2 u - \Delta^2 u_t \right) u + |u_t|^2 \right] dx + 2\int_\Omega \Delta u \Delta u_t dx \\ &= 2\int_\Omega u^2 \ln|u|^2 dx - 2\int_\Omega u\Delta^2 u dx - 2\int_\Omega u\Delta^2 u_t dx \\ &\quad +2\int_\Omega |u_t|^2 dx + 2\int_\Omega \Delta u \Delta u_\tau dx \\ &= 2\int_\Omega u^2 \ln|u|^2 dx - 2\int_\Omega |\Delta u|^2 dx + 2\int_\Omega |u_t|^2 dx \\ &= -2I(u) + 2\int_\Omega |u_t|^2 dx. \end{aligned} \tag{47}$$

Then, we have

$$\begin{aligned} G''(t)G(t) - (G'(t))^2 = 2G(t)\left(\int_\Omega |u_t|^2 dx - \int_\Omega |\Delta u|^2 dx + \int_\Omega u^2 \ln|u|^2 dx \right) \\ + 4\left[\eta(t) - \left(G(t) - (T-t)\|\Delta u_0\|^2 \right) \right. \\ \left. . \left(\|u_t(t)\|^2 + \int_0^t \|\Delta u_\tau(\tau)\|^2 d\tau \right) \right] \end{aligned} \tag{48}$$

where

$$\begin{aligned} \eta(t) &= \left(\|u\|^2 + \int_0^t \|\Delta u_\tau(\tau)\|^2 d\tau \right) . \left(\|u_t\|^2 + \int_0^t \left\| \Delta u_\tau(\tau)^2 \right\| d\tau \right) \\ &\quad - \left(\int_\Omega u u_t dx + \int_0^t \int_\Omega \Delta u \Delta u_\tau dx d\tau \right)^2. \end{aligned}$$

Using the following Cauchy-Schwarz inequality

$$(ab + cd)^2 \leq \left(a^2 + c^2\right) . \left(b^2 + d^2\right),$$

we get

$$\begin{aligned} &\left(\int_\Omega u u_t dx + \int_0^t \int_\Omega \Delta u \Delta u_\tau dx d\tau \right)^2 \\ \leq\ &\left(\|u\|^2 + \int_0^t \|\Delta u_\tau(\tau)\|^2 d\tau \right) \left(\|u_t\|^2 + \int_0^t \|\Delta u_\tau(\tau)\|^2 d\tau \right). \end{aligned}$$

Then, it is not difficult to verify that $\eta(t) \geq 0$ for $0 \leq t \leq T$. Then by (48), we have

$$G''(t)G(t) - \left(G'(t)\right)^2 \geq G(t)\xi(t) \tag{49}$$

where

$$\begin{aligned}
\xi(t) &= -2\int_0^t |u_t|^2 dx - 2\int_\Omega |\Delta u|^2 dx \\
&\quad +2\int_\Omega |u|^2 \ln |u|^2 dx - 4\int_0^t \|\Delta u_\tau(\tau)\|^2 d\tau \\
&= -4E(t) + 2\int_\Omega |u|^2 dx - 4\int_0^t \|\Delta u_\tau(\tau)\|^2 d\tau \\
&\leq -4E(0) + 2\int_\Omega |u|^2 dx.
\end{aligned} \tag{50}$$

Since $E(0) \leq 0$ and $I(u_0) < 0$, by virtue of (50), one can know that $\xi(t) \geq 0$, then we have

$$G''(t)G(t) - \left(G'(t)\right)^2 \geq 0. \tag{51}$$

On the other hand, by direct calculation, we get

$$\left(\ln |G(t)|\right)' = \frac{G'(t)}{G(t)}, \tag{52}$$

$$\begin{aligned}
\left(\ln |G(t)|\right)'' &= \left(\frac{G'(t)}{G(t)}\right)' \\
&= \frac{G''(t)G(t) - [G'(t)]^2}{G^2(t)} > 0.
\end{aligned} \tag{53}$$

By (53), we know that $\left(\ln |G(t)|\right)'$ is increasing, using this fact, integrating (52) over (t_0, t), we get

$$\begin{aligned}
\ln |G(t)| - \ln |G(t_0)| &= \int_{t_0}^t \left(\ln |G(\tau)|\right)' d\tau \\
&= \int_{t_0}^t \frac{G'(\tau)}{G(\tau)} d\tau \\
&\geq \frac{G'(t_0)}{G(t_0)}(t - t_0).
\end{aligned}$$

Then, we have

$$G(t) \geq G(t_0) e^{\frac{G'(t_0)}{G(t_0)}(t-t_0)}. \tag{54}$$

If we choose t_0 sufficiently small such that $G(t_0) > 0$ and $G'(t_0) > 0$, then we have

$$\lim_{t\to\infty} G(t) = \infty.$$

References

[1] J. Barrow and P. Parsons, *Inflationary models with logarithmic potentials*, Physical Review D, **52** (1995), 5576-5587.

[2] I. Bialynicki-Birula and J. Mycielski, *Nonlinear wave mechanies*, Annals of Physics, **100**(1-2) (1976), 62-93.

[3] P. Gorka, *Logarithmic Klein-Gordon equation*, Acta Physica Polonica B, **40**(1) (2009), 59-66.

[4] T. Hiramatsu, M. Kawasaki and F. Takahashi, *Numerical study of Q-ball formationin gravity-mediation*, Journal of Cosmology and Astroparticle Physics, **6**(8) (2010), 1-34.

[5] L. Ma and Z.B. Fang, *Energy decay estimates and infinite blow-up phenomena for a strongly damped semilinear wave equation with logarithmic nonlinear source*, Mathematical Methods in the Applied Sciences, *41* (2018), 2639-2653.

[6] M.M. Al-Gharabli and S.A. Messaoudi, *Existence and a general decay result for a plate equation with nonlinear damping and a logarithmic source term*, Journal of Evolution Equations,**18**(1) (2018), 105-125.

[7] S.A. Messaoudi, *Global existence and nonexistence in a system of Petrovsky*, Journal of Mathematical Analysis and Applications**, 265**(2) (2002), 296-308.

[8] E. Pişkin and N. Polat, *On the decay of solutions for a nonlinear Petrovsky equation*, Mathematical Sciences Letters, **3**(1) (2014), 43-47.

[9] S.T. Wu and L.Y. Tsai, *On global solutions and blow up of solutions for a nonlinearly damped Petrovsky system*, Taiwanese Journal of Mathematics, **13**(2A) (2009), 545-558.

[10] M.M. Al-Gharabli, A. Guesmia and S.A. Messaoudi, *Well-posedness and asymptotic stability results for a viscoelastic plate equation with a logarithmic nonlinearity*, Applicable Analysis, **99**(1) (2020), 50-74.

[11] S. Boulaaras, A. Draifia and M. Alnegga, *Polynomial Decay Rate for Kirchhoff Type in Viscoelasticity with Logarithmic Nonlinearity and Not Necessarily Decreasing Kernel*, Symmetry, **11** (2019), 1-24.

[12] H. Chen, P. Luo and G.W. Liu, *Global solutions and blow-up of a semilinear heat equation with logarithmic nonlinearity*, Journal of Mathematical Analysis and Applications, **422** (2015), 84-98.

[13] Q. Hu, H. Zhang and G. Liu, *Asymptotic behaviour for a class of logarithmic wave equations with linear damping*, Applied Mathematics and Optimization, **79** (2019), 131-144.

[14] W. Lian, M.S. Ahmed and X. Runzhang, *Global existence and blow up of solution for semilinear hyperbolic equation with logarithmic nonlinearity*, Nonlinear Analysis, **184** (2019), 239-257.

[15] R. Xu, W. Lian, X. Kong and Y. Yang, *Fourth order wave equation with nonlinear strain and logarithmic nonlinearity*, Applied Numerical Mathematics, **141** (2019), 185-205.

[16] Y. Yang, J. Li and T. Yu, *Qualitative analysis of solutions for a class of Kirchhoff equation with linear strong damping term, nonlinear weak damping term and power-type logarithmic source term*, Applied Numerical Mathematics, **141** (2019) 263-285.

[17] L. Gross, *Logarithmic Sobolev inequalities*, American Journal of Mathematics, **97**(4) (1975) 1061-1083.